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# AN ANALYSIS OF THE PRIMARY AND SUPERHARMONIC CONTACT RESONANCES – PART 2

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This paper presents results of investigations of non-linear normal contact micro-vibrations excited by a harmonic force in a system of two bodies in planar contact. The system models, for instance, slide units of machine tools or their positioning systems. The main aim of the computational analysis is to present resonance graphs and time histories obtained with numerical and perturbation methods. Good agreement between the perturbation and numerical results leads to the conclusion that the perturbation solution is correct. The obtained perturbation solution describes well both the primary resonance and the superharmonic resonances. Characteristic phenomena typical for non-linear vibrations are depicted, viz. asymmetry of vibrations, multi-harmonic vibrations, non-elliptical phase portraits, loss of contacts, bending resonance peaks, bi-stabilities, and multi-stability.

*Key words:* non-linear contact, vibrations, superharmonic resonance, multi-stability

## 1. Introduction

The contact of rough surfaces is flexible, which affects behaviour of machines. The contact flexibility particularly affects the static deflections and natural frequencies of machine tools; because contact deflections of sliding guideways can be larger than distortions of the machine parts (Chlebus and Dybala, 1999; Fan *et al.*, 2012; Gutowski, 2003; Kaminskaya *et al.*, 1960; Marchelek, 1974; Shi and Polycarpou, 2005; Thomas, 1999). In consequence, contact flexibility affects vibrations of machine tools (Dhupia *et al.*, 2007; Fan *et al.*, 2012; Gutowski, 2003; Huo *et al.*, 2010; Marchelek, 1974; Moradi *et al.*, 2010; Neugebauer *et al.*, 2007). In this context, contact vibrations and contact resonances are important.

The primary contact resonance and superharmonic resonance were studied experimentally (Chajkin *et al.*, 1939; Hess and Soom, 1992; Grigorova and Tolstoi, 1966; Kostek, 2004; Rigaud and Perret-Liaudet, 2003; Perret-Liaudet and Rigad, 2007; Tolstoi, 1967). The obtained experimental results show the non-linear nature of contact vibrations, bi-stability and multi-harmonic vibrations. Moreover, the contact vibrations were simulated with numerical methods (Grudziński and Kostek, 2007; Kligerman, 2003; Kostek, 2004, 2013b; Rigaud and Perret-Liaudet, 2003). These methods provide an opportunity to study superharmonic resonances, bifurcations, and chaotic vibrations, which are difficult to observe experimentally.

Finally, the contact vibrations can be studied with the perturbation method. This method provides an opportunity to study periodic vibrations, hence the primary contact resonance and superharmonic resonances can be computed. Still, a few works present the perturbation solution of contact vibrations or similar vibrations (Hess and Soom, 1991a,b; Kostek, 2013a; Nayak, 1972; Nayfeh, 1985; Perret-Liaudet, 1998, Perret-Liaudet and Rigad, 2007). Usually the perturbation solutions hold only for the primary contact resonance or one superharmonic resonance. Thus, there is a need to test for a wide range of excitation frequencies and amplitudes, the perturbation solution of the fourth order, which is presented in the first part of this article (Kostek, 2013a). The results obtained with the perturbation method are compared with numerical results.

In addition, the second aim of this paper is to present and analyse the contact vibrations excited for various amplitudes and frequencies of the excitation force.

## 2. Theoretical fundamentals

The studied system has been presented previously in the first part of this publication, thus certain information is repeated briefly in this section. The considered system consists of two bodies in planar contact – a rigid block (slider), resting on a massive rigid base (slideway) (Fig. 1). The interface of the bodies, which represents the planar contact of rough surfaces, is modelled with a non-linear spring and damper (Hunt and Crossley, 1975; Kostek, 2004; Martins *et al.*, 1990). Four forces act on the slider, they are: the exciting force $P$, spring force $F_s$, damping force $F_d$, and terrestrial gravity force $Q$. These forces and motion of the slider are described by the following equations

$$\begin{array}{llllll} \text{IF} & y > 0 & \text{THEN} & F_s = -Sc_n y^{m_2}, \quad \delta = y & \text{ELSE} & F_s = 0, \quad \delta = 0 \\ \text{IF} & y > 0 & \text{THEN} & F_d = -Sh_n y^l \dot{y} & \text{ELSE} & F_d = 0 \end{array} \tag{2.1}$$

and

$$\begin{aligned} &P = P_a \cos(2\pi f_e t) \qquad Q = Mg \qquad F_k = F_s + Q \\ &\ddot{y} = M^{-1}\Big(F_k(y) + F_d(y, \dot{y}) + P(t)\Big) \end{aligned} \tag{2.2}$$

where $y$ denotes the displacement, $F_s$ the spring force, $F_d$ the damping force, $S$ the apparent contact area $S = 0.0009\,\mathrm{m}^2$, $c_n$, $m_2$, $h_n$ and $l$ are the parameters of the contact interface $c_n = 4.52693E16\,\mathrm{N/m}^4$, $m_2 = 2$, $h_n = 3.5E11\,\mathrm{Ns/m}^4$, $l = 1$, $\delta$ is the normal contact deflection, whereas $P$ denotes the exciting force, $P_a$ the amplitude of the exciting force, $f_e$ the frequency of excitation, $t$ time, $Q$ the terrestrial gravity force, $M$ mass of the slider $M = 0.2106\,\mathrm{kg}$, and $g$ the acceleration of gravity $g = 9.81\,\mathrm{m/s}^2$, and finally $F_k$ is the conservative force. The sum of the spring force $F_s$ (Fig. 1b) and the terrestrial gravity force $Q$ is the conservative force $F_k$. The graph of the conservative force is non-linear (Fig. 1c). More information is presented in the first article.

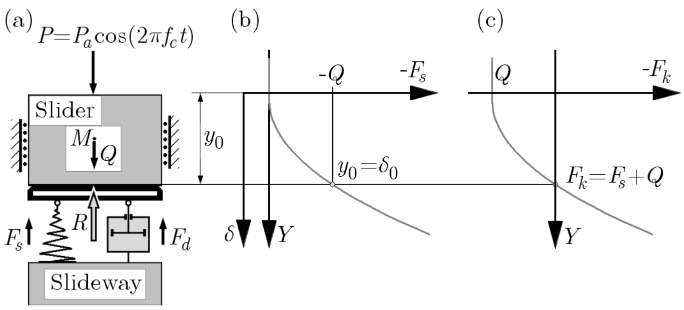


Fig. 1. Model of the considered dynamical system (a), characteristic of the spring force $F_s$ (b), characteristic of the conservative force $F_k$ (c), $y_0$ and $\delta_0$ denotes the equilibrium position

## 3. Contact resonances

The resonance characteristics obtained for the studied system are depicted in Fig. 2. In this figure, local minima and maxima of a time history are presented against excitation frequency. A number of resonance peaks is observed, for the studied single degree-of-freedom non-linear system. This is common for non-linear systems, because many resonances are excited (Awrejcewicz, 1996; Fyrillas and Szeri, 1998; Grudziński and Kostek, 2007; Kostek, 2004; Nayfeh and Mook, 1995; Parlitz and Lauterborn, 1985; Thompson and Stewart, 2002). The primary resonance and the superharmonic resonances are visible in Fig. 2. The resonances take place when the exciting frequency $f_e$ is in the following relation to the natural frequency $f_{n0}$

$$f_e \approx \frac{1}{m} f_{n0} \tag{3.1}$$

where $f_{n0}$ denotes the natural frequency of a linearised system, whereas $m$ is a positive integer. If $m = 1$, then the primary resonance is excited, whereas for $m > 1$ the superharmonic resonances are excited. If the frequency of the $m$-th harmonic of vibrations approaches the natural frequency, then the amplitude of the $m$-th harmonic is rising; this is the nature of superharmonic resonances. Thus for 1/2 superharmonic resonance the second harmonic is amplified, $f_e \approx (1/2)f_{n0}$.

The resonance characteristics obtained with numerical and perturbation methods are depicted in Fig. 2. The perturbation solution is simplified because the linear model of damping is adopted; moreover, the loss of contact is neglected, and therefore the numerical solution is more accurate than the perturbation solution. Nevertheless, the perturbation solution describes well the contact vibrations. The contact vibrations are asymmetrical to the equilibrium position $y_0$, thus the resonance graphs are asymmetrical to the equilibrium position too. Moreover, the natural frequency is a function of the vibration amplitude, hence the primary resonance peak bends, and bi-stability area is visible in Fig. 2. Further, the perturbation solution describes well the primary resonance, even for loss of contact – "gapping". Furthermore, the 1/2, 1/3 and 1/4 superharmonic resonances are well reflected. Nevertheless, the solution of order higher than fourth is necessary to reflect the well bending of the 1/2 superharmonic resonance peak (Fig. 2c). Finally, the perturbation and numerical solutions show good agreement for under resonant (quasistatic) vibrations. This issue will be described in the next article and was discussed in Skrodzewicz (2003).

Summarising, good agreement between the perturbation and numerical solutions leads to the conclusion that the perturbation solution is correct. The time histories of contact vibrations are analysed in the next section, which provides valid information about contact resonances.

## 4. An analysis of contact vibrations

The primary resonance is sensitive to non-linearity. Its peak rises quickly, and with the amplitude of excitation being $P_a = 0.05Q$, a bi-stability area appears (Fig. 2a). The contact micro-vibrations of the primary resonance excited at frequency $f_e = 1410\,\mathrm{Hz}$ are asymmetrical. The feature is visible in both time histories of the displacement (Fig. 3a) and acceleration (Fig. 3c). Accordingly, the phase portrait is not elliptical and has a slightly asymmetrical shape (Fig. 3b), which is coupled with the multi-harmonic character of the vibrations (Fig. 3d). The phenomena result from the non-linearity of the system. Finally, the perturbation solution well describes the time history of displacement (Fig. 3a).

The 1/2 superharmonic resonance excited by the external force of the frequency $f_e = 743\,\mathrm{Hz}$ and amplitude $P_a = 0.05Q$ is not very clear in the resonance graph (Fig. 2a) nor in the time history of the displacement (Fig. 4a). Nevertheless, the phenomenon is visible in the spectrum, where the second harmonic appears (Fig. 4d). Moreover, this resonance is visible in the time history of acceleration (Fig. 4c), where two local minima and maxima take place during one period of vibration. At the same time, the shape of the phase portrait is far from the ellipsoidal one (Fig. 4b), which is typical for non-linear vibrations. The 1/2 superharmonic resonance is an effect of the non-linearity and grows with the increase of the excitation amplitude. Summarising,

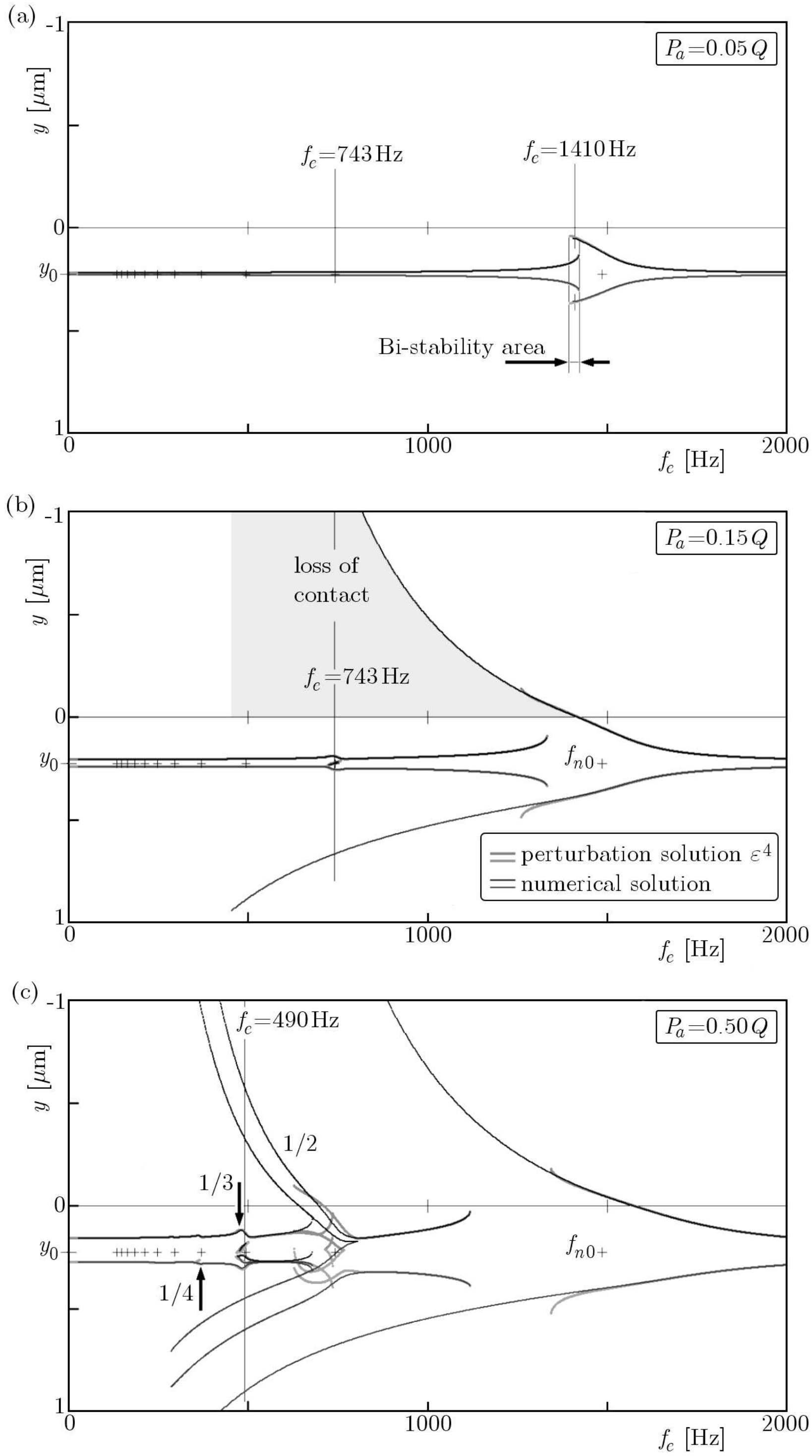


Fig. 2. Graphs of the contact resonances; local extrema of time histories against the frequency of excitation $f_e$; plus signs (+) represent resonant frequencies $f_e \approx f_{n0}/m$

the perturbation solution shows excellent agreement with the numerical results for this excitation (Fig. 4a).

The area of bi-stability becomes larger with a further increase of the excitation amplitude. This is a result of bending of the primary resonance peak and an increase of the primary resonance amplitude. Therefore, for the excitation amplitude being $P_a = 0.15Q$, the primary resonance and the 1/2 superharmonic resonance can be excited at the same frequency, $f_e = 740\,\text{Hz}$

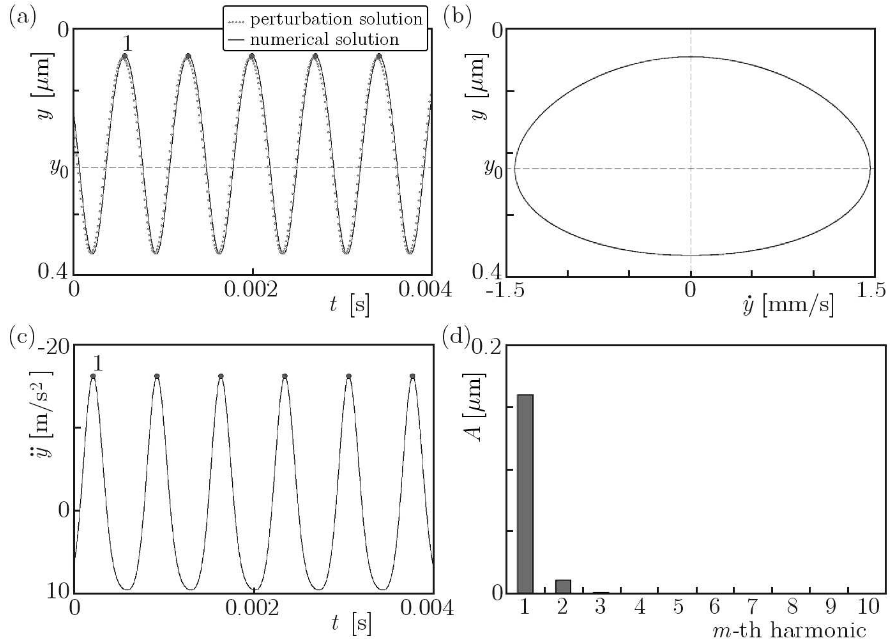


Fig. 3. Time histories of the displacement (a), and acceleration (c), phase portrait (b), and spectrum (d) of the primary resonance vibrations excited by a harmonic force of the parameters $P_a = 0.05Q$, $f_e = 1410\,\mathrm{Hz}$

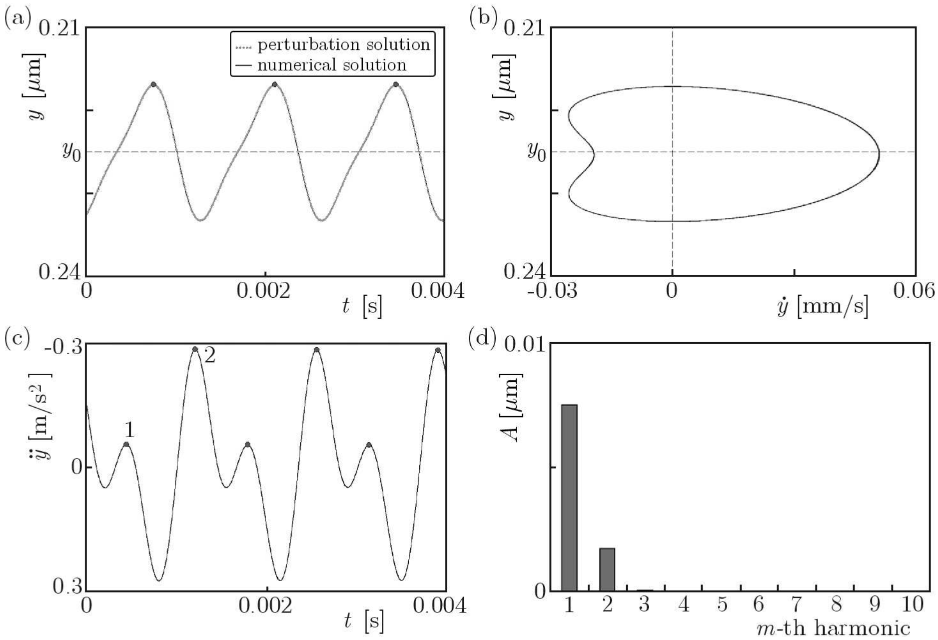


Fig. 4. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the 1/2 superharmonic resonance vibrations excited by a harmonic force of the parameters $P_a = 0.05Q$, $f_e = 743\,\mathrm{Hz}$

(Fig. 2b). Asymmetry and "gapping" of the primary resonance vibrations are visible both in the resonance graph (Fig. 2b) and in the time history of displacement (Fig. 5a). The loss of contact is coupled with high acceleration of the slider (Fig. 5c) and significant loading of the slideway, which result from hitting the slider in the slideway. The phase portrait of the vibrations has a characteristic shape, typical for the contact vibrations (Fig. 5b), whereas the spectrum (Fig. 5d) clearly reveals the increase of higher harmonics amplitudes.

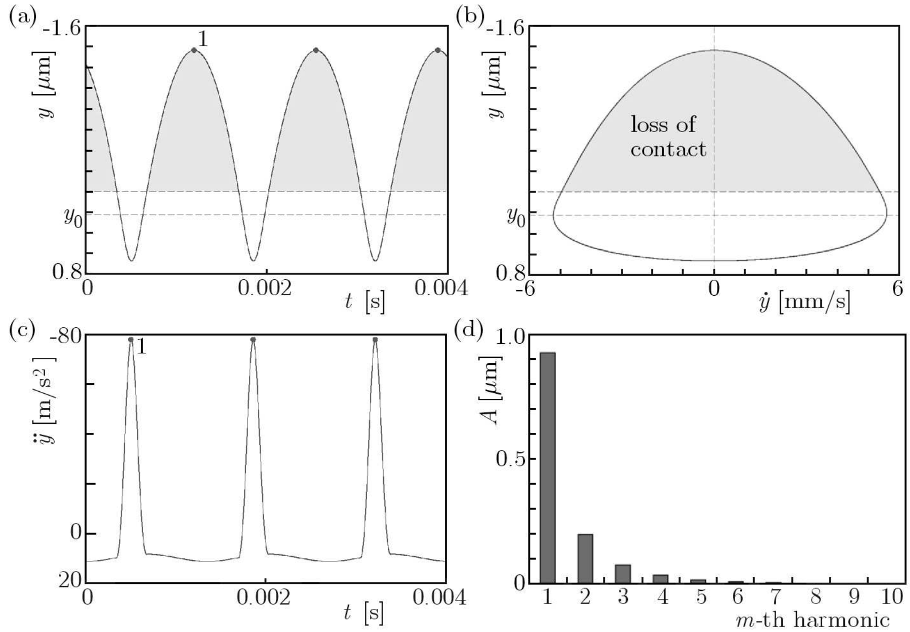


Fig. 5. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the primary resonance vibrations excited by a harmonic force of the parameters $P_a = 0.15Q$, $f_e = 740\,\mathrm{Hz}$

The afore-mentioned 1/2 superharmonic resonance grows with the increase of the excitation amplitude, thus the two local minima and maxima are observed during one period, both in the time histories of displacement (Fig. 6a) and acceleration (Fig. 6c). Consequently, the second harmonic is larger than previously shown (Fig. 6d), the phase portrait has a more sophisticated shape (Fig. 6b), and four curves present the resonance in Fig. 2b. In spite of the fact that the vibration amplitude is larger, the perturbation solution holds (Fig. 6a).

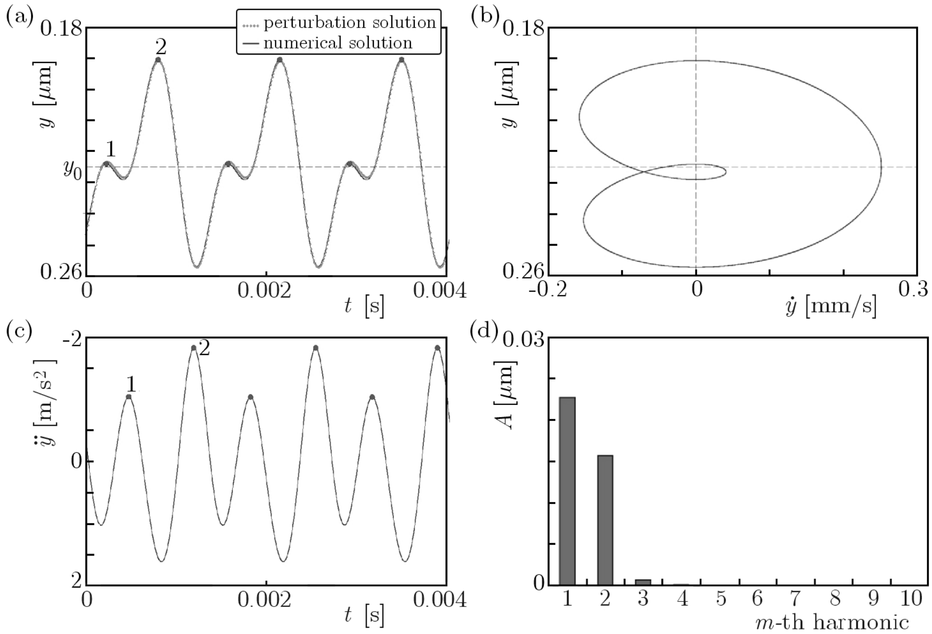


Fig. 6. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the 1/2 superharmonic resonance vibrations excited by a harmonic force of the parameters $P_a = 0.15Q$, $f_e = 740\,\mathrm{Hz}$

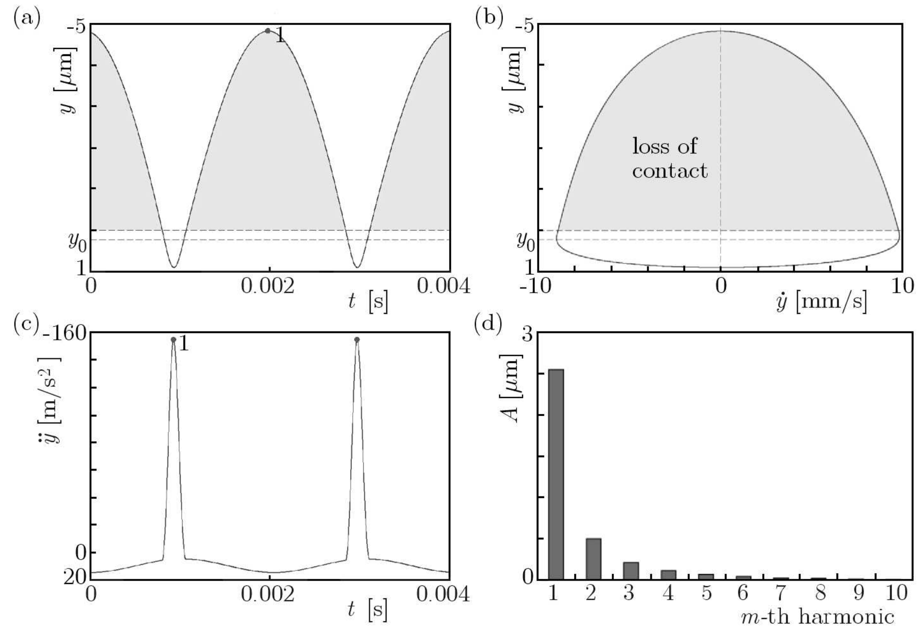


Fig. 7. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the primary resonance vibrations excited by a harmonic force of the parameters $P_a = 0.50Q$, $f_e = 490\,\mathrm{Hz}$

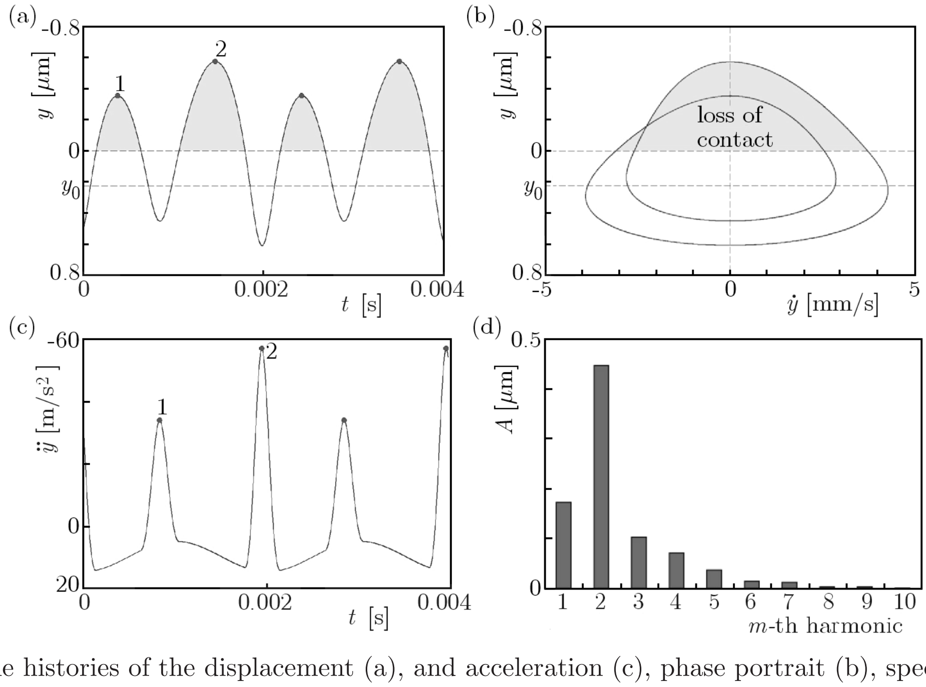


Fig. 8. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the 1/2 superharmonic resonance vibrations excited by a harmonic force of the parameters $P_a = 0.50Q$, $f_e = 490\,\mathrm{Hz}$

The further increase of the excitation amplitude leads to bending of the 1/2 superharmonic resonance peak, which in turn introduces multi-stability to the system. In this case, three solutions are possible: the three resonances are excited, viz. the primary resonance, the 1/2 and 1/3 superharmonic resonances for $P_a = 0.50Q$ and $f_e = 490\,\mathrm{Hz}$ (Fig. 2c). Analysis of the vibrations reveals the complexity of their kinematics. The time histories of the displacement and acceleration show that a number of minima and maxima during one period is linked with the integer $m$ (Figs. 7a, 7c, 8a, 8c, 9a, 9c). Consequently, for the primary resonance one minimum and maxi-

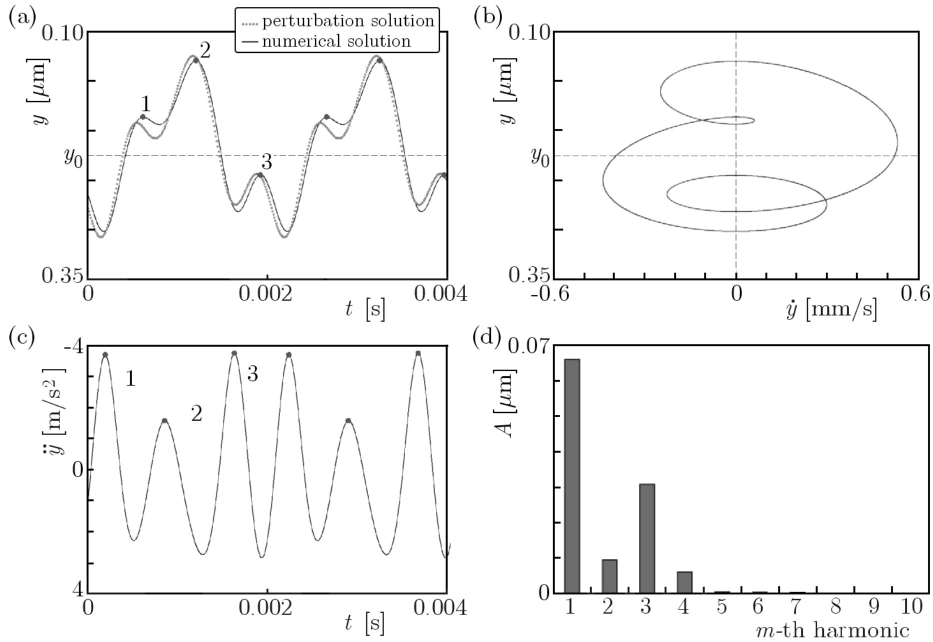


Fig. 9. Time histories of the displacement (a), and acceleration (c), phase portrait (b), spectrum (d) of the 1/3 superharmonic resonance vibrations excited by a harmonic force of the parameters $P_a = 0.50Q$, $f_e = 490\,\text{Hz}$; three local minima (1, 2, 3) are observed over one cycle of the excitation (a), (c)

mum are observed during one period, thus two curves are present in the primary resonance in Fig. 2c. Whereas for the 1/2 superharmonic resonance, two minima and maxima are observed during one period, thus four lines are present in the resonance in Fig. 2c. Finally, six curves are presented in the 1/3 superharmonic resonance in Fig. 2c. In addition, analysis of the spectrums shows that the amplitude $A$ of the $m$-th harmonic is significant (Figs. 7d, 8d, 9d). Also, at the same time, the shape of the phase portraits becomes more sophisticated with the increase of the integer $m$. For the primary resonance, a point rounds the origin once during one period (Fig. 7b); while for the 1/2 superharmonic resonance one rounds the origin twice (Fig. 8b). Additionally, for the 1/3 superharmonic resonance, two additional loops are observed in the phase portrait (Fig. 9b). Summarising this, the contact micro-vibrations are complex and their non-linearity can lead to "gapping", superharmonic resonances, bi-stability and multi-stability. Finally, the perturbation solution reflects the nature of the 1/3 superharmonic resonance (Fig. 9a).

## 5. Conclusions

The presented study has been focused on comparison of the results obtained with the perturbation and numerical methods. The perturbation solution well describes both the primary resonance and the superharmonic resonances, which is an important feature. Good agreement between the perturbation and numerical results leads to the conclusion that the perturbation solution is correct.

The contact vibrations are non-linear, thus characteristic phenomena are observed. The asymmetrical and non-linear conservative force introduces asymmetry of contact vibrations. Hence, the notion of the amplitude should be precisely defined. Moreover, for large excitation amplitudes, the loss of contact is observed. Beside, non-linearity of the conservative force introduces higher harmonics, thus contact vibrations are multi-harmonic. In consequence, kinematics of contact vibrations is complex, shapes of the phase portraits are non-elliptical, and local minima and maxima are observed in the time histories of displacement. Finally, the multi-harmonic nature of contact vibrations introduces superharmonic resonances, which result from magnification of higher harmonics.

For the studied system, the natural frequency is a function of the vibration amplitude, which introduces bending of the resonance peaks, bi-stability and finally multi-stability. The bi-stability is a result of bending of the primary resonance peak, whereas the multi-stability is a result of bending of the superharmonic resonance peaks.

Summarising, the contact vibrations are far more complex than linear vibrations. Thus, non-linear models of contact should be used in machine tools dynamics. Moreover, non-linear systems should be investigated under a wide range of excitation signals, because many resonances can be excited.

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