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# VIBRATION AND STABILITY OF A COLUMN SUBJECTED TO THE GENERALISED LOAD BY A FORCE DIRECTED TOWARDS THE POLE

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A new loading of slender systems, which is a generalised load by a force directed towards the positive or negative pole, is presented in the paper. A new constructional scheme of the loading and receiving heads is introduced. Theoretical considerations connected with the formulation of boundary conditions are shown in this paper. Values of the critical force and the course of natural frequency in relation to the external load for given geometry and physical constants of the column are determined dependending on the constructional solution of the loading and receiving heads. The results of theoretical and experimental research are compared one with another.

*Key words:* elastic column, divergence instability, natural frequency

## 1. Introduction

### 1.1. Generalised load

Let us take into account the slender system shown in Fig. 1. The flexural rigidity of the prismatic column is denoted as $EJ$, mass density as $\rho_1 A$ (where $E$ is the longitudinal modulus of elasticity, $J$ – moment of inertia related to the neutral axis in the bending plane, $\rho_1$ – material density, $A$ – cross-section area) $\varphi$ – angle between the direction of the force $P$ and tangent to the column in the point $x = l$, $W(x,t)$ – transverse displacement of the column, $m$ – concentrated mass at the free end of the column.

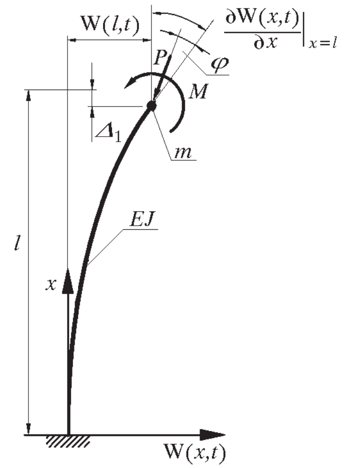


Fig. 1. Scheme of a cantilever column subjected to a generalised load

The column is rigidly fixed $x = 0$. It is loaded by a concentrated force $P$ and bending moment $M = Pe$. The boundary conditions for this structure at the free end of the column are as follows

$$
\begin{aligned}
&EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} + Pe = 0 \\
&EJ\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} + P\varphi - m\frac{\partial^2 W(x,t)}{\partial t^2}\Big|_{x=l} = 0
\end{aligned}
\tag{1.1}
$$

The conditions of the column fixing are

$$
W(0,t) = 0 \qquad \frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{1.2}
$$

The load of the cantilever column (without giving a constructional solution to the forcing and leading heads), which recalls the following dependence between the transverse force and bending moment at the end of the column $x = l$ and the displacement and deflection angle of this end

$$
e = \rho\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \nu W(l,t) \qquad \varphi = \mu\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \gamma W(l,t) \tag{1.3}
$$

where $\rho$, $\nu$, $\mu$, $\gamma$ are determined coefficients.

Such a load is so-called the generalised load (compare Gajewski and Życzkowski, 1970, 1988; Kordas, 1963).

It should be stated that all known loads of slender systems have determined values of the $\rho$, $\nu$, $\mu$, $\gamma$ coefficients. It applies both to conservative and non-conservative loads. Only the load by a follower force directed towards a pole makes an exception, which will be presented in a further part of this paper.

The load is conservative if the gradient rotation of its vector field is equal to zero, which leads to the relation (compare Gajewski and Życzkowski, 1970, 1988; Kordas, 1963)

$$\nu + \mu - 1 = 0 \tag{1.4}$$

The load is non-conservative if the above relation is not fulfilled.

The extended conservative condition, resulted from the self-adjointness condition of differential operators for the generalised load, was obtained in the paper by Tomski *et al.* (1996) in the form

$$(\mu + \nu - 1)\Big(W_n(l,t)\frac{\partial W_m(x,t)}{\partial x}\Big|_{x=l} - W_m(l,t)\frac{\partial W_n(x,t)}{\partial x}\Big|_{x=l}\Big) = 0 \tag{1.5}$$

where the indices $n$, $m$ denote the $n$th and $m$th form of vibrations.

If $W(l,t)$ and $(\partial W(x,t)/\partial x)\Big|_{x=l}$ are linearly dependent, the second factor in relationship (1.5) is set at zero.

### 1.2. Hirtherto existing nomenclature related to loads of slender systems

The following definitions related to loads of slender systems, which will be used in the nomenclature of a characteristic load, are be given in this part of the paper.

**(01)** Generalised load – transverse force and bending moment at the free end of the column depending on the displacement and deflection angle of this end (compare Gajewski and Życzkowski, 1969, 1988; Kordas, 1963).

**(02)** Follower force load – angle of the direction of this force action is equal to the deflection angle at the free end of the column (compare Beck, 1953; Bogacz and Janiszewski, 1986; Bolotin, 1963).

**(03)** Load developed by a force directed towards the positive pole – fixed point, through which the direction of the force action passes, is placed below the free end of the column (compare Gajewski and Życzkowski, 1969; Timoshenko and Gere, 1961).

**(04)** Load developed by a force directed towards the negative pole – fixed point, through which the direction of the force action passes, is placed above the free end of the column (compare Gajewski and Życzkowski, 1969).

### 1.3. Shapes of natural frequency curves versus applied load in stender systems

Slender elastic systems (columns, frames) can loose their stability due to buckling (compare Gajewski and Życzkowski, 1969; Leipholz, 1974; Timoshenko and Gere, 1961; Ziegler, 1968). These systems are called divergence systems. Systems, which loose the stability due to growing amplitudes of oscillatory vibrations, are called flutter systems (compare Beck, 1953; Bogacz and Janiszewski, 1986; Bolotin, 1963). Hybrid systems also exist. They loose the stability in a flutter or divergence way due to a combination of certain geometrical and physical parameters (compare Argyris and Symeonidis, 1981; Dzhanelidze, 1958; Sundararajan, 1973, 1976; Tomski *et al.*, 1985, 1990).

The mentioned above systems are characterised by a specific course of curves in the plane: load $P$ – natural frequency $\omega$ (Fig. 2) (where $P_c$ is the critical load).

It was proved in the work by Leipholz (1974) that for Euler's load (load with a fixed point of its application and fixed point of action the curve of natural frequency in the plane: load-natural frequency always has a negative slope (Fig. 2a).

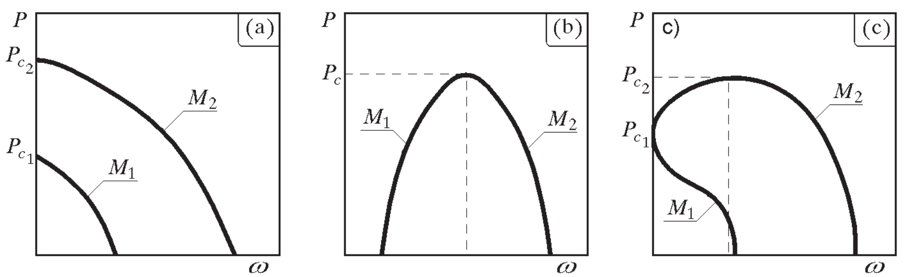


Fig. 2. (a) Divergence system, (b) flutter system, (c) hybrid system

### 1.4. Scope of the paper

The following topics are to be addressed throughout the paper:

- Names of characteristics loads within the range of nomenclature related to loads of slender system known in the literature are determined. The characteristic load is the load which was found at the Institute of Mechanics and Machine Design Foundations at the Technical University of Częstochowa.

- Realisation of the specific load in two alternative constructions of the loading and receiving heads, namely heads built from linear and circular elements (constant curvature) is presented.
- The geometric boundary parameters and resulting of natural frequencies are presented in the case of heads built from the circular elements.
- It is impossible to determine the boundary conditions on the basis of differential relationships between external and internal forces in systems with specific loads. It is the reason for a detailed diagram presentation of how to reveal the potential energy to determine the standard equation of vibrations and boundary conditions according to Hamilton's principle.
- Theoretical and simulation examinations, concerning the natural frequency and column stability with the generalised load by the force directed towards the negative and positive pole in the case of heads built from the circular elements, are the main part of the paper. The results of the theoretical and simulation studies of a column subjected to the generalised load by a force directed towards the positive pole are verified experimentally.

## 2. Specific load

### 2.1. Nomenclature

With respect to the properties of loads acting on the slender system (compare Section 1.2), the authors considered in their investigations the loads whose nomenclature originated from the scheme shown in the following diagram (Fig. 3).

Denotations $(\cdot)^*$ and $(\cdot)^{**}$ in the above diagram are connected with different constructions of the characteristic load heads. This type of load could be realised by the linear elements $(\cdot)^*$ or circular elements $(\cdot)^{**}$. Appropriate schematic diagrams concerning the way of application will be presented in the further part of this paper.

### 2.2. Realisation of load – schematic diagrams

Schematic diagrams of the heads realising the characteristic load are presented in Fig. 4 and Fig. 5. The loading heads of the column, shown in Fig. 4, are made of the linear elements with $l_c$ and $l_d$ in length. The heads, shown in Fig. 5, are built of component elements with the radii $R$ and $r$.

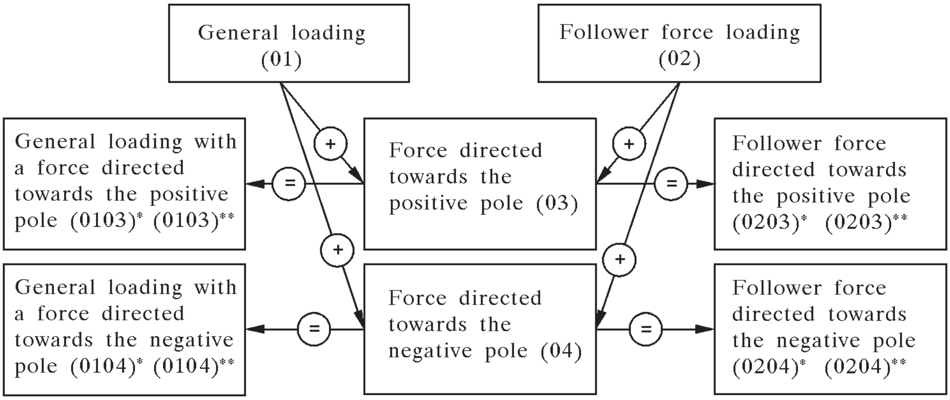


Fig. 3. Diagram specifying the origin of names of the characteristic load

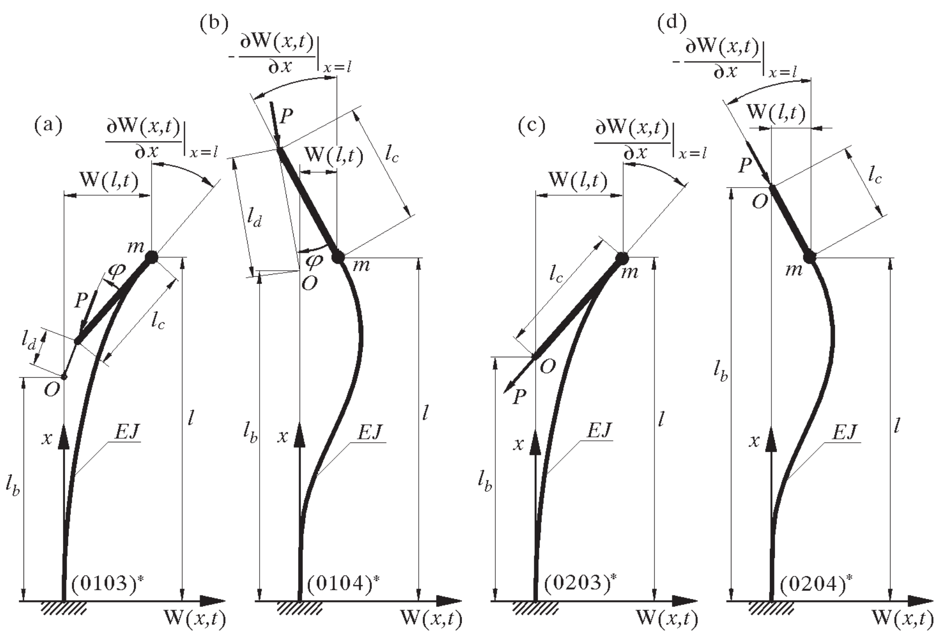


Fig. 4. Schemes of heads realising characteristic loads ((0103)*, (0104)*, (0203)*, (0204)*)

The basic features of the structures loading the columns are:

- The head built of the linear elements $(0103)^{*}$, $(0104)^{*}$ – Fig. 4):

  – part of the length $l_c$ has infinite flexural rigidity,
  – part of the length $l_d$ is loaded by the external force $P$ which is directed along the axis of the underformed column (point "0"),
  – point "0" is placed at distance $l_b$ from the place of the column mounting ($l_b < l$ – positive pole, $l_b > l$ – negative pole),
  – in a particular case $l_d = 0$ ($(0203)^{*}$, $(0204)^{*}$) – Fig. 4.

- The head built of the linear elements (constant curvature) $(0103)^{**}$, $(0104)^{**}$ – Fig. 5:

  – segments of the heads with the radius $R$ (loading head) and $r$ (receiving head) are infinitely rigid,
  – the centre of the element with the radius $R$ is in the undeformed axis of the column,
  – direction of the loading force passes through point "0" which lies in the undeformed axis of the column at the distance $l_b$ from the place of the column mounting ($l_b < l$ – positive pole, $l_b > l$ – negative pole).

In a particular case, the elements of the head can have the same radii $R = r$, $(0203)^{**}$, $(0204)^{**}$ – Fig. 5) which is equivalent to $l_d = 0$.

The element with infinite flexural rigidity, which length is denoted as $l_0$, is built in the structure of the forcing heads with the elements of radii $R$ and $r$ due to constructional reasons. It should be mentioned that the element with infinite flexural rigidity exists also in the heads built of the linear elements (Fig. 4), but the length $l_0$ of this element is included in the element of the length $l_c$.

Schematic diagrams of slender systems, subjected to a specific load, correspond with the diagram describing the possibility of load manifestation with respect to construction of the heads realising the load. Schematic diagrams are shown in Fig. 6a (heads built of the linear elements – $(\cdot)^{*}$) or in Fig. 6b (heads built of the circular elements – $(\cdot)^{**}$).

Three-component parts $X$, $Y$, $Z$, which fulfil the given tasks, can be distinguished:

- in the $X$, system load 1 and internal forces 3a (externalised) are balanced by head 2, which takes place in rolling guides. The force action and placement of internal forces 3 depend on the shape of head 2 in section I-I,

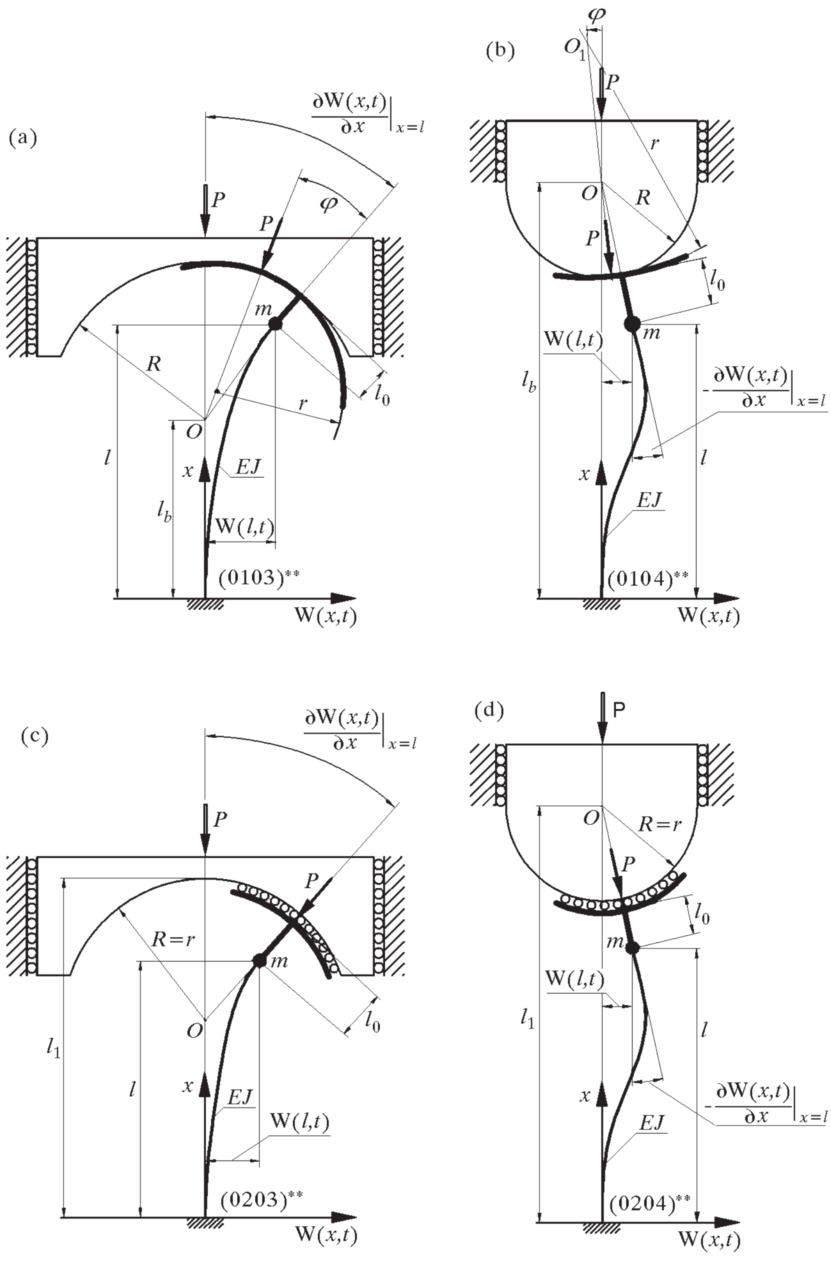


Fig. 5. Schemes of heads realising characteristic loads ((0103)**, (0104)**, (0203)**, (0204)**)

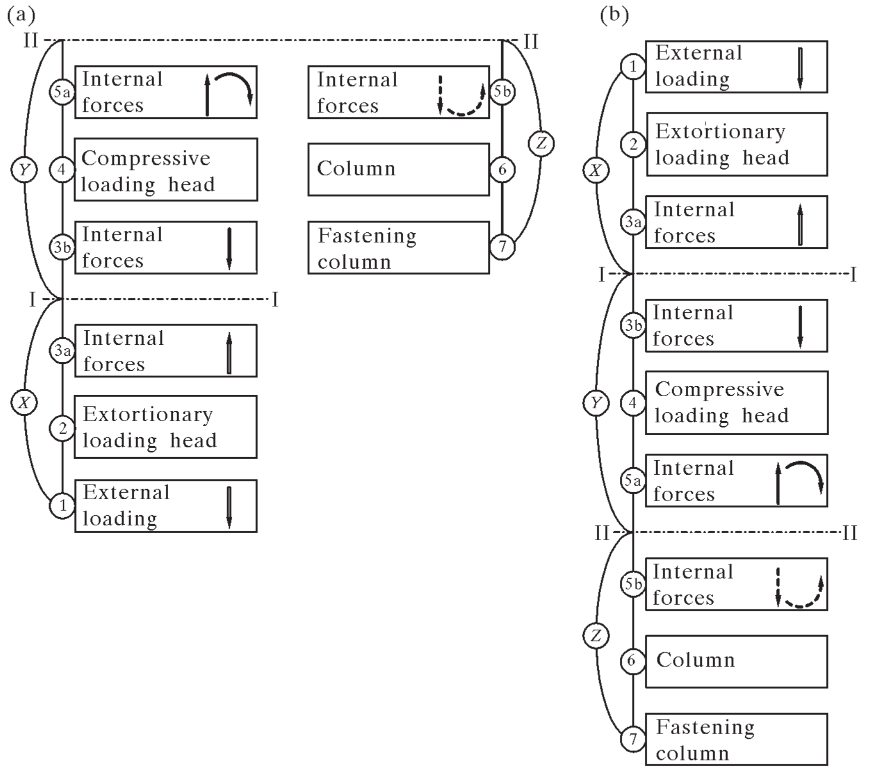


Fig. 6. Diagram of systems realising the characteristic load in the case of: (a) heads built of the linear elements – $(\cdot)^*$; (b) heads built from circular elements – $(\cdot)^{**}$

- in the $Y$ and $Z$ system, manifestation of the load in section I-I makes it possible to specify the boundary conditions on the basis of mechanical or potential energy balance,
- in the $Y$ system, forces 3b and 5a are in balance owing to head 4,
- manifestation of the load in the $Z$ system in section II-II makes it possible to specify the boundary conditions on the basis of mechanical or potential energy balance and also on the differential dependence of internal load 5b on the internal forces in column 6.

### 2.3. Investigation of slender systems subjected to the specific load

Taking into account the nomenclature of the characteristic load (which was not fully known during publication of previous papers), the results of experimental, theoretical and simulation studies, concerning natural frequencies of the slender system, can be summarised in the following way:

- generalised load directed towards the positive pole (Tomski *et al.*, 1984, 1985, 1996, 1999) in alternative designs of heads built of the linear elements and heads built of the linear and circular (constant curvature) elements (Tomski *et al.*, 1999),
- load forced by the follower force directed towards the positive or negative pole:
  - Tomski *et al.* (1998) – in an alternative design of the heads built of the linear elements,
  - Tomski *et al.* (2000) – in an alternative design of the heads built of the circular element (constant curvature).

In the above publications, the specified constructions of tested systems fulfil three basic features:

- connection of elements, making the forcing heads, are infinitely rigid,
- sliding friction does not occur in the rotational nodes of the forcing heads,
- rolling friction is present in the rotational nodes.

The potential of the generalised load with the force directed towards the pole is determined by the first factor, while the load with the follower force towards the pole is determined by the second factor of condition (1.5).

### 2.4. Divergence-pseudo-flutter systems

An uncommon phenomenon is that some of the specific loads of the slender systems give a new course of the natural frequency in the plane: load ($P$)-natural frequency ($\omega$). Such a system was recognised by Tomski *et al.* (1994), for which the course of the curve in the plane $P$-$\omega$ is shown in Fig. 7.

The $P(\omega)$ function for this system has the following course:

- for loads $P \in < 0, P_c)$ ($P_c$ is the critical load) the angle of the tangent to the $P(\omega)$ curve can take a positive, zero or negative value,
- for load $P \approx P_c$, the slope of the curve in the $P$-$\omega$ plane is always negative,
- change of the natural vibration form (from the first to the second and inversely) takes place along the curve, which determines the $P(\omega)$ function for the basic frequency ($M_1$, $M_2$ – denote the first and second forms of vibrations, respectively).

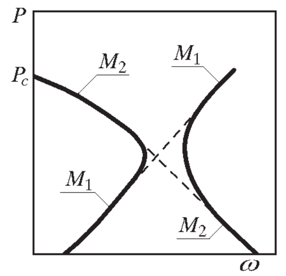


Fig. 7. Divergence-pseudo-flutter system

Such a course of the curve in the $P$-$\omega$ plane with the generalised load with the force directed towards the positive pole was proved by Tomski *et al.* (1996).

A physical interpretation of the conservative condition of the load according to the field theory was also reported by Tomski *et al.* (1996). The course of the curve of the natural frequency can refer to the systems subjected to the generalised load by the force directed towards the positive pole and load by the follower force directed towards the positive pole, and also can refer to a slender system subjected to the force directed towards the positive pole.

The system, which is characterised by the course of the curve in the $P$-$\omega$ plane, shown in Fig. 7, can be called a divergence-pseudo-flutter type system.

Results of experimental research on the natural frequency of the column, which agree to certain extent with the course of the curve of the natural frequency shown in Fig. 7, are presented by Bogacz and Mahrenholtz (1983), Danielski and Mahrenholtz (1991). Neither the boundary conditions for the tested column nor solution to the boundary problem was given there. An attempt to optimise a column subjected to the generalised load by the force directed towards the positive pole was made by Bogacz *et al.* (1998).

The results of experimental investigations, connected with changes of the free vibration form along the curve in the plane: load-natural frequency, are presented by Bogacz *et al.* (1998).

## 3. Constructional schemes of the heads loading the columns

Constructional schemes of the loading and receiving heads in systems subjected to the generalised load by the force directed towards the negative pole (0104)** or positive pole (0103)** (whose schematic diagrams are presented in

Figs 5a,b are shown in Figs 8a,b. These systems are composed of enclosure (2), which ends with rolling guides (1). Enclosure (2) is characterised at its end by the radius of curvature $R > 0$ (Fig. 8b) or rolling element (3) with radius $R < 0$ (Fig. 8a) mounted on the enclosure dependending on the constructional solution.

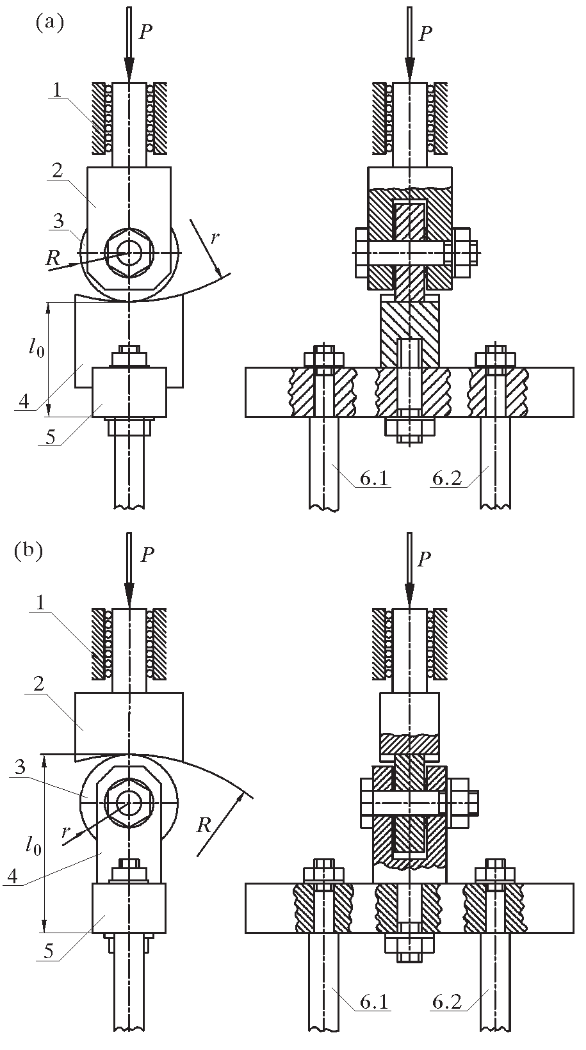


Fig. 8. Constructional schemes of heads realising the column load: (a) for $R < 0$, $l_0 > 0$ – column (0104)**, (b) for $R > 0$, $l_0 > 0$ – column (0103)**

Element (4) is made in an analogous way. It is connected to cube (5) in which two column rods are mounted. It is assumed that the elements of the length $l_0$ (element (4) and cube (5)) are infinitely rigid. This relates to constructional considerations. Element (2) makes up the forcing head, while (4,5) – the loading head.

Rolling element (3) is numbered among the loading head (Fig. 8a) or receiving head (Fig. 8b) dependending on the method of the system loading.

The column consists of two rods (6.1), (6.2) with the bending rigidity $(EJ)_1$ and $(EJ)_2$, respectively, and mass per unit length $(\rho_1 A)_1$ and $(\rho_1 A)_2$ (and $(EJ)_1 = (EJ)_2$, $(\rho_1 A)_1 = (\rho_1 A)_2$, $(EJ)_1 + (EJ)_2 = EJ$, $(\rho_1 A)_1 + (\rho_1 A)_2 = \rho_1 A$). The rods have the same cross-sections and are made of the same material. The rods and their physical and geometrical parameters are distinguished by "1,2" indices, which are only needed to calculate symmetric natural frequencies and to determine the corresponding forms of vibration.

Thus we can assume the global bending rigidity $EJ$ and elementary mass of the column $\rho_1 A$ in all following considerations thoughout this paper.

## 4. Physical model of the system $(0103)^{**}$

The physical model of the considered column, in the constructional variant of loading and receiving heads shown in Fig. 8b – column $(0103)^{**}$, is presented in Fig. 9. The manifestation of the load in sections I-I and II-II is in accordance with the diagram shown in Fig. 6b.

The column could be mounted in an elastic, hinged or rigid way ($c_1$ – is a coefficient of the mounting elasticity). At the free end, the column is connected with the loading head, which transmits the load through the stiff element of the length $l_0$. The concentrated mass $m$ takes into account the total reduced mass of (3), (4), (5) elements – Fig. 8b.

The total potential energy of the system $V_k^j$ (where $k = 1,2,3,4,5$; $j = *, **, ***$) depicted in Fig. 9 is examined depending on the place of its manifestation (Table 1).

The kinetic energy of the considered system is as follows

$$T = T_1 + T_2 = \frac{1}{2}\rho_1 A \int\limits_0^l \Big(\frac{\partial W(x,t)}{\partial t}\Big)^2\, dx + \frac{m}{2}\Big(\frac{\partial W(x,t)}{\partial t}\Big|_{x=l}\Big)^2 \tag{4.2}$$

**Table 1.** Potential energy of the system

| Manifestation of the load in section | | |
|---|---|---|
| I-I | II-II | I-I and II-II |
| Energy of elastic strain | | |
| $V_1^* = \frac{1}{2}EJ\int\limits_0^l \mathcal{A}_2^2\,dx$ | $V_1^{**} = \frac{1}{2}EJ\int\limits_0^l \mathcal{A}_2^2\,dx$ | $V_1^{***} = 0$ |
| Potential energy of the vertical force $P$ component | | |
| $V_2^* = P\Delta_1 + P\Delta_2 + P\Delta_3$ | $V_2^{**} = P\Delta_1$ | $V_2^{***} = P\Delta_2 + P\Delta_3$ |
| Potential energy of the horizontal force $P$ component | | |
| $V_3^* = \frac{1}{2}P\beta(W^* + r\beta)$ | $V_3^{**} = \frac{1}{2}P\beta W(l,t)$ | $V_3^{***}$ |
| Potential energy of bending moment | | |
| $V_4^* = 0$ | $V_4^{**} = -V_4^{***}$ | $V_4^{***}$ |
| Potential energy of mounting elasticity | | |
| $V_5^* = \frac{1}{2}c_1\mathcal{A}_1^2\Big\vert_{x=0}$ | $V_5^{**} = \frac{1}{2}c_1\mathcal{A}_1^2\Big\vert_{x=0}$ | $V_5^{***} = 0$ |

where

$$
\begin{aligned}
V_3^{***} &= \frac{1}{2}P\beta[W^* + r\beta - W(l,t)] \\
V_4^{***} &= -\frac{1}{2}P(\mathcal{A}_1\big\vert_{x=l} - \beta)(r - l_0)\mathcal{A}_1\big\vert_{x=l} \\
\mathcal{A}_1 &= \frac{\partial W(x,t)}{\partial x} \qquad \mathcal{A}_2 = \frac{\partial^2 W(x,t)}{\partial x^2} \\
\Delta_1 &= -\frac{1}{2}\int\limits_0^l \mathcal{A}_1^2\,dx \qquad \Delta_2 = -\frac{l_0}{2}\mathcal{A}_1^2\Big\vert_{x=l} \\
\Delta_3 &= \frac{r}{2}\Big(\mathcal{A}_1^2\Big\vert_{x=l} - \beta^2\Big) \\
\varphi &= \frac{1}{R-r}\Big[(R - l_0)\mathcal{A}_1\Big\vert_{x=l} - W(l,t)\Big] \\
\beta &= \frac{1}{R-r}\Big[W(l,t) - (r - l_0)\mathcal{A}_1\Big\vert_{x=l}\Big] \\
W^* &= W(l,t) - (r - l_0)\mathcal{A}_1\Big\vert_{x=l}
\end{aligned}
\tag{4.1}
$$

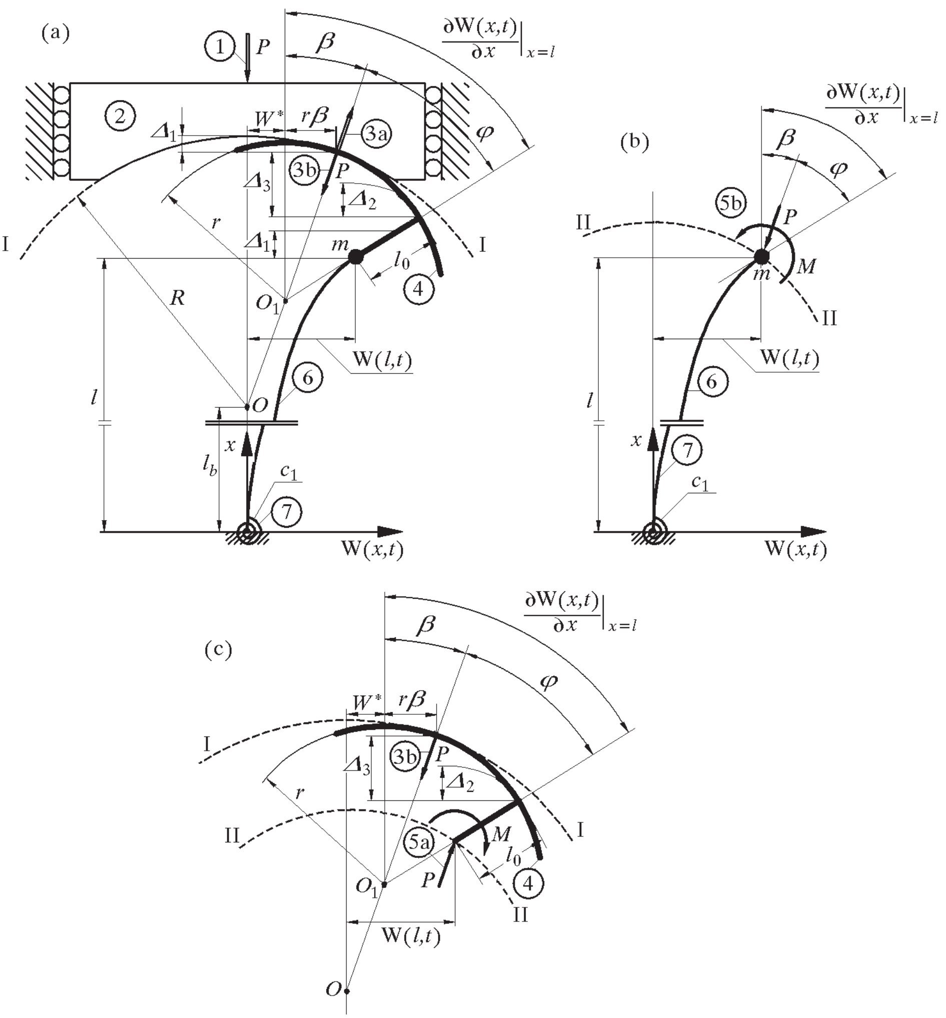


Fig. 9. Physical model of the system

In this paper, the formulation of the problem takes place with the use of Hamilton's principle (Goldstein, 1950)

$$\delta \int\limits_{t_1}^{t_2} \Big(T - \sum_{k=1}^{5} V_k\Big)\, dt = 0 \tag{4.3}$$

Boundary conditions for the considered system are obtained by determining the potential energy variations with the load manifestation in section I-I $(V_k^*)$

or in section II-II ($V_k^{**}$). The commutation of integration (with respect to $x$ and $t$) and the calculation itself is used within Hamilton's principle.

The equation of motion, after taking into account the commutation of variation and differentiation operators and after integrating the kinetic and potential energies of the system, is obtained in the form

$$EJ\frac{\partial^4 W(x,t)}{\partial x^4} + P\frac{\partial^2 W(x,t)}{\partial x^2} + \rho_1 A\frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{4.4}$$

Additionally, giving consideration to the conditions of system mounting in the form

$$W(0,t) = 0 \qquad\qquad EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=0} - c_1\frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{4.5}$$

the missing boundary conditions at the free end of the system are imposed

$$\begin{aligned} &EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} + P\Big[\rho\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \nu W(x,t)\Big|_{x=l}\Big] = 0 \\ &EJ\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} + P\Big[\mu\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \gamma W(x,t)\Big|_{x=l}\Big] - m\frac{\partial^2 W(l,t)}{\partial t^2} = 0 \end{aligned} \tag{4.6}$$

Relations (4.6) determine the boundary conditions at the free end of the system of columns loaded by the generalised load with the force directed towards the negative pole ($R < 0$, $r < 0$) – (0104)** as well as loaded by the generalised load with the force directed towards the positive pole ($R > 0$, $r > 0$) – (0103)**.

Values of the $\rho$, $\nu$, $\mu$, $\gamma$ parameters for the considered columns and systems, whose heads realise boundary conditions with respect to radii $R$ and $r$ of the loading and receiving heads, are given in Table 2.

Schematic diagrams of the systems whose heads realise boundary conditions related to the radii $R$ and $r$ are shown in Fig. 10. The results of simulations for the considered systems, together with the systems whose heads realise boundary parameters related to the critical load and changes of the natural frequency in relation to the external load, are presented in the further part of the paper on the basis of the solution to the boundary problem.

## 5. Solution to the boundary problem

The equations of motion for the considered column, after separating the time and space variables of the $W_i(x,t)$ function in the form

$$W_i(x,t) = y_i(x)\mathrm{e}^{j\omega t} \qquad\qquad i = 1,2 \tag{5.1}$$

**Table 2.** Values of $\rho$, $\nu$, $\mu$, $\gamma$ parameters in considered systems

| | | | | | $\rho$ | $\nu$ | $\mu$ | $\gamma$ |
|---|---|---|---|---|---|---|---|---|
| Column subjected to load | generalised load with force directed towards the negative pole | (0104)** Fig. 5b | $R<0,\ r<0$ | | $\mathcal{B}_1$ | $-\dfrac{r-l_0}{R-r}$ | $\dfrac{R-l_0}{R-r}$ | $\dfrac{-1}{R-r}$ |
| | | Fig. 10e | | $\dfrac{1}{r}=0$ | $l_0-R$ | 1 | 0 | 0 |
| | | Fig. 10f | | $R=0$ | $\dfrac{(r-l_0)l_0}{r}$ | $\dfrac{r-l_0}{r}$ | $\dfrac{l_0}{r}$ | $\dfrac{1}{r}$ |
| | follower force directed towards the negative pole | (0204)** Fig. 5d | | $R=r$ | $W(l,t)=(R-l_0)\mathcal{A}_1\Big\vert_{x=l}$ ; Eq A | | | |
| | generalised load with force directed towards the positive pole | (0103)** Fig. 5a | $R>0,\ r>0$ | | $\mathcal{B}_1$ | $-\dfrac{r-l_0}{R-r}$ | $\dfrac{R-l_0}{R-r}$ | $\dfrac{-1}{R-r}$ |
| | | Fig. 10a | | $\dfrac{1}{R}=0$ | $r-l_0$ | 0 | 1 | 0 |
| | | Fig. 10b | | $r=0$ | $\dfrac{-l_0(R-l_0)}{R}$ | $\dfrac{l_0}{R}$ | $\dfrac{R-l_0}{R}$ | $\dfrac{-1}{R}$ |
| | force directed towards the positive pole | Fig. 10c | | $r=l_0$ | 0 | 0 | 1 | $\dfrac{-1}{R-r}$ |
| | | Fig. 10d | | $r=R=l_0$ | 0 | 0 | 1 | $\infty$ |
| | follower force directed towards the positive pole | (0203)** Fig. 5c | | $R=r$ | $W(l,t)=(R-l_0)\mathcal{A}_1\Big\vert_{x=l}$ ; Eq A | | | |

where

$$\mathcal{B}_1=\frac{(r-l_0)(R-l_0)}{R-r}$$

$$\text{Eq A:}\qquad \frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l}-\frac{1}{R-l_0}\mathcal{A}_2\Big|_{x=l}-\frac{m}{EJ}\frac{\partial^2 W(l,t)}{\partial t^2}=0$$

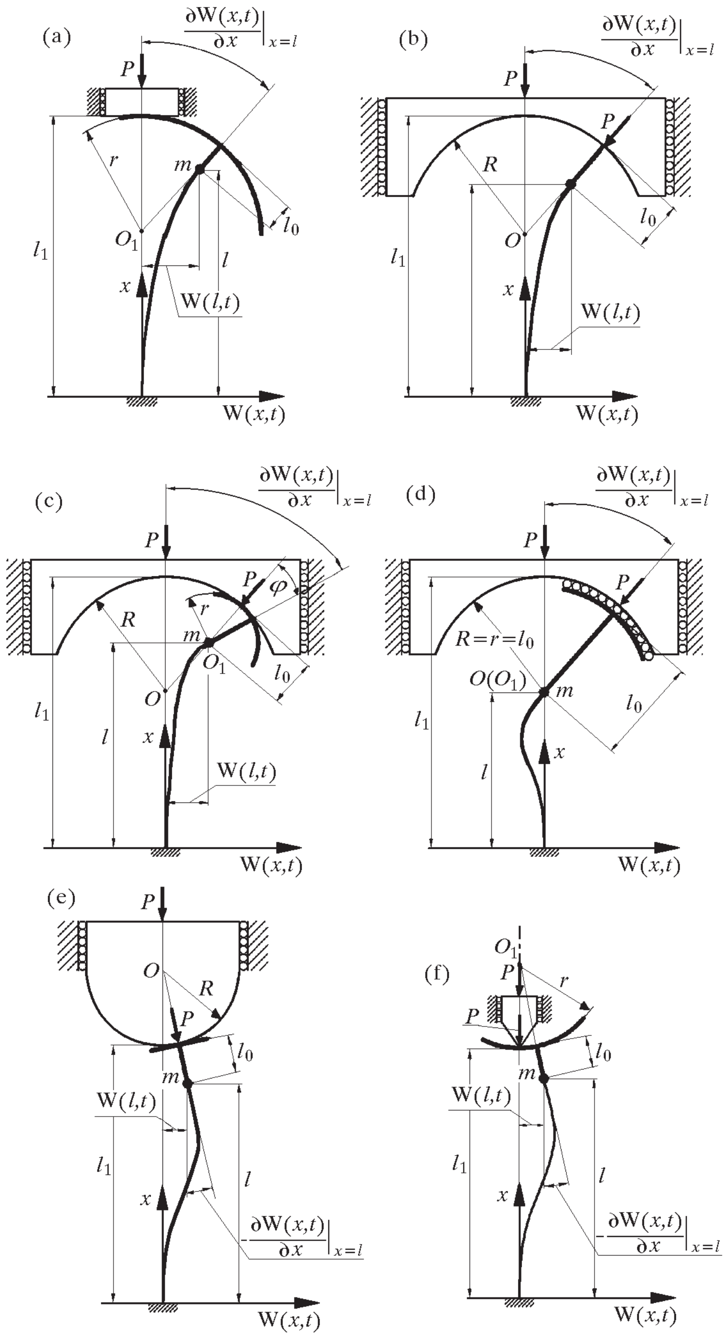


Fig. 10. Schematic diagrams of heads realising boundary parameters

are as follows

$$
\begin{aligned}
&(EJ)_i y_i^{IV}(x) + S_i y_i^{II}(x) - (\rho_1 A)_i \omega^2 y_i(x) = 0 \\
&\sum_{i=1}^{2} S_i = P
\end{aligned}
\tag{5.2}
$$

taking into account the symmetrical distribution of the bending rigidity and mass per unit length.

The boundary conditions at the fixed point and at the free end of the column with consideration of relationships (4.5), (4.6) take the following form

$$
\begin{aligned}
&y_1(0) = y_2(0) = 0 \qquad\qquad y_1^I(0) = y_2^I(0) \\
&y_1(l) = y_2(l) \qquad\qquad y_1^I(l) = y_2^I(l) \\
&y_1^{II}(0) + y_2^{II}(0) - c_1^* y_1^I(0) = 0 \\
&y_1^{II}(l) + y_2^{II}(l) + \beta^2[\rho y_1^I(l) + \nu y_1(l)] = 0 \\
&y_1^{III}(l) + y_2^{III}(l) + \beta^2[\mu y_1^I(l) + \gamma y_1(l)] + \frac{m\omega^2}{(EJ)_1} y_1(l) = 0
\end{aligned}
\tag{5.3}
$$

where

$$
c_1^* = \frac{c_1}{(EJ)_1} \qquad\qquad \beta^2 = \frac{P}{(EJ)_1}
\tag{5.4}
$$

The general solution to Eqs (5.2) is

$$
y_i(x) = C_{1i}\cosh(\alpha_i x) + C_{2i}\sinh(\alpha_i x) + C_{3i}\cos(\beta_i x) + C_{4i}\sin(\beta_i x)
\tag{5.5}
$$

where $C_{ni}$ are integration constants $(n = 1, 2, 3, 4)$ and

$$
\alpha_i^2 = -\frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \Omega_i^{*2}} \qquad\qquad \beta_i^2 = \frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \Omega_i^{*2}}
\tag{5.6}
$$

while

$$
\Omega_i^{*2} = \frac{(\rho_1 A)_i \omega^2}{(EJ)_i} \qquad\qquad k_i = \sqrt{\frac{S_i}{(EJ)_i}}
$$

Substituting solutions (5.5) into boundary conditions (5.3), yields a transcendental equation to eigenvalues of the considered system.

## 6. Experimental stand

The experimental stand designed for the research of the considered systems (compare Tomski *et al.*, 1998, 1999; Tomski and Gołębiowska-Rozanow, 1998; Tomski and Szmidla, 2003) is shown in Fig. 11. It consists of two loading heads 1(1) and 1(2). Head 1(1) can be transversely shifted along guides 2(1). Head 1(2) can be horizontally shifted along guides 2(2) and 2(3) in the transverse and longitudinal directions, adequately.

The load is applied to the tested column by means of screw systems belonging to the heads. The loading force is measured by dynamometers 3(1) and 3(2). The supports of systems can be mounted on a plate. They realise requested boundary conditions. The tested column together with the head exerting a load is clamped to supports 4(1) and 4(3). Support 4(1) enables mounting of the loading head.

A hinged or rigid mounting of the system is realised in support 4(3) dependending on the considered case. Invesigations of natural frequencies are performed with the use of a two-channel vibration analyser made by Brüel and Kjaer (Denmark).

## 7. Numerical and experimental results

For the considered system, numerical computations were accomplished on the basis of the solution to the boundary value problem. Then, the course of natural frequencies in relation to the external loads (for systems whose physical and geometrical parameters are given in Table 3) are experimentally verified on the stand (Fig. 11).

**Table 3.** Geometrical and physical parameters of the considered columns

| Column | $EJ$ [Nm$^2$] | $\rho_1$A [kg/m] | $l$ [m] | $R$ [m] | $r$ [m] | $l_0$ [m] | $m$ [kg] |
|---|---|---|---|---|---|---|---|
| $D_1$ | 152.68 | 0.631 | 0.605 | 0.040 | 0.02 | 0.13 | 0.41 |
| $D_2$ | 152.68 | 0.631 | 0.735 | 0.040 | 0.0275 | 0 | 0.39 |
| $D_3$ | 152.68 | 0.631 | 0.605 | 0.085 | 0.02 | 0.13 | 0.41 |
| $D_4$ | 152.68 | 0.631 | 0.625 | 0.085 | 0.02 | 0.13 | 0.41 |
| $D_5$ | 152.68 | 0.631 | 0.710 | 0.085 | 0.02 | 0.04 | 0.33 |
| $D_6$ | 152.68 | 0.631 | 0.730 | 0.085 | 0.02 | 0.04 | 0.33 |

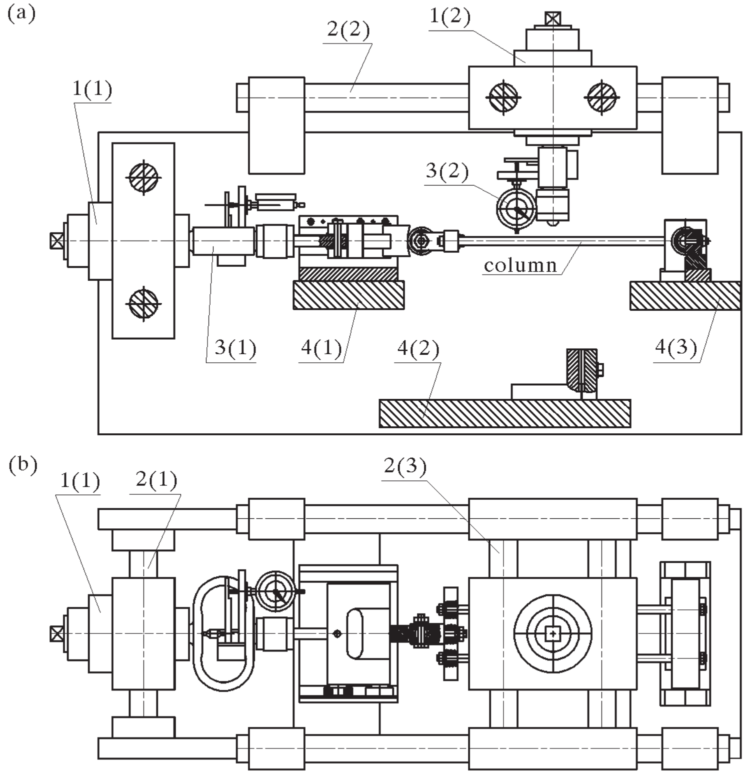


Fig. 11. Test stand for experimental investigations of the column system

Parameters of the loading and receiving heads are also given in the above table. The constructional scheme of the loading head (column (0103)$^{**}$) is presented in Fig. 8b.

The results, obtained from experimental investigations (point) and numerical computations (lines), are presented in Fig. 12 for columns $D_4$, $D_6$ with hinged mounting $c_1^* = 0$ at $x = 0$. The rigid mounting $1/c_1^* = 0$ was applied in the remaining cases. The results are limited to the first two basic natural frequencies $M_1$, $M_2$ and additional frequency $M_2^e$ characterised by symmetry of vibrations (compare Tomski *et al.*, 1998).

Both numerical and experimental results are in good agreement.

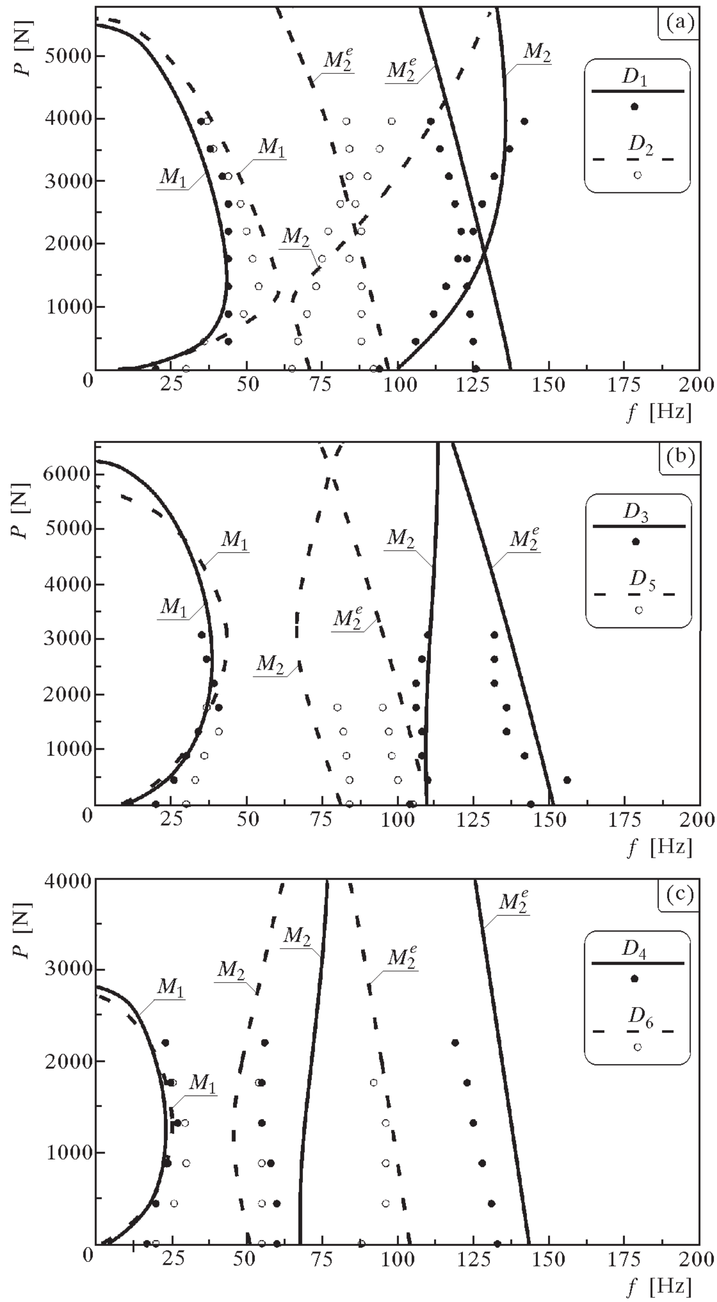


Fig. 12. Frequency curve in the plane: load-natural frequency for $(0103)^{**}$ column – $D_1$, $D_2$ (a); $D_3$, $D_5$ (b); $D_4$, $D_6$ (c)

For the system, considered in this work, additional numerical computations were carried out for the following columns:

- loading by the generalised force directed towards the negative pole $(0104)^{**}$,
- loading by the generalised force directed towards the positive pole $(0103)^{**}$,

related to changes of the critical load and natural frequencies taking into account the rigid mounting of the system $(1/c_1^* = 0)$.

The changes of the critical load $P_c$, in the full range of the radius $R$ of the loading head for three chosen lengths of the rigid element $l_0$ and values of the radius $r$ of the receiving head, are determined in Fig. 13.

The value of the critical load and $R$, $r$, $l_0$ parameters are related to the overall length of the system $l_1$

$$\begin{aligned} \lambda_c &= \frac{P_c l_1^2}{EJ} \qquad & R^* &= \frac{R}{l_1} \qquad & l_0^* &= \frac{l_0}{l_1} \\ r^* &= \frac{r}{l_1} \qquad & \mathit{\Delta} r &= \frac{R-r}{l_1} \end{aligned} \tag{7.1}$$

while

$$l_1 = l_0 + l = \text{idem}$$

The curves $B_q$, $C_q$ $(q = 1, 2, 3)$ from the $(B_{q\infty}, C_{q\infty})$ point to $(B_q', C_q')$ represent the value of the critical load parameter for the column loaded by the generalised force directed towards the negative pole $(0104)^{**}$.

In the remaining range, it means from the point $(B_q'', C_q'')$ to $(B_{q\infty}, C_{q\infty})$, the system is loaded by the generalised force directed towards the positive pole $(0103)^{**}$.

In the particular case, when $R^* = r^*$ $(\mathit{\Delta} r = 0)$, the course of critical load changes stays in accordance with the curves $A_q$.

Such a character of the critical load changes corresponds with the column loaded by the follower force directed towards the negative pole ($(0204)^{**}$ – Fig. 5d) for $R^* < 0$ or the positive pole ($(0203)^{**}$ – Fig. 5c) for $R^* > 0$, independently of the length $l_0$ of the rigid element of the loading head.

The course of each of the curves (broken line) from the point $(B_q', C_q')$ to $(B_q'', C_q'')$ was not determined. The relationship, resulting from the construction of the loading heads $\mathit{\Delta} r > 0$, is not fulfilled for $R^*$, $r^*$ from the above range.

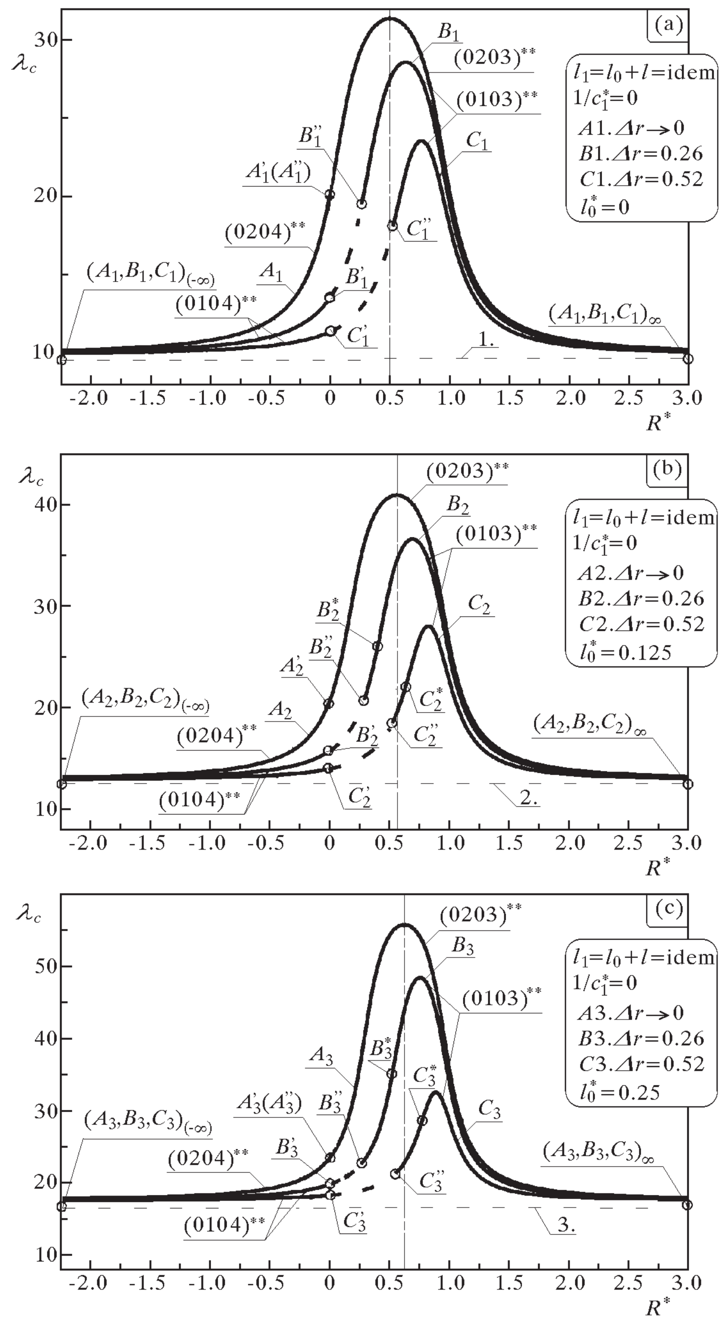


Fig. 13. Change of the critical load parameter $\lambda_c$ in relation to the radius $R$ of the loading head for $l_0^* = 0$ (a); $l_0^* = 0.125$ (b); $l_0^* = 0.25$ (c)

For the considered values of the radius $R^*$, every curve of critical load changes is characterised by occurrence of the maximum value of the critical load. The extreme value is determined for $R^*$, $r^*$, $l_0^*$ fulfilling the dependence

$$\frac{\frac{1}{2}(R^* + r^*) - l_0^*}{1 - l_0^*} = \frac{1}{2} \tag{7.2}$$

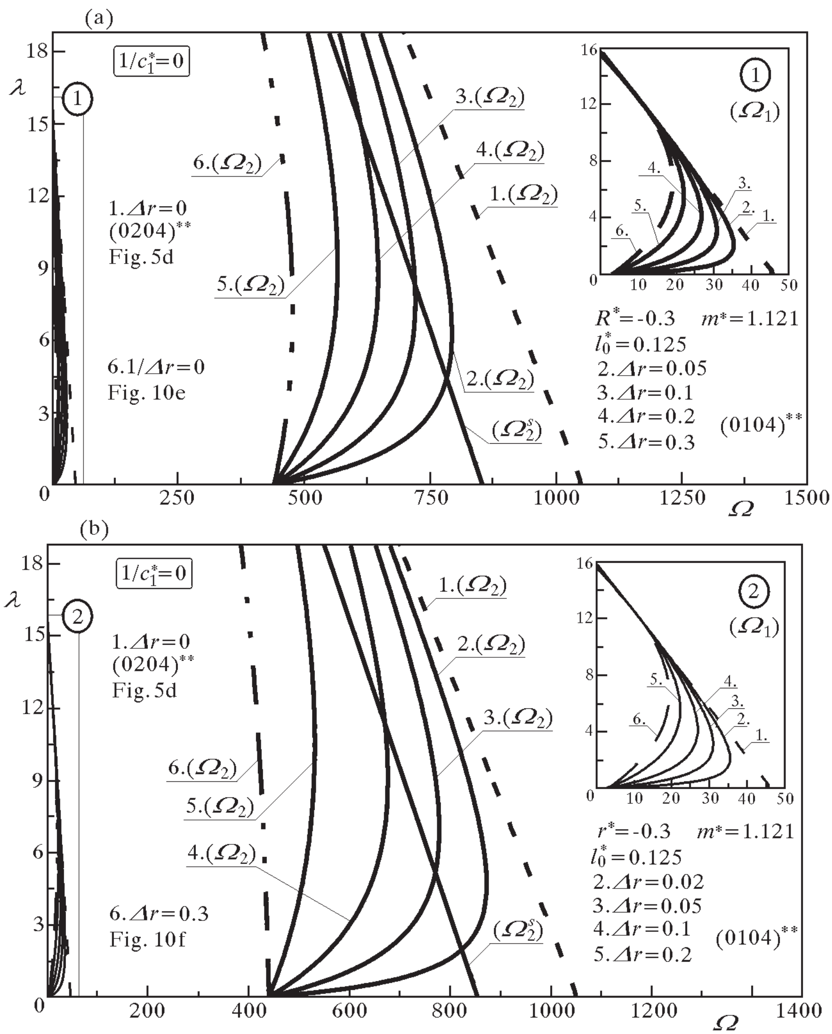


Fig. 14. Frequency curves in the plane: load parameter $\lambda$ – natural frequency $\Omega$ for column $(0104)^{**}$ when $R^* = -0.3$ (a); $r^* = -0.3$ (b)

The value of the critical load parameter corresponding with $R^* \to \pm\infty$ is specified by lines (1,2,3). Additionally, points $A'_q$ describe the value of the critical force characterised by the hinged mounting at the free end of the system. The scheme of this system is shown in Fig. 10d. The critical load parameter corresponds with the system subjected to the force directed to the positive pole when $r^* = l_0^*$, see points $B_q^*$, $C_q^*$ in Fig. 13b,c. The scheme of the system is shown in Fig. 10c.

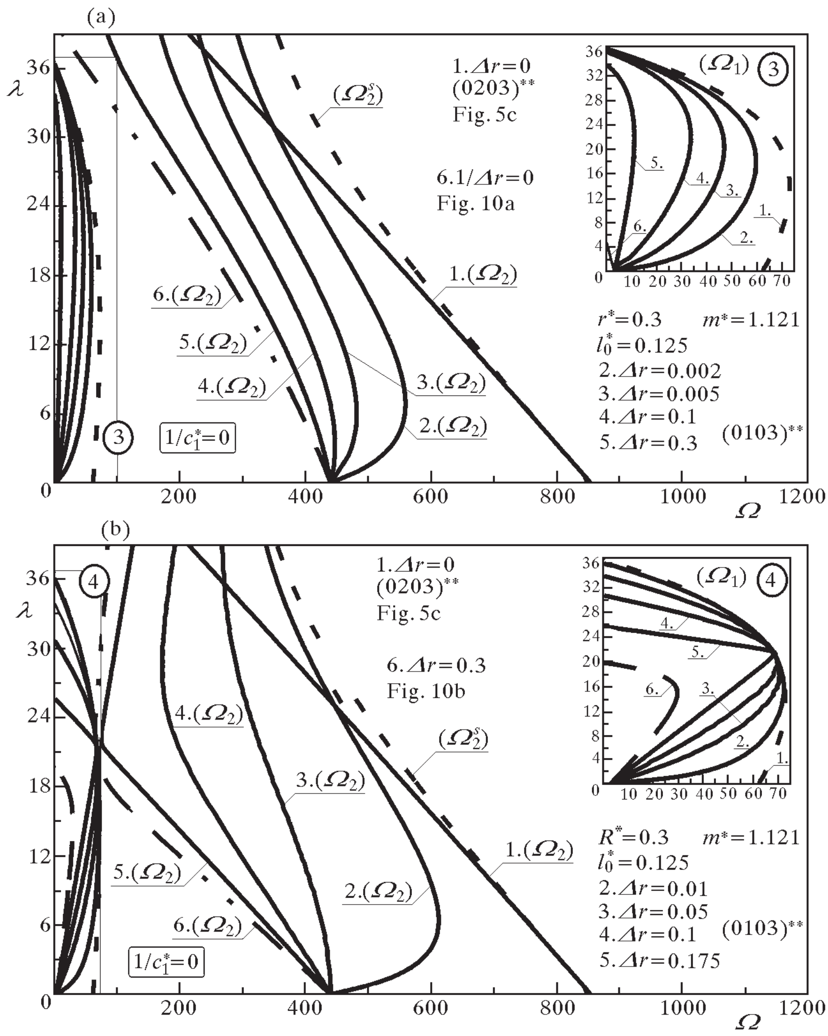


Fig. 15. Frequency curves in the plane: load parameter $\lambda$ – natural frequency $\Omega$ for column $(0103)^{**}$ when $r^* = 0.3$ (a); $R^* = 0.3$ (b)

The course of free vibration frequencies in relation to the external load was determined for columns subjected to the generalised force directed towards the negative pole $(0104)^{**}$ and subjected to the generalised force directed towards the positive pole $(0103)^{**}$. The presentation was limited (Fig. 14, Fig. 15) to the determination of character of the two first basic natural frequencies in a dimensionless form $\Omega_2^s$ in relation to the dimensionless loading parameter $\lambda$ for chosen head parameters.

The following parameters describe operation of the heads: the forcing $R^*$ and loading, $r^*$, $l_0^*$ taking into account the rigid mounting of the system $1/c_1^* = 0$ and the constant mass $m$ concentrated at the free end of the system.

It is assumed

$$\lambda = \frac{Pl_1^2}{EJ} \qquad \Omega = \frac{\rho_1 A\omega^2 l_1^4}{EJ} \qquad m^* = \frac{m}{\rho_1 A l_1} \tag{7.3}$$

The influence of the parameter $r^*$ on the course of the natural frequency for a constant value of the $R^*$ parameter is illustrated in Fig. 14a and Fig. 15b. Changes of geometry of the loading head $R^*$ at a constant value of the radius of the receiving head $r^*$ are represented by curves of free vibration frequencies in Fig. 14b and Fig. 15a. For the presented diagrams, the course of natural frequencies is limited by curves (1) and (6), plotted for the boundary parameters $R^*$ and $r^*$.

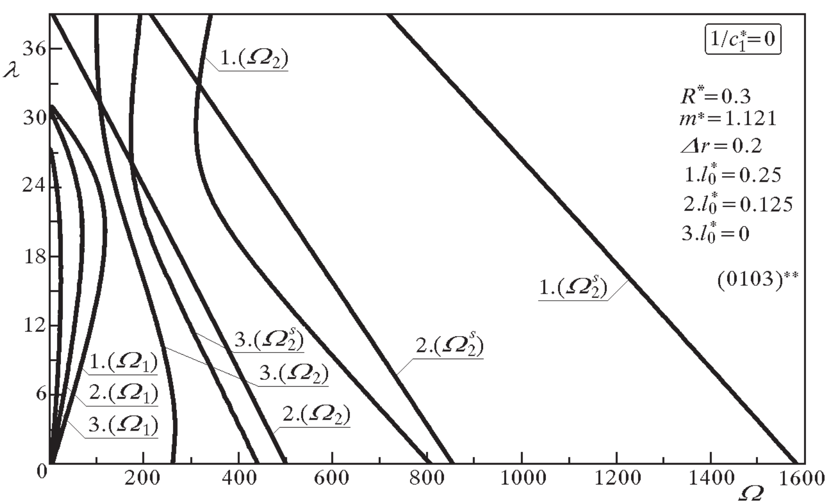


Fig. 16. Frequency curves in the plane: load parameter $\lambda$ – free vibration frequency $\Omega$ for column $(0103)^{**}$ for various $l_0^*$

Adequate schematic diagrams of the columns and heads realising boundary parameters $R^*$ and $r^*$ are presented in diagrams and Fig. 10 and Fig. 5c,d.

The numbering of the considered columns for one of the two types of systems, namely divergence or divergence-pseudo-flutter, can be done according to the presented courses of natural frequencies. The influence of changes in the length of the rigid element $l_0^*$ on the course of free vibration frequencies in relation to the external load for a constant value of the parameters $m^*$, $R^*$ and $r^*$ ($\Delta r = R^* - r^*$) is shown in Fig. 16.

The value of the critical load for all presented curves of free vibration frequencies (Fig.14-Fig. 16) is determined for $\Omega = 0$.

## 8. Summary

- The nomenclature of the characteristic load, which is the generalised load by a force directed towards a pole or load by a follower force directed towards a pole, is determined with taking into account the properties of load acting on a slender system.
- The generalised load by a force directed towards a pole and load by a follower force directed towards a pole can be realised in two kinds of constructions of the loading and receiving heads, namely heads built of linear and circular elements (constant curvature).
- Different cases of the column loading can be obtained for various geometrical parameters of the head: loading and receiving, built of the circular elements.
- It was stated, on the basis of carried out numerical simulations dealing with the load by the generalised force directed towards the negative or positive pole, that there were such values of geometrical parameters of the forcing and loading heads for which the maximum of the critical load was obtained.
- There are such values of the geometrical parameters of the heads realising the load for which the tested systems appear to be of the divergence-pseudo-flutter type.
- The numerical and experimental results, describing the free vibration frequencies in relation to the external load, are in good agreement.

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## Drgania i stateczność kolumny poddanej obciążeniu uogólnionemu z siłą skierowaną do bieguna

Streszczenie

W pracy prezentuje się nowe obciążenie układów smukłych, które jest obciążeniem uogólnionym z siłą skierowaną do bieguna dodatniego lub ujemnego. Przedstawia się nowe rozwiązania konstrukcyjne głowic realizujących i przejmujących to obciążenie. Prezentuje się rozważania teoretyczne dotyczące sformułowania warunków brzegowych. W zależności od rozwiązania konstrukcyjnego głowicy obciążającej i przejmującej obciążenie określa się wartość siły krytycznej oraz przebieg częstości drgań własnych w funkcji obciążenia zewnętrznego dla zadanej geometrii i stałych fizycznych kolumny. Wyniki badań teoretycznych porównuje się z wynikami badań eksperymentalnych.

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