STRESS DISTRIBUTION IN TWO-LAYERED HALF-SPACE WITH PERIODICAL STRUCTURE CAUSED BY HERTZ PRESSURE

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The paper presents analytical solutions for displacements and stresses in an elastic layered half-space with periodic structure in the case of an axi-symmetrical contact problem. The solutions are arrived at employing the Hankel transform. Finally, a comparison is made between the results obtained using the classical theory of elasticity and those obtained within the framework of the homogenized model with microlocal parameters.

Key words: layered structures, contact, stress distribution

1. Introduction

Extensive applications of layered composite materials in both civil and mechanical engineering, geophysics as well as their widespread presence in nature (varved clays, sandstone-slates, thin layered limestones) have made composite materials an object of intensive research. To be able to create or verify strength hypotheses of the materials, adequate knowledge of stress distribution in the composites caused by mechanical factors, eg. pressure distribution, is essential.

The formulation of contact problems for periodically layered composites based on the classical theory of elasticity leads to boundary value problems for partial differential equations with discontinuous, strongly oscillating coefficients. As this approach is rather complicated, the classical formulation of the problem is often replaced with approximated models (Sun *et al.*, 1967; Achenbach, 1975; Bensoussan *et al.*, 1978; Christensen, 1980; Sanchez-Palenzia, 1980) in which mechanical properties of the medium are determined on the basis of mechanical and geometrical properties of the analyzed composite components.

In some problems, a periodically layered structure is replaced with a homogeneous transversely isotropic medium (Postma, 1955). In this model, calculated strains and stresses are averaged in the fundamental layer. On the other hand, the homogenized model with microlocal parameters (Matysiak and Woźniak, 1987; Woźniak, 1987) makes it possible to evaluate local values of strains and stresses in each sublayer. Owing to the simplification of the mathematical solution, the homogenized model with microlocal parameters has been applied to resolve some contact problems of mechanics and geomechanics (Kaczyński and Matysiak, 1988, 1993, 2001; Kulchytsky-Zhyhailo and Matysiak, 1995; Matysiak and Pauk, 1995).

However, the application of the homogenized model to a periodically layered structure results in errors. To estimate these errors, it is necessary to compare solutions for the periodically layered structure based on two approaches:

(i) homogenized model with microlocal parameters

(ii) non-homogeneous medium with separate elastic layers.

In the case of heat transfer and theory of elasticity, such investigations were carried out in Kulchytsky-Zhyhailo and Kołodziejczyk (2004, 2007), Kulchytsky-Zhyhailo and Matysiak (2005a,b). It was shown that the solutions obtained using these models were consistent if the

ratio δ of thickness of the fundamental layer to the size of heating area or loading area was small ($\delta < 0.2$).

One of the fundamental issues of contact mechanics is the problem of pressing a rigid sphere into an elastic half-space. In the case of a homogeneous half-space, the problem is reduced to the loading of the surface by Hertz pressure

$$p(r) = \frac{3}{2}p_0\sqrt{1 - r^2} \tag{1.1}$$

where: p_0 – mean contact pressure, r – dimensionless cylindrical coordinate ($r < 1$) with respect to the radius a of the contact area.

Similar results were obtained for a periodically layered half-space using the homogenized model with microlocal parameters (Kulchytsky-Zhyhailo and Matysiak, 1995). For a non-homogeneous half-space, expression (1.1) is a good description of pressure distribution for the case when $\delta < 0.2$ (Kulchytsky-Zhyhailo and Kołodziejczyk, 2007). It should be noted here that the authors focused their attention on two parameters only, i.e. pressure distribution and size of the contact area.

In the present paper, the authors focus on stress analysis in a layered medium with periodical structure caused by pressure (1.1) and compare the results obtained using the two models.

2. Formulation of the problem

Let us consider a two-layered elastic half-space with a periodic structure loaded by Hertz pressure (Fig. 1). A repeated fundamental layer is composed of two homogeneous elastic sublayers with thickness l_1 and l_2. Mechanical properties of the sublayers are characterized by Young's moduli E_1 and E_2 and Poisson's ratios ν_1 and ν_2.
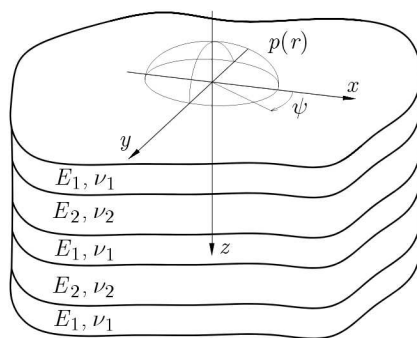

Fig. 1. Diagram of the analyzed problem

3. Homogenized model

Initially, the basis for consideration is the homogenized model with microlocal parameters. The governing equations of the homogenized model given in terms of macro-displacements can be written in the following form (Kaczyński and Matysiak, 2001):
— equations

$$A_1\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\Big) + (A_3 + A_5)\frac{\partial^2 w}{\partial r \partial z} + A_5\frac{\partial^2 u}{\partial z^2} = 0$$

$$(A_3 + A_5)\Big(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\Big) + A_4\frac{\partial^2 w}{\partial z^2} + A_5\Big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\Big) = 0 \tag{3.1}$$

— boundary conditions on the surface of the half-space

$$\sigma_{zz} = -p(r)H(1 - r) \qquad \sigma_{rz} = 0 \qquad z = 0 \tag{3.2}$$

— regularity conditions in infinity

$$u, w \to 0 \qquad r^2 + z^2 \to \infty \tag{3.3}$$

where

$$A_1 = \left(\widetilde{\lambda} + 2\widetilde{\mu} - \frac{[\lambda]^2}{\widehat{\lambda} + 2\widehat{\mu}}\right) \qquad A_3 = \widetilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\widehat{\lambda} + 2\widehat{\mu}} \qquad A_5 = \widetilde{\mu} - \frac{[\mu]^2}{\widehat{\mu}}$$

$$A_4 = (\widetilde{\lambda} + 2\widetilde{\mu}) - \frac{([\lambda] + 2[\mu])^2}{\widehat{\lambda} + 2\widehat{\mu}} \qquad \widetilde{\lambda} = \eta\lambda_1 + (1 - \eta)\lambda_2 \qquad [\lambda] = \eta(\lambda_1 - \lambda_2)$$

$$\widehat{\lambda} = \eta\lambda_1 + \frac{\eta^2}{1 - \eta}\lambda_2 \qquad \widetilde{\mu} = \eta\mu_1 + (1 - \eta)\mu_2 \qquad [\mu] = \eta(\mu_1 - \mu_2)$$

$$\widehat{\mu} = \eta\mu_1 + \frac{\eta^2}{1 - \eta}\mu_2 \qquad \lambda_i = \frac{E_i\nu_i}{(1 + \nu_i)(1 - 2\nu_i)} \qquad \mu_i = \frac{E_i}{2(1 + \nu_i)}$$

and $\eta = l_1/l$, u, w are dimensionless displacements, $\boldsymbol{\sigma}$ – stress tensor, $H(\cdot)$ – Heviside's unit step function.

Relations between the stresses and macro-displacements are as follows (Kaczyński, 1994)

$$\sigma_{zz} = A_3\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + A_4\frac{\partial w}{\partial z} \qquad \sigma_{rr}^{(i)} = K_i\frac{\partial u}{\partial r} + L_i\frac{u}{r} + M_i\frac{\partial w}{\partial z}$$

$$\sigma_{\psi\psi}^{(i)} = L_i\frac{\partial u}{\partial r} + K_i\frac{u}{r} + M_i\frac{\partial w}{\partial z} \qquad \sigma_{rz} = A_5\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{3.4}$$

and

$$K_i = \lambda_i + 2\mu_i - h_i\lambda_i\frac{[\lambda]}{\widehat{\lambda} + 2\widehat{\mu}} \qquad L_i = \lambda_i - h_i\lambda_i\frac{[\lambda]}{\widehat{\lambda} + 2\widehat{\mu}}$$

$$M_i = \lambda_i - h_i\lambda_i\frac{[\lambda] + 2[\mu]}{\widehat{\lambda} + 2\widehat{\mu}} \qquad i = 1, 2 \tag{3.5}$$

$$h_1 = 1 \qquad h_2 = -\frac{\eta}{1 - \eta}$$

where i is the sublayer number in the fundamental layer. From relations (3.4), it follows that the presented model yields separate equations for computation of stress tensor components in the first and second sublayer of the fundamental layer. However, for σ_{zz} and σ_{rz} they are identical (index i can be omitted). For the stress components σ_{rr} and $\sigma_{\psi\psi}$ having jump discontinuities on boundaries between the layers we have two formulas for each.

Solutions of the set of equations (3.1) can be expressed as follows (Elliot, 1949a,b)

$$u = \frac{\partial \Psi_1}{\partial r} + \frac{\partial \Psi_2}{\partial r} \qquad w = \kappa_1\frac{\partial \Psi_1}{\partial z} + \kappa_2\frac{\partial \Psi_2}{\partial z} \tag{3.6}$$

where

$$\kappa_k = \frac{\gamma_k^2 A_1 - A_5}{A_3 + A_5} \tag{3.7}$$

Ψ_k are elastic potentials satisfying equations

$$\frac{\partial^2 \Psi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi_i}{\partial r} + \frac{\partial^2 \Psi_i}{\partial z^2}\gamma_i^2 = 0 \qquad i = 1, 2 \tag{3.8}$$

γ_i^2 are roots of the characteristic equation

$$A_1A_5\gamma^4 + (A_3^2 + 2A_3A_5 - A_1A_4)\gamma^2 + A_4A_5 = 0 \tag{3.9}$$

In order to find a solution to equations (3.8), Hankel's transform is used

$$\widetilde{f}(s,z) = \int\limits_0^\infty rf(r,z)J_0(sr)\,dr \tag{3.10}$$

where J_0 is the Bessel function. A general solution to the set of equations (3.8) satisfying conditions (3.3) takes the form

$$\widetilde{\Psi}_i(s,z) = C_i(s)\exp\left(-\frac{sz}{\gamma_i}\right) \qquad i = 1,2 \tag{3.11}$$

Expressing (3.4)-(3.6) in Hankel's transform domain and satisfying boundary conditions (3.2), we obtain

$$C_1 = C_{-1}^{(p)}\widetilde{p}(s)\frac{1}{s^2} \qquad C_2 = C_0^{(p)}\widetilde{p}(s)\frac{1}{s^2} \tag{3.12}$$

where

$$\widetilde{p}(s) = \frac{3}{2}p_0\sqrt{\frac{\pi}{2}}\frac{J_{3/2}(s)}{\sqrt{s^3}} \qquad C_{-1}^{(p)} = \frac{\gamma_1}{A_5(\gamma_2 - \gamma_1)}\frac{A_3 + A_5}{A_3 + A_1\gamma_1^2}$$

$$C_0^{(p)} = -\frac{\gamma_2}{A_5(\gamma_2 - \gamma_1)}\frac{A_3 + A_5}{A_3 + A_1\gamma_2^2}$$

Taking into account (3.4), (3.6), (3.11), we obtain

$$u(r,z) = \int\limits_0^\infty U_r(s,z)\widetilde{p}(s)J_1(sr)\,ds \qquad w(r,z) = \int\limits_0^\infty U_z(s,z)\widetilde{p}(s)J_0(sr)\,ds$$

$$\sigma_{rr}^{(i)} = \int\limits_0^\infty S_{r1}^{(i)}(s,z)\widetilde{p}(s)J_0(sr)s\,ds - \frac{1}{r}\int\limits_0^\infty S_{r2}^{(i)}(s,z)\widetilde{p}(s)J_1(sr)\,ds \qquad i = 1,2$$

$$\sigma_{\psi\psi}^{(i)} = \int\limits_0^\infty (S_{r1}^{(i)} - S_{r2}^{(i)})\widetilde{p}(s)J_0(sr)s\,ds + \frac{1}{r}\int\limits_0^\infty S_{r2}^{(i)}(s,z)\widetilde{p}(s)J_1(sr)\,ds \qquad i = 1,2 \tag{3.13}$$

$$\sigma_{zz} = \int\limits_0^\infty S_{z1}(s,z)\widetilde{p}(s)J_0(sr)s\,ds \qquad \sigma_{rz} = \int\limits_0^\infty S_{z2}(s,z)\widetilde{p}(s)J_1(sr)\,ds$$

where

$$U_r(s,z) = -\sum_{k=1}^2 C_{k-2}^{(p)}\exp\left(-\frac{sz}{\gamma_k}\right) \qquad U_z(s,z) = -\sum_{k=1}^2 \kappa_k\gamma_k^{-1}C_{k-2}^{(p)}\exp\left(-\frac{sz}{\gamma_k}\right)$$

$$S_{r1}^{(i)}(s,z) = \sum_{k=1}^2\left(M_i\frac{\kappa_k}{\gamma_k^2} - K_i\right)C_{k-2}^{(p)}\exp\left(-\frac{sz}{\gamma_k}\right) \qquad S_{r2}^{(i)}(s,z) = 2\mu_i\sum_{k=1}^2 C_{k-2}^{(p)}\exp\left(-\frac{sz}{\gamma_k}\right)$$

$$S_{z1}(s,z) = \sum_{k=1}^2\left[A_4\frac{\kappa_k}{\gamma_k^2} - A_3\right]C_{k-2}^{(p)}\exp\left(-\frac{sz}{\gamma_k}\right) \qquad S_{z2}(s,z) = A_5\sum_{k=1}^2(1+\kappa_k)\gamma_k^{-1}C_{k-2}^{(p)}\exp\left(\frac{sz}{\gamma_k}\right)$$

4. Classical model

Let us consider a layered half-space as n fundamental layers ($2n$ sublayers) joined with a homogeneous part of the half-space (Fig. 2). It was shown (Kulchytsky-Zhyhailo and Kołodziejczyk, 2007) that over a fixed number of separately considered layers, the mechanical properties of the homogeneous part of the half-space can be quite arbitrarily chosen. They may include the properties of either the first or the second layer. However, considering the fact that the homogeneous part of the half-space possesses averaged properties of the homogenized medium described above, it is possible to obtain a sufficiently exact approximation solution of the problem for a significantly smaller number of separately considered layers.
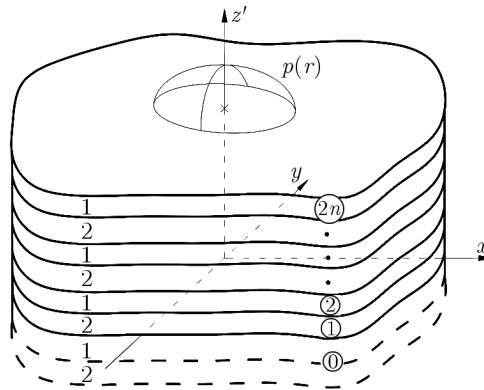

Fig. 2. Diagram of the analyzed problem

The equations for separately considered layers can be written as follows

$$\Delta u_k - \frac{u_k}{r^2} + d_k \frac{\partial \theta_k}{\partial r} = 0 \qquad \Delta w_k + d_k \frac{\partial \theta_k}{\partial z'} = 0 \qquad k = 1, 2, \dots, 2n \qquad (4.1)$$

where $\theta_k = \partial u_k/\partial r + u_k/r + \partial w_k/\partial z'$, k – layer number, $d_{2j} = 1/(1 - 2\nu_1)$, $d_{2j-1} = 1/(1 - 2\nu_2)$, $j = 1, 2, \dots, n$.

Differential equations (3.1) for the homogenized medium "0" should be attached to the above system of equations. The solution to the equations should satisfy the following conditions:
— boundary conditions on the surface $z' = n\delta$

$$\sigma_{zz}^{(n)} = -p(r)H(1 - r) \qquad \sigma_{rz}^{(n)} = 0 \qquad z' = n\delta \qquad (4.2)$$

— boundary conditions between the layers and between the layers and the homogenized medium

$$\begin{aligned} u_k &= u_{k+1} & w_k &= w_{k+1} & \sigma_{rz}^{(k)} &= \sigma_{rz}^{(k+1)} \\ \sigma_{zz}^{(k)} &= \sigma_{zz}^{(k+1)} & z' &= z_k & k &= 0, 1, \dots, 2n - 1 \end{aligned} \qquad (4.3)$$

— regularity conditions in infinity

$$u_k, w_k \to 0 \quad \text{for} \quad r^2 + z'^2 \to \infty \qquad (4.4)$$

where z_k is the coordinate z' of the point of intersection of the upper plane limiting the k-th layer with z' axis.

The solution to the classical elasticity theory problem formulated in this way was obtained using the Hankel transform. Applying the algorithm described in Kulchytsky-Zhyhailo and Kołodziejczyk (2007), we obtain expressions to calculate displacements and stresses in form (3.13),

where functions U_r, U_z, S_{z1}, S_{z2}, S_{r1}, S_{r2} should be replaced with the following

$$2U_{z'}^{(k)}(s,z') = D_k s_k C_{4k-3}^{(p)}(s) + D_k c_k C_{4k-2}^{(p)}(s) + 2ss_k C_{4k-1}^{(p)}(s) + 2sc_k C_{4k}^{(p)}(s)$$

$$2U_r^{(k)}(s,z') = \frac{\mu_1}{\mu_k} S_{r2}^{(k)}(s,z') = ((2+d_k)s_k + D_k c_k) C_{4k-3}^{(p)}(s)$$

$$+ ((2+d_k)c_k + D_k s_k) C_{4k-2}^{(p)}(s) + 2sc_k C_{4k-1}^{(p)}(s) + 2ss_k C_{4k}^{(p)}(s)$$

$$\frac{\mu_1}{\mu_k} S_{r1}^{(k)}(s,z') = ((1+2d_k)s_k + D_k c_k) C_{4k-3}^{(p)}(s) + ((1+2d_k)c_k + D_k s_k) C_{4k-2}^{(p)}(s)$$

$$+ 2sc_k C_{4k-1}^{(p)}(s) + 2ss_k C_{4k}^{(p)}(s) \tag{4.5}$$

$$\frac{\mu_1}{\mu_k} S_{z'1}^{(k)}(s,z') = (s_k + D_k c_k) C_{4k-3}^{(p)}(s) + (c_k + D_k s_k) C_{4k-2}^{(p)}(s)$$

$$+ 2sc_k C_{4k-1}^{(p)}(s) + 2ss_k C_{4k}^{(p)}(s)$$

$$\frac{\mu_1}{\mu_k} S_{z'2}^{(k)}(s,z') = ((1+d_k)c_k + D_k s_k) C_{4k-3}^{(p)}(s) + ((1+d_k)s_k + D_k c_k) C_{4k-2}^{(p)}(s)$$

$$+ 2ss_k C_{4k-1}^{(p)}(s) + 2sc_k C_{4k}^{(p)}(s)$$

where

$$c_k = \cosh(s(z_k - z')) \qquad s_k = \sinh(s(z_k - z')) \qquad D_k = d_k s(z_k - z') \qquad \mu_k = \frac{E_k}{2(1+\nu_k)}$$

Expressions (4.5) and the corresponding equations for the homogenized half-space "0" include $(8n+2)$ unknown functions $C_k(s)$, $k = -1, 0, 1, \ldots, 8n$. Satisfying boundary conditions (4.2) and (4.3), we obtain a system of algebraic equations allowing us to determine the unknown functions (Kulchytsky-Zhyhailo and Kołodziejczyk, 2007).

5. Numerical results and discussion

The stress distribution arising in the non-homogeneous half-space obtained within the framework of the homogenized model depends on four dimensionless parameters: ratio E_1/E_2 of Young's moduli of the materials of individual layers, Poisson's ratios ν_1 and ν_2, the thickness ratio of the upper layer to the thickness of the periodicity cell $\eta = l_1/l = \delta_1/\delta$. When considering the layered medium, the stress distribution is additionally dependent on the parameter δ. In order to confine the range of the considered parameters, it was assumed that $\nu_1 = \nu_2 = 0.3$. To emphasize the differences that arise between the solutions obtained from both models the ratio of Young's modulus was assumed $E_1/E_2 = 8$. Since the results change only insignificantly within the considered range of δ parameter when altering the sequence of layers in the periodicity cell, we restrict our analysis to the case when $E_1/E_2 > 1$.

The calculations show that the stress distributions σ_{zz} and σ_{rz} for the homogenized model are very close to the corresponding stress distribution for the layered medium. All the distributions are similar to those obtained for a homogeneous medium in the classical Hertz contact problem. However, significantly different results can be observed for the stress distribution of σ_{rr} and $\sigma_{\psi\psi}$. These stress distributions along z-axis are shown in Fig. 3 (at z-axis $\sigma_{rr} = \sigma_{\psi\psi}$). For a non-homogeneous medium on each boundary between layers these stress components undergo a jump. This discontinuity is illustrated by black vertical lines. The two gray lines are obtained on the basis of the homogenized model, and they describe the stress distribution in the layers with smaller and greater Young's modulus respectively. As seen in Fig. 3, the stress distribution obtained from the homogenized model describes the stress distribution in the layered medium with sufficient accuracy. The difference between the results decreases with reduction of δ parameter. Thus, a suggestion can be made that a detailed analysis of stress distribution in a

non-homogeneous medium can be based on the homogenized model, which will radically reduce the time of computation.
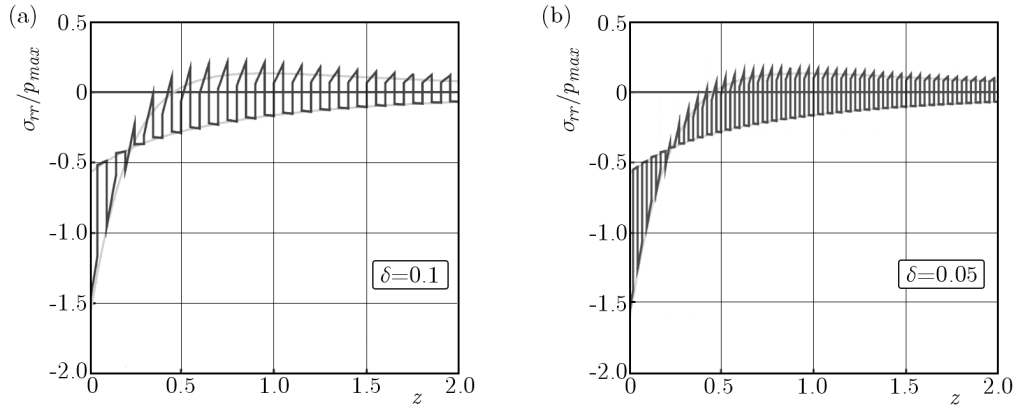

Fig. 3. Stress distribution σ_{rr}/p_{max} along z-axis for different values of δ; black line – solution for the layered medium, gray lines – solution for the homogenized model, dashed lines – boundaries between layers; $E_1/E_2 = 8$, $\nu_1 = \nu_2 = 0.3$, $\eta = 0.5$

When analyzing the stress level, special attention should be paid to the existence of regions of tensile stresses that are determined on the basis of principal stress σ_1 as well as to the distribution of the second invariant of the deviatoric stress tensor

$$J_2 = \frac{1}{\sqrt{6}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \qquad (5.1)$$

Typical distributions of dimensionless parameters σ_1 and J_2 with respect to parameter $p_{max} = (3/2)p_0$ are shown in Figs. 4 and 5. In layers with greater Young's modulus two regions of tensile stresses occur. The first is found in the vicinity of the points of the unloaded surface of the half-space (Fig. 4a). Maximum of the σ_1 stress, similarly to the contact problem for the homogeneous medium, occurs on the boundary of the loaded circle. The stress can be determined from the expression

$$\sigma_1^{(1)}(1,0) = \frac{p_0 \mu_1}{\sqrt{A_1 A_4} + A_3} \qquad (5.2)$$

The other region arises at a certain depth. Tensile stresses in this region are generally much smaller than the stress at points $z = 0$, $r = 1$.
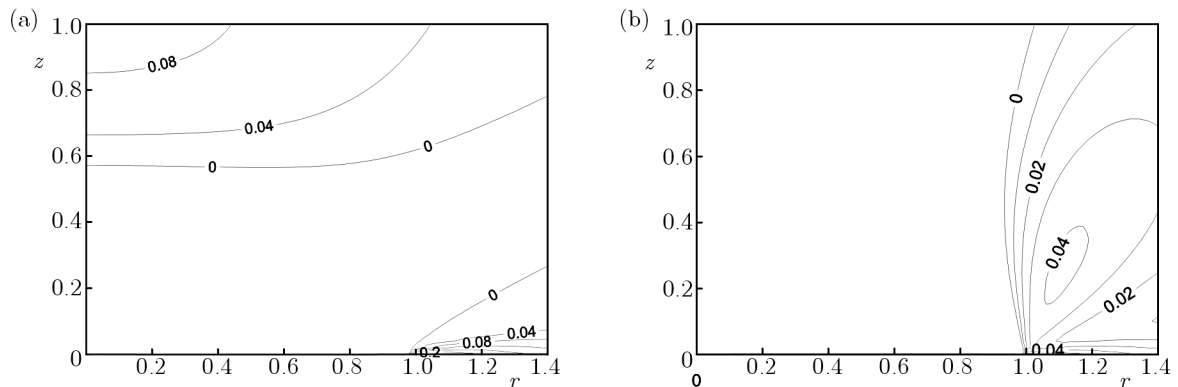

Fig. 4. Tensile stress distribution σ_1 in layers with: (a) greater Young's modulus, (b) smaller Young's modulus

In layers with smaller Young's modulus, the region of tensile stresses occurs beneath the unloaded surface of the half-space (Fig. 4b). In this case, there exist two local maxima of σ_1

stresses. The first at the boundary of the loading area, the other at a certain small depth in the vicinity of the boundary of the contact surface. It should be noted that for points $z = 0$, $r = 1$ we have a simple relation $\sigma_1^{(1)}/\sigma_1^{(2)} = \mu_1/\mu_2$.

The distribution of J_2 parameter in the layers with greater Young's modulus has three local maxima (Fig. 5a). The first one occurs at some depth on z-axis, the second one – at point $z = 0$, $r = 0$, the third one – at points $z = 0$, $r = 1$. The distribution of J_2 parameter in layers with smaller Young's modulus (Fig. 5b) is similar the one that occurs for the homogeneous isotropic medium in the classical Hertz contact problem. The only maximum arises on z-axis at some depth beneath the surface.
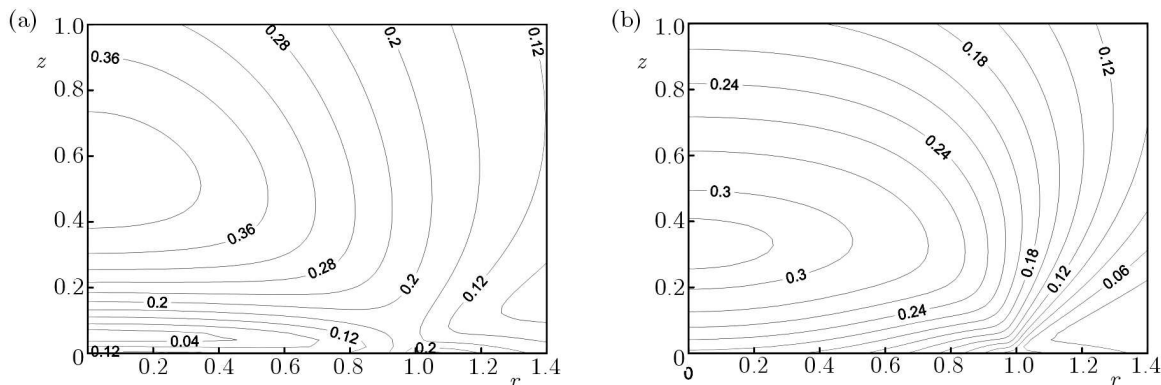

Fig. 5. Distribution of J_2 in layers with: (a) greater Young's modulus, (b) smaller Young's modulus

Figures 6 and 7 show maximum values of relative parameters $\sigma_1/\sigma_{1,H}^{(i)}$ and $J_2/J_{2,H}^{(i)}$ as a function of η for three values $E_1/E_2 = 2, 4, 8$ described by lines 1,2,3 respectively. $\sigma_{1,H}^{(i)}$, $J_{2,H}^{(i)}$ are corresponding parameters given by the classical Hertz contact problem for the homogeneous medium with mechanical properties of the i-th layer $(i = 1, 2)$ (Timoshenko and Goodier, 1951; Johnson, 1985)

$$\sigma_{1,H}^{(i)} = \frac{1}{3}p_{max,H}^{(i)}(1 - 2\nu_i) \qquad J_{2,H}^{(i)}(\nu_i = 0.3) = 0.356 p_{max,H}^{(i)} \qquad (5.3)$$

where $p_{max,H}$ is the maximum pressure in the classical Hertz contact problem of pressing of a sphere into a homogeneous isotropic half-space and

$$\frac{p_{max,H}^{(i)}}{p_{max}} = \left[\frac{A_1(\gamma_1 + \gamma_2)\mu_i}{(A_1 A_4 - A_3^2)(1 - \nu_i)} \right]^{2/3} \qquad (5.4)$$

Black lines in Figs. 6a and 6b show values of the parameter on the boundary of load circle. Gray lines show maximum values on z-axis.

It can be seen in Fig. 6a that tensile stresses on z-axis and on the boundary of the circle of contact achieve comparable values for small values of η and relatively large difference in values of Young's moduli. For η close to 1, the tensile stress does not appear at all.

It follows from Fig. 6b that the local maximum J_2 existing on z-axis dominates the local maximum at the boundary of the loading circle. One exception is the small range $0.37 < \eta < 0.47$ $(E_1/E_2 = 8)$. It should be noted that gray line 3 consists of two parts. The left hand part corresponds to the case when the maximum J_2 at the center of the loading circle prevails in the maximum J_2 on z-axis at some depth. The part on the right refers to the opposite situation. When $\eta \to 1$, the maximum on z-axis is at the same depth as in the classical Hertz contact problem, i.e. $z = 0.48$ $(\nu_1 = 0.3)$. With decreasing η, the depth of maximum J_2 increases (e.g. $E_1/E_2 = 4$ from 0.48 $(\eta \to 1)$ to 0.66 $(\eta = 0.01)$).
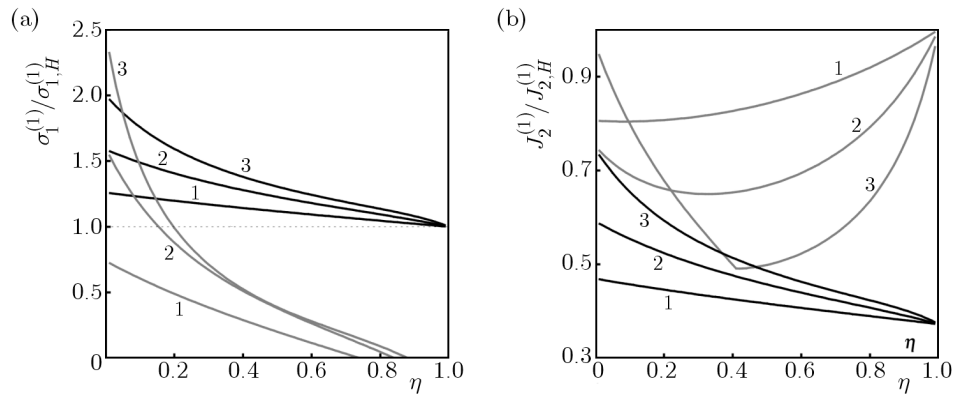
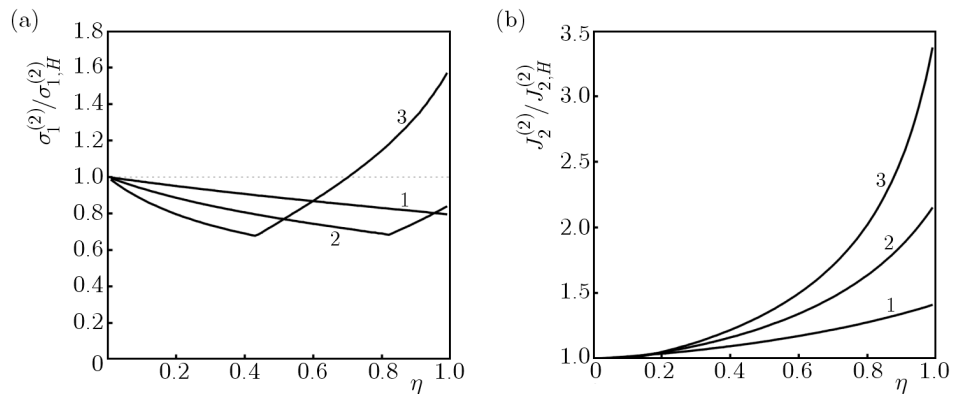Fig. 6. Distribution of (a) σ_1, (b) J_2 in layers with greater Young's modulus as function of η

Fig. 7. Distribution of (a) σ_1, (b) J_2 along z-axis in layers with smaller Young's modulus as function of η

Lines in Figs. 7a and 7b show maximum σ_1 and J_2 in the layers with smaller Young's modulus. Curves 2 and 3 in Fig. 7a consist of two parts. The one on the left corresponds to the case when maximum σ_1 is on the boundary of the loaded circle, the one on the right – when maximum σ_1 is at some depth under the unloaded surface.

When $\eta \to 0$, the maximum J_2 in layers with smaller Young's modulus occurs at the same depth as in the Hertz contact problem. With increasing η, the depth of maximum J_2 decreases (i.e $E_1/E_2 = 4$ from 0.48 ($\eta \to 0$) to 0.25 ($\eta = 0.99$).

6. Conclusions

The paper presents the stress state in a layered half-space with periodical structure caused by Hertz pressure. Two different models of the layered medium have been considered: one – a homogenized model in which mechanical properties of the medium are determined using both the mechanical and geometrical properties of the analyzed composite components, and the other – where a certain number of layers are considered as separate homogeneous and interacting entities satisfying the conditions of ideal mechanical contact. In conclusion, it is demonstrated that stresses in layers of different mechanical properties are quite different. Additionally, the location of tensile stress regions σ_1 as well as the distribution of the second invariant of the deviatoric stress tensor J_2 are presented.

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