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MODELLING OF TURBULENT FLOW IN THE NEAR-WALL REGION USING PDF METHOD

Marta Wacławczyk[1]
Jacek Pozorski

Institute of Fluid-Flow Machinery, Polish Academy of Sciences, Gdańsk
e-mail: mw@imp.gda.pl

The paper presents near-wall turbulence models which incorporate the idea of elliptic relaxation. The simplified elliptic blending model is applied in the Lagrangian probability density function (PDF) approach. The PDF method is extended to compute near-wall viscous momentum transport. Computations are performed for fully developed turbulent channel flow and validated against available DNS data.

Key words: near-wall turbulence, PDF method, elliptic relaxation

1. Introduction

One of the inherent difficulties in modelling the turbulent flow is related to the near-wall regions. At the same time, most of the technically important turbulent flows are bounded, at least in part, by solid surfaces. In the immediate vicinity of the wall experimental investigations and the DNS results show the existence of complicated vortical structures of considerable kinetic energy (Aubry *et al.*, 1988). DNS computations give insight into the dynamics of the turbulent eddies, mechanisms of their generation, and interactions between them. However, due to high numerical cost of such simulations, engineering applications to date are limited to the Reynolds averaged Navier-Stokes (RANS) methods which provide statistical description of turbulent flows. The mean (ensemble averaged) variables are also affected by the presence of walls. In particular, the wall effects should be accounted for within RANS models.

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First, the molecular transport of heat and momentum becomes important and cannot be neglected, as is sometimes done for high-Re turbulent flows far from solid boundaries. Due to the no-slip condition large gradients of mean statistics occur in the vicinity of the wall. Another effect is the lack of separation of macro- and microscales of turbulence which also arises from the viscosity action in the near-wall region. Moreover, the Reynolds stresses are strongly anisotropic which is caused by the blocking of wall-normal fluctuations.

Most often, the modelling of near-wall flows is performed in the Eulerian approach, and so-called low-Re models are introduced for the purpose. They can be based on the damping function method (Rousseau *et al.*, 1997) or elliptic relaxation model of Durbin (1993). The functions which damp particular terms in the equations are derived from comparison with experiments, and often involve wall distance as an argument. For this reason, the damping function approach is likely to fail in more complex geometries or complicated flow cases (e.g. with separation or reattachment zones). The elliptic relaxation method is based on the Poisson equation for pressure fluctuations. The method accounts for the non-local character of pressure fluctuations and is therefore sounder from the physical point of view in comparison to the damping functions approach.

The paper presents a model for near-wall turbulent flows derived for the Lagrangian (PDF) approach; there, the non-local wall effects should also be included. A PDF model for low-Re numbers was derived by Dreeben and Pope (1998). In the model, viscosity was introduced through the Brownian motion in physical space and some additional terms in the equation for velocity. Non-local effects were originally modelled by the full six-equation elliptic relaxation method. However, in our work we apply a simplified approach of Manceau and Hanjalić (2002), derived for the Eulerian Reynolds stress transport model (RSM). The method is adapted here to the Lagrangian PDF approach. Due to numerical problems with down-to-the-wall integration, the previous scheme developed for high-Re turbulent flows (Minier and Pozorski, 1999) had to be changed and is now based on the exponential form of stochastic equations. The computations have been performed for the fully developed channel flow. The DNS data of Moser *et al.* (1999) are used for comparison.

2. Modelling of near-wall flows: elliptic relaxation method

The instantaneous turbulent velocity field is influenced considerably by the wall proximity. As a consequence, the wall effects have also an impact on the flow statistics, like the mean velocity $\langle U_i \rangle$, the turbulent kinetic energy k, or the turbulent stresses $\langle u_i u_j \rangle$. Let us recall here that, according to the Reynolds decomposition, the instantaneous velocity can be written as a sum of its mean and fluctuation parts $U_i = \langle U_i \rangle + u_i$. The wall boundary condition $\langle U_i \rangle = 0$ leads to large velocity gradients and consequently to large values of the turbulence production term

$$\mathcal{P} = \frac{\partial \langle U_i \rangle}{\partial x_j} \langle u_i u_j \rangle \tag{2.1}$$

(with summation over repeating indices) in the near-wall region. Due to the no-slip and impermeability conditions all components of velocity fluctuation are zero at the wall. Hence, in its proximity the components can be written as the Taylor series expansion (see eg. Manceau *et al.*, 2001)

$$\begin{aligned} u &\sim a_1 y + a_2 y^2 + ... \\ v &\sim b_1 y + b_2 y^2 + ... \\ w &\sim c_1 y + c_2 y^2 + ... \end{aligned}$$

where the streamwise, wall-normal and spanwise fluctuation components are denoted by u, v and w, respectively. Applying the above formulae to the continuity equation we obtain

$$0 = \frac{\partial v}{\partial y}\bigg|_{y=0} \sim b_1 + 2b_2 y \tag{2.2}$$

hence $b_1 = 0$ and

$$v \sim b_2 y^2 + ...$$

This result reveals the effect of kinematic blocking of the wall-normal fluctuations, which is felt even far from the wall and consequently introduces strong anisotropy to the Reynolds stress tensor. On the other hand, the enhancement of the pressure fluctuations, due to their reflection from the surface, induces isotropisation of the Reynolds stresses. However, this effect is weaker than the kinematic blocking. As the wall is approached and the viscous transport becomes dominant, characteristic length and time scales of turbulent eddies become comparable with those of dissipative eddies. Hence, the Kolmogorov hypothesis is not valid in this region; this fact irrevocably limits the validity of standard turbulence models.

In the case of incompressible flows considered here, the kinematic effects of wall blocking and pressure reflection are definitely of non-local elliptic nature and are immediately felt far from the wall. They represent a major challenge for turbulence models where only one-point closures (i.e. functions of only one point $\boldsymbol{x}$ of the flow) are involved. Let us recall here the transport equations for the Reynolds stresses (cf. Pope, 2000) which take the following form

$$\frac{D\langle u_i u_j\rangle}{Dt} = -\underbrace{\frac{\partial\langle u_i u_j u_k\rangle}{\partial x_k}}_{D^T_{ij}} \underbrace{-\langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_k} - \langle u_j u_k\rangle\frac{\partial\langle U_i\rangle}{\partial x_k}}_{\mathcal{P}_{ij}} + \Pi_{ij} + \underbrace{\nu\nabla^2\langle u_i u_j\rangle}_{D^\nu_{ij}} \tag{2.3}$$

where D/Dt stands for the material derivative along mean streamlines, the diffusion tensor D^T_{ij} is connected with turbulent transport, D^ν_{ij} stands for the viscous transport and $\mathcal{P}_{ij}$ is the production of turbulent stresses $\langle u_i u_j\rangle$. The non-locality of the flow field is represented by the tensor Π_{ij} which contains mean velocity-pressure gradient correlations and dissipation

$$\Pi_{ij} = -\frac{1}{\varrho}\Big\langle u_i\frac{\partial p}{\partial x_j}\Big\rangle - \frac{1}{\varrho}\Big\langle u_j\frac{\partial p}{\partial x_i}\Big\rangle - 2\nu\Big\langle\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\Big\rangle = \phi_{ij} + \epsilon_{ij} \tag{2.4}$$

The dissipation ϵ_{ij} is a function of the fluctuating velocity gradient which can be interpreted as a quantity describing a "short-range" non-locality, connected with the length scales of dissipative eddies. Phenomena occurring at the smallest turbulent scales are difficult to model, especially when the hypothesis of separation of integral and viscous scales is no longer valid (Bradshaw, 1994). The Kolmogorov assumption breaks down in this region, as the rate of energy transfer from the large to the smaller eddies is not equal to the rate at which the energy is being dissipated by the smallest vortices. The pressure fluctuations p, present in the RHS of Eq. (2.4), can be computed from the elliptic Poisson equation (cf. Pope, 2000). Hence, they represent a long-range non-locality of turbulence.

It is evident that non-local effects should be somehow included in turbulence models, whereas usual hypotheses applied to derive the basic version of closure are: high Reynolds number, local isotropy and quasi-homogeneity of turbulence. The tensor Π_{ij} is then a function of one-point statistics like the dissipation rate of the kinetic energy ϵ, turbulent stresses $\langle u_i u_j\rangle$ and mean velocity gradients. The dissipation ϵ is computed from its own transport equation. In order to derive a physically sound closure for near-wall flows, Durbin (1993) proposed a model which is based on an integral form of the tensor ϕ_{ij}. From the Poisson equation for pressure fluctuations the tensor ϕ_{ij} can be written as an integral containing a function of two-point statistics (i.e. non-local information)

$$\varrho\phi_{ij}(\boldsymbol{x}) = \int\limits_{\Omega} \Psi_{ij}(\boldsymbol{x},\boldsymbol{x}')G_{\Omega}(\boldsymbol{x},\boldsymbol{x}')\, dV(\boldsymbol{x}') \tag{2.5}$$

where G_{Ω} is a Green function of the flow domain Ω, further replaced by its free-space form $G = -1/(4\pi r)$, with $r = |\boldsymbol{x} - \boldsymbol{x}'|$. The exact form of the function Ψ_{ij} is detailed in e.g. Manceau (2000) and Pope (2000). As evidenced by experiments, the two-point correlations can be approximated, for a wide range of r values, by exponential functions. Durbin proposed the following form for the function Ψ_{ij}

$$\Psi_{ij}(\boldsymbol{x},\boldsymbol{x}') = k(\boldsymbol{x})\frac{\Psi_{ij}(\boldsymbol{x}',\boldsymbol{x}')}{k(\boldsymbol{x}')}\exp\Big(-\frac{r}{L}\Big) \tag{2.6}$$

where L is the characteristic length scale defined as the maximum of the turbulent length scale and the scale connected with dissipative eddies (valid close to the wall)

$$L = C_L \max\Big\{\frac{k^{3/2}}{\epsilon}, C_T\Big(\frac{\nu^3}{\epsilon}\Big)^{1/4}\Big\} \tag{2.7}$$

where C_L and C_T are model constants. Now, integral (2.5) becomes

$$\varrho\frac{\phi_{ij}(\boldsymbol{x})}{k(\boldsymbol{x})} = -\int\limits_{\Omega} \frac{\Psi_{ij}(\boldsymbol{x}',\boldsymbol{x}')}{k(\boldsymbol{x}')}\frac{\exp(-r/L)}{4\pi r}\, dV(\boldsymbol{x}') \tag{2.8}$$

The term $G' = -\exp(-r/L)/(4\pi r)$ which appears inside the integral, is the Green function connected with the operator $1/L^2 - \nabla^2$; therefore ϕ_{ij} is the solution of the following elliptic Helmholz equation

$$L^2\nabla^2\frac{\phi_{ij}}{k} - \frac{\phi_{ij}}{k} = -\frac{\phi_{ij}^h}{k} \tag{2.9}$$

Above, ϕ_{ij}^h denotes a standard quasi-homogeneous model used to compute turbulent fields far from walls, e.g. Rotta's return to isotropy or isotropisation of production (IP) model. It is assumed that far from walls the Laplacian term in Eq. (2.9) disappears and then ϕ_{ij} is equal to its quasi-homogeneous form. The same elliptic equation is solved also for the dissipation tensor ϵ_{ij} and hence for the tensor Π_{ij}.

A simplified elliptic relaxation approach was specified by Manceau and Hanjalić (2002). They state that six elliptic equations (2.9) of the original model of Durbin are somewhat redundant and unnecessarily increase the computational cost. Manceau and Hanjalić solve only one additional elliptic equation for the so-called blending function α

$$L^2\nabla^2\alpha - \alpha = -\frac{1}{k} \tag{2.10}$$

The velocity-pressure-gradient tensor ϕ_{ij} is then found from an interpolation between its near-wall and quasi-homogeneous limits

$$\phi_{ij} = (1-k\alpha)\phi_{ij}^w + k\alpha\phi_{ij}^h \tag{2.11}$$

It follows from the above formula that the required near-wall value of $k\alpha$ is 0; in the core region of the flow we expect $k\alpha = 1$. Hence, Eq. (2.10) is solved with the following boundary conditions

$$\alpha\Big|_{y=0} = 0 \qquad\qquad \alpha\Big|_{y=H} = \frac{1}{k} \tag{2.12}$$

The same blending method is used for the dissipation tensor ϵ_{ij}.

3. Turbulence modelling using PDF method

The modelling of a turbulent field can be performed in two basic approaches, i.e. in the Eulerian or Lagrangian point of view. In the first one, most often used, flow variables are connected with a certain point in space (x,y,z) and time t. Thus, discretized equations can be solved on a space-time grid. A good example of the Eulerian approach are the Reynolds stress equations (2.3), presented in the previous section. In the Lagrangian approach flow parameters are related to a certain element of fluid which has the initial position (x_0,y_0,z_0) in the time instant t_0. In the Lagrangian approach used in the paper we solve equations for stochastic particles which model fluid elements.

3.1. High-Reynolds models

In the modelling of high-Reynolds number flows, viscosity is not accounted for explicitly, the viscous action is modelled only by the dissipation ϵ. Transport equations for stochastic particles take the general form (Pope, 2000)

$$\begin{aligned} dX_i &= U_i\,dt \\ dU_i &= -\frac{1}{\varrho}\frac{\partial\langle P\rangle}{\partial x_i}\,dt - G_{ij}(U_j - \langle U_j\rangle)\,dt - \frac{1}{2}\frac{\epsilon}{k}(U_i - \langle U_i\rangle)\,dt + \sqrt{B}\,dW_i \end{aligned} \tag{3.1}$$

where

$$B = \frac{2}{3} G_{kl} \langle u_k u_l \rangle$$

Above, dW is an increment of the Wiener process, written in the discrete form as $\Delta W = \sqrt{\Delta t}\,\xi$ where ξ is a standard random Gaussian number. The turbulence model is introduced by a specific form of the tensor G_{ij} which is a function of mean turbulence statistics and model constants. In the Monte Carlo simulation stochastic differential equations (3.1) are solved for a large set of stochastic particles. In order to compute the mean statistics the flow domain is first discretized and then parameters connected with particles within one cell of the spatial grid are averaged. PDF computations in the high Reynolds number approach for fully developed channel flow were performed by Minier and Pozorski (1999). Boundary conditions for stochastic particles were placed in the logarithmic region.

It is important to note that from Lagrangian equations (3.1) one can deduce corresponding Eulerian equations for the one-point statistics, namely the mean velocity $\langle U_i \rangle$, the Reynolds stresses $\langle u_i u_j \rangle$, as well as for higher order moments, e.g. triple correlations $\langle u_i u_j u_k \rangle$. For this purpose we write the evolution equation for the probability density function $f = f(\boldsymbol{V}; \boldsymbol{x}, t)$, also called the Fokker-Planck formula (van Kampen, 1990). For the sake of an example, let us consider the following stochastic equations

$$\begin{aligned} dX &= U\,dt \\ dU &= A\,dt + B\,dW \end{aligned}$$

where A and B are constants. The probability density function connected with the above equations is denoted by $f(V; x, t)$, where x, V belong to a sample space of the position X and the velocity U. The expression $f(V; x, t)\,dV$ is the probability that the variable U connected with a stochastic particle takes a value within the bounds $V \leqslant U \leqslant V + dV$. The evolution equation for the probability density function writes

$$\frac{\partial f}{\partial t} + V \frac{\partial f}{\partial x} = -A \frac{\partial f}{\partial V} + \frac{1}{2} B \frac{\partial^2 f}{\partial V^2}$$

Similarly, stochastic equations (3.1) correspond to the following formula for the PDF

$$\begin{aligned} \frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\varrho} \frac{\partial \langle P \rangle}{\partial x_i} \frac{\partial f}{\partial V_i} + \frac{\partial}{\partial V_i} \Big[G_{ij} (V_j - \langle U_j \rangle) f \Big] + \\ + \frac{1}{2} \frac{\epsilon}{k} \frac{\partial}{\partial V_i} [(V_i - \langle U_i \rangle) f] + \frac{1}{2} B \frac{\partial^2 f}{\partial V_j^2} \end{aligned} \tag{3.2}$$

The mean velocity and other statistics can be computed from integration over the sample space

$$\begin{aligned}\langle U_i\rangle(\boldsymbol{x},t) &= \int\limits_{-\infty}^{+\infty} V_i f(\boldsymbol{V};\boldsymbol{x},t)\,d\boldsymbol{V}\\ \langle u_i u_j\rangle(\boldsymbol{x},t) &= \int\limits_{-\infty}^{+\infty} (V_i-\langle U_i\rangle)(V_j-\langle U_j\rangle) f(\boldsymbol{V};\boldsymbol{x},t)\,d\boldsymbol{V}\end{aligned} \tag{3.3}$$

In order to derive the transport equation for the mean velocity, formula (3.2) is multiplied by V_i and then integrated over $\boldsymbol{V}$. Equations for the Reynolds stresses are obtained after multiplying (3.2) by $(V_i-\langle U_i\rangle)(V_j-\langle U_j\rangle)$ and integrating. As a result we get

$$\begin{aligned}\frac{D\langle U_i\rangle}{Dt} &= -\frac{\partial\langle u_i u_j\rangle}{\partial x_j} - \frac{1}{\varrho}\frac{\partial\langle P\rangle}{\partial x_i}\\ \frac{D\langle u_i u_j\rangle}{Dt} &= -\frac{\partial\langle u_i u_j u_k\rangle}{\partial x_k} - \langle u_i u_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_k} - \langle u_j u_k\rangle\frac{\partial\langle U_i\rangle}{\partial x_k} - \\ &\quad -G_{jk}\langle u_i u_k\rangle - G_{ik}\langle u_j u_k\rangle - \frac{\epsilon}{k}\langle u_i u_j\rangle + B\delta_{ij}\end{aligned} \tag{3.4}$$

At this stage, the PDF method corresponds to the Eulerian high-Re mean velocity equation and the Reynolds stress transport (RSM) models; however the turbulent transport term $\partial\langle u_i u_j u_k\rangle/\partial x_k$ is exact and does not require modelling.

The particular form of G_{ij} depends on an assumed turbulence model (Pope, 1994). As an example, $G_{ij}=\epsilon/2k(1-2\widetilde{C})\delta_{ij}$ where $\widetilde{C}$ is a constant, corresponds to the Eulerian return-to-isotropy model derived by Rotta.

3.2. Low-Reynolds models

In the modelling of low-Reynolds numbers flows it is important to include viscous transport of momentum. This poses no particular difficulty in the Eulerian approach, where viscosity appears explicitly in terms containing Laplacians of mean quantities. A low-Re model for the Lagrangian approach was derived by Dreeben and Pope (1998); first, the viscous diffusion term was represented through a random motion of stochastic particles. Hence, the equation for a particle position writes

$$dX_i = U_i\,dt + \sqrt{2\nu}\,dW_i^X \tag{3.5}$$

This form is chosen to retrieve the $\nu\nabla^2 f$ term in the evolution equation for the PDF; this further leads to the Laplacian terms in the mean velocity and stress transport equations. Next, the expression for velocity increment is derived via the Ito equation (van Kampen, 1990)

$$dU_i = \frac{\partial U_i}{\partial t}dt + \frac{\partial U_i}{\partial x_j}dX_j + \frac{1}{2}\frac{\partial^2 U_i}{\partial x_k^2}(dX_k)^2 \tag{3.6}$$

After substituting formula (3.5) the above equation takes the form

$$dU_i = \Big(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j}\Big)\,dt + \sqrt{2\nu}\frac{\partial U_i}{\partial x_j}\,dW_j^X + \nu\frac{\partial^2 U_i}{\partial x_k^2}\,dt \tag{3.7}$$

Noting that the expression in parentheses is the RHS of Navier-Stokes equations, we get

$$dU_i = \Big(-\frac{1}{\varrho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_k^2}\Big)\,dt + \sqrt{2\nu}\frac{\partial U_i}{\partial x_j}\,dW_j^X + \nu\frac{\partial^2 U_i}{\partial x_k^2}\,dt \tag{3.8}$$

The instantaneous velocity and pressure can be written according to the Reynolds decomposition as a sum of their mean and fluctuation parts. However, gradients of u_i and p are unknown and require modelling. Hence, for these terms we apply the same closure as for high Reynolds numbers

$$\begin{aligned} dU_i = {} & \Big(-\frac{1}{\varrho}\frac{\partial\langle P\rangle}{\partial x_i} + 2\nu\frac{\partial^2\langle U_i\rangle}{\partial x_k^2}\Big)\,dt + \sqrt{2\nu}\frac{\partial\langle U_i\rangle}{\partial x_j}\,dW_j^X + \\ & + G_{ij}(U_j - \langle U_j\rangle)\,dt - \frac{1}{2}\frac{\epsilon}{k}(U_i - \langle U_i\rangle)\,dt + \sqrt{B}\,dW_i^V \end{aligned} \tag{3.9}$$

Next, a formula for the PDF corresponding to equations (3.5) and (3.9) can be derived. After the proper integration we obtain the following transport equations

$$\begin{aligned} \frac{D\langle U_i\rangle}{Dt} &= -\frac{\partial\langle u_iu_j\rangle}{\partial x_j} - \frac{1}{\varrho}\frac{\partial\langle P\rangle}{\partial x_i} + \nu\frac{\partial^2\langle U_i\rangle}{\partial x_k^2} \\ \frac{D\langle u_iu_j\rangle}{Dt} &= -\frac{\partial\langle u_iu_ju_k\rangle}{\partial x_k} - \langle u_iu_k\rangle\frac{\partial\langle U_j\rangle}{\partial x_k} - \langle u_ju_k\rangle\frac{\partial\langle U_i\rangle}{\partial x_k} + \\ & \quad + \nu\frac{\partial^2\langle u_iu_j\rangle}{\partial x_k^2} - G_{jk}\langle u_iu_k\rangle - G_{ik}\langle u_ju_k\rangle - \frac{\epsilon}{k}\langle u_iu_j\rangle + B\delta_{ij} \end{aligned} \tag{3.10}$$

It should be noted that Reynolds equation $(3.10)_1$ is exact and all the transport equations contain the required Laplacians of mean quantities.

3.3. Modelling of near-wall flows

Apart from the viscosity action, the modelling of near-wall flows should also account for non-local kinematic effects which were described in Section 2. For the purpose, Dreeben and Pope (1998) solved elliptic relaxation equations for all components of the tensor G_{ij}. In our work we applied the simplified version of the method (Manceau and Hanjalić, 2002) where only one additional equation for elliptic blending function (2.10) is solved. Components of the tensor G_{ij} are then computed from the relation

$$G_{ij} = (1 - k\alpha)G^w_{ij} + k\alpha G^h_{ij} \tag{3.11}$$

The elliptic blending method was initially derived for the Eulerian approach. Here, we apply it to the Lagrangian equations. One of the differences is concerned with the proper near-wall form G^w_{ij}. It should assure a proper scaling of Reynolds stresses in the near-wall region, namely $\langle u^2\rangle \sim y^2$, $\langle v^2\rangle \sim y^4$, $\langle w^2\rangle \sim y^2$ and $\langle uv\rangle \sim y^3$. Here, we recall a derivation proposed by Dreeben and Pope (1998). Near the wall, the turbulent transport, convective and production terms become negligible and Reynolds-stress equations $(3.10)_2$ reduce to

$$\nu\frac{\partial^2\langle u_i u_j\rangle}{\partial y^2} - \frac{\langle u_i u_j\rangle}{k}\epsilon = G_{ik}\langle u_j u_k\rangle + G_{jk}\langle u_i u_k\rangle - \frac{2}{3}G_{kl}\langle u_k u_l\rangle\delta_{ij} \tag{3.12}$$

It is assumed that near the wall the tensor G_{ij} takes the form $G_{ij} = C\epsilon/k$ where C is a constant. From the near-wall balance of terms in the kinetic energy equation $\nu\nabla^2 k = \epsilon$ we derive the scaling formula $\epsilon/k = 2\nu/y^2$ which is further used in (3.12), leading to the following differential equation

$$\nu\frac{\partial^2\langle u_i u_j\rangle}{\partial y^2} - a_{ij}\frac{\langle u_i u_j\rangle}{y^2} = \mathcal{O}(y) \tag{3.13}$$

(no summation over repeating indices). The solution to the above equation is

$$\langle u_i u_j\rangle = A_{ij}y^{(1-\sqrt{1+4a_{ij}})/2} + B_{ij}y^{(1+\sqrt{1+4a_{ij}})/2} + C_{ij}y^3 \tag{3.14}$$

The value of exponent in the first term is negative, hence the no-slip boundary condition forces $A_{ij} = 0$. In order to assure that $\langle u^2\rangle \sim y^2$ and $\langle w^2\rangle \sim y^2$ we should have $a_{11} = a_{33} = 2$; with $a_{ij} > 6$ the last term on the RHS will dominate the solution and $\langle u_i u_j\rangle \sim y^3$. The boundary form G^w_{ij} used in the present computations

$$G^w_{22} = \frac{9}{2}\frac{\epsilon}{k} \qquad G^w_{ij} = 0 \qquad \text{for } i \neq 2 \quad \text{or} \quad j \neq 2 \tag{3.15}$$

gives $a_{11} = a_{33} = 2$, $a_{12} = 11$, $a_{22} = 14$, which assure the proper scaling of all Reynolds stresses except for $\langle v^2 \rangle$, which should be of the order y^4. It is a drawback of the elliptic relaxation method that it does not provide the proper scaling of all Reynolds stresses. We also state that with the above definition, G_{ij}^w does not contract to zero when the equation for the kinetic energy is derived, however the remaining term is of a smaller order and can therefore be neglected.

4. Numerical results

The computations have been performed for the case of fully developed turbulent channel flow at $\mathrm{Re}_\tau = 395$. In the presence of large mean velocity gradients it was necessary to use a turbulence model that accounts for the rapid pressure term (Pope, 2000). For this reason, we implemented a model that corresponds to the Eulerian basic pressure-strain model (Rotta+IP) used by Durbin (1993), with the same values of constants. The tensor G_{ij}^h is

$$G_{ij}^h = \frac{1}{2}\frac{\epsilon}{k}(C_1 - 1)\delta_{ij} - C_2 \frac{\partial \langle U_i \rangle}{\partial x_j} \tag{4.1}$$

with $C_1 = 1.5$ and $C_2 = 0.6$. For this case, we do not solve a separate equation for the turbulent frequency ω, but the values of the turbulent time scale $T = 1/\langle \omega \rangle$ are read from a file with the DNS data of Moser *et al.* (1999). It is left for further work to set the proper values of coefficients in the ω equation and solve it together with the IP model for velocity. The dissipation which appears in the model equations is a sum of two components

$$\epsilon = \langle \omega \rangle k + C_T^2 \nu \langle \omega \rangle^2 \tag{4.2}$$

the first of them stands for the turbulent time scale and the second one is connected with the scale of dissipative eddies. The results illustrate the need of a specific near-wall treatment, as actually done through the elliptic blending equation. Both the mean velocity (Fig. 1a) and the turbulent kinetic energy (Fig. 1b) are in better accordance with the DNS data when the elliptic relaxation model is applied. In Fig. 1b, a sharp maximum of the kinetic energy is observed at $y/H = 0.03$ (around $y^+ \approx 12$) as evidenced by the DNS data, although it is still somewhat underpredicted.

Contrary to the streamwise component u, the fluctuations of v are damped. This is illustrated by scatter plots of near-wall streamwise, spanwise and

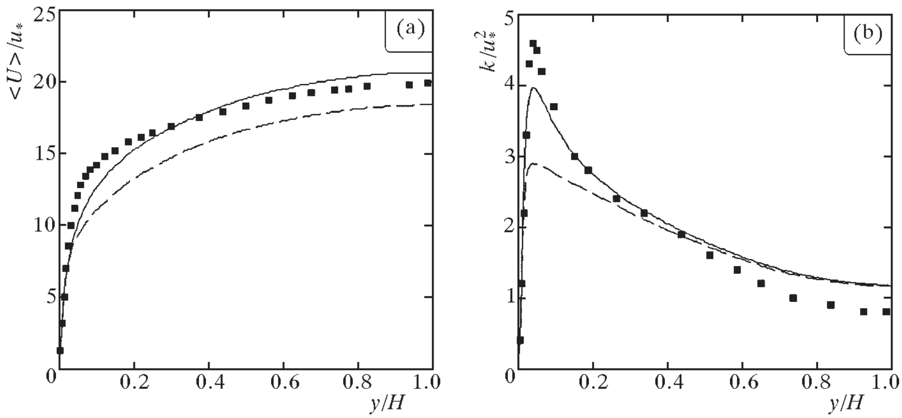

Fig. 1. Turbulent channel flow at $\mathrm{Re}_\tau = 395$, the IP model: (a) mean velocity $\langle U\rangle^+$, (b) turbulent kinetic energy k^+. DNS data (Moser *et al.*, 1999) ■; PDF computations: without elliptic relaxation (– – –), with elliptic relaxation (———)

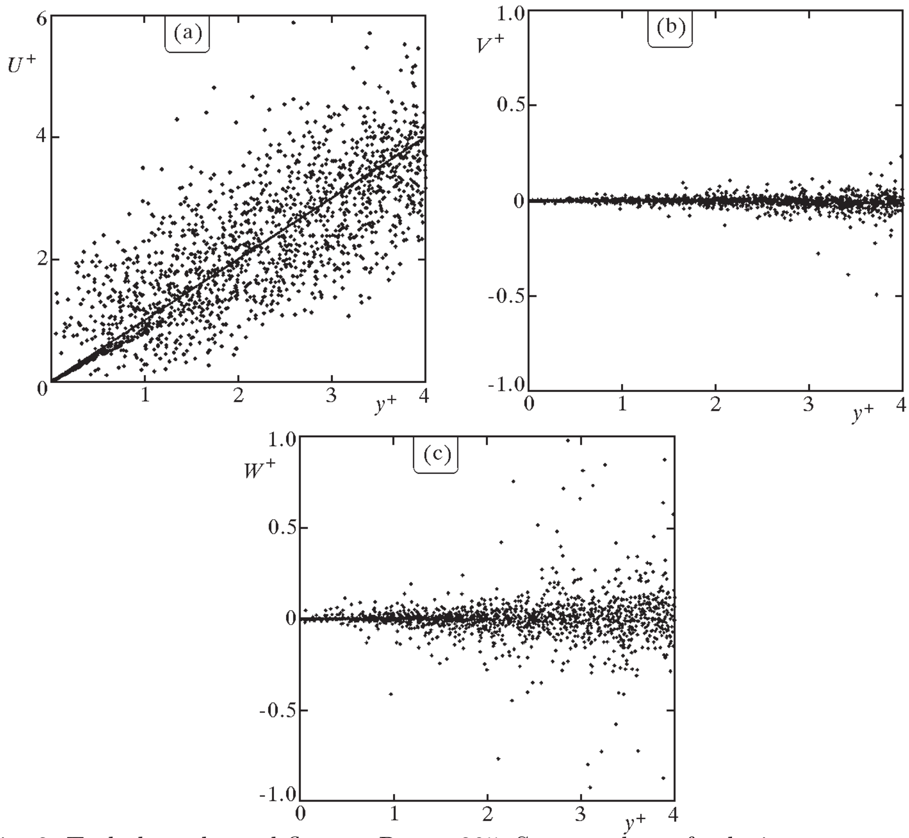

Fig. 2. Turbulent channel flow at $\mathrm{Re}_\tau = 395$; Scatter plots of velocity components near the wall: (a) streamwise (solid line – theoretical profile $U^+ = y^+$), (b) wall-normal, (c) spanwise

wall-normal velocity components presented in Fig. 2. The variance of wall-normal fluctuations is much smaller than that of the two other components.

After applying the elliptic relaxation method also the shear stresses $\langle uv \rangle$ are in a good overall agreement with the DNS data (cf. Fig. 3a). This is also clearly seen in the near-wall scaling presented in Fig. 3b. Although the elliptic relaxation improves the results, we do not obtain exactly $\langle uv \rangle \sim y^3$. This can be caused by numerical problems with the near-wall integration.

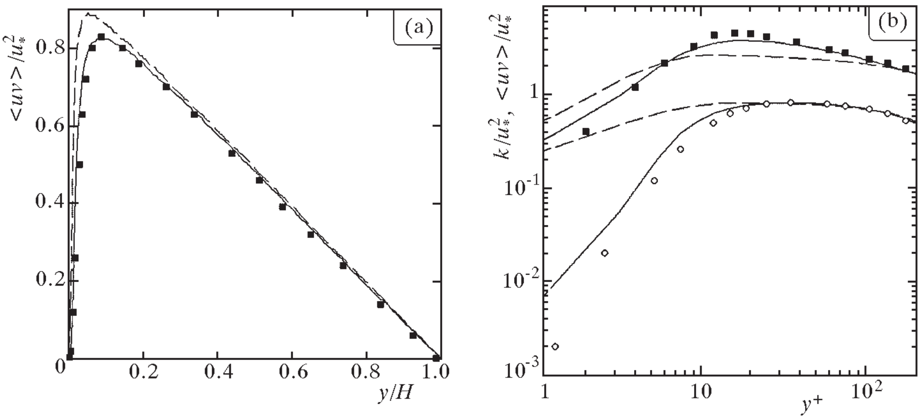

Fig. 3. Turbulent channel flow at $\mathrm{Re}_\tau = 395$; (a) turbulent shear stress $\langle uv \rangle$, (b) near-wall scaling of turbulent shear stress and kinetic energy. DNS data (Moser *et al.*, 1999): symbols; PDF computations: without elliptic relaxation (– – –), with elliptic relaxation (———)

Let us only note here that in the vicinity of the wall, the turbulent kinetic energy k tends to 0 as y^2, while its dissipation rate attains a constant value. At the same time the streamwise and spanwise velocity components scale as y. As a consequence, when the Euler discrete scheme is used to solve stochastic differential equation (3.9), two of its components tend to infinity with $y \to 0$

$$\frac{\epsilon}{k} u_i \Delta t \sim \frac{1}{y} \to \infty \qquad \text{for} \quad i \neq 2 \tag{4.3}$$

This causes serious numerical problems in the near-wall integration, unless the time step Δt is very small. This is the reason for introducing another numerical scheme based on the exponential solution to equations (3.9). The numerical scheme presented in the work of Minier *et al.* (2001) has been further developed here to account for a non-diagonal form of the matrix G_{ij} and included in the numerical algorithm.

5. Conclusions and perspectives

The elliptic relaxation method was used to model the non-local effects connected with the presence of the wall. The derivation of the method was briefly recalled in the paper. An advantage of the elliptic relaxation in comparison to the damping function approach is that there is no explicit dependence on the wall distance y or local Reynolds number. Moreover, the method is more physically sound and does not depend on flow geometry. Simplified variants of the method like e.g. k-v^2-f (Durbin, 1995) or elliptic blending model (Manceau and Hanjalić, 2002) are interesting for engineering applications due to a reduced numerical cost. Another promising perspective for the near-wall RANS models is to solve the equations in conjunction with the Large Eddy Simulation (LES) approach for the outer layer of the flow (Piomelli and Balaras, 2001). This makes it possible to perform high-Re LES computations at a reasonable cost.

In the paper we applied the elliptic blending approach in the Lagrangian PDF model to compute velocity statistics in a fully developed turbulent channel flow. Reasonable accuracy has been achieved in comparison with the available DNS data of Moser *et al.* (1999) at $\mathrm{Re}_\tau = 395$. The near-wall turbulence modelling in the PDF approach is, to a certain degree, related to the Eulerian second-moment closure. However, the turbulent transport term is closed and does not require modelling. The PDF method is often used to model chemical reactions (Libby and Williams, 1994) due to the closed source term. For the purpose, either a joint velocity-scalar PDF approach can be used or velocity statistics can be taken from external data with only scalar dynamics computed by the PDF method (Pozorski, 2002).

When supplemented with a suitable scalar transport equation, the approach presented in the paper can be applied to the case of near-wall turbulence with heat transfer to model the thermal fluctuations in the vicinity of the wall. Ultimately, such a model will serve as a building block to be used in a coupled solid-fluid case with the aim of predicting the thermal stresses in the wall material.

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Modelowanie przepływu turbulentnego w obszarze przyściennym metodą PDF

Streszczenie

W pracy przedstawiono modele turbulencji dla przepływów przyściennych wykorzystujące metodę relaksacji eliptycznej. Metodę tę zastosowano do obliczeń w podejściu Lagrange'a. Zaprezentowano przy tym model dla funkcji gęstości prawdopodobieństwa (ang. PDF — *Probability Density Function*) stosowany do przepływów o niskich liczbach Reynoldsa. Wykonano obliczenia dla przypadku przepływu turbulentnego w kanale płaskim; wyniki porównano z dostępnymi danymi DNS.

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