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FUNDAMENTAL SOLUTIONS RELATED TO THERMAL STRESS INTENSITY FACTORS OF MODES I AND II – THE AXIALLY SYMMETRIC PROBLEM

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This elaboration considers the crack problems for infinite thermoelastic solids subjected to steady temperature or heat flux. The crack faces are assumed to be insulated. Green's functions are obtained for the thermal stress intensity factors of modes I and II. The Green's functions are defined as a solution to the problem of a thermoelastic transversely isotropic solid with a penny-shaped or an external crack under general axisymmetric thermal loadings acting along a circumference on the plane parallel to the crack plane.

Key words: thermoelasticity, anisotropy, crack problems, Green's functions, stress intensity factors of mode I and II

1. Introduction

The penny-shaped crack in a temperature field was treated by Olesiak and Sneddon (1960); the problem was symmetrical with respect to the crack plane. The features of antisymmetry were presented by Florence and Goodier (1963) in the linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack.

In this paper, we consider the steady thermal stress in a cracked solid. The problems of the crack treated here are solved by using two types of axisymmetric ring thermal loadings as fundamental solutions: a uniform heat flux and temperature. The research is aimed at the assessing of the effect of dissimilar thermal conditions on the stress intensity factors. The stress intensity factors of modes I and II are derived in this study in terms of elementary functions. The results presented for general cases are new, but some of those related to special cases of isotropic or transversely isotropic solids with crack surface thermal loadings have been already known (cf. Olesiak and Sneddon, 1960; Florence and Goodier, 1963; Rogowski, 1984).

2. Basic equations

The basic equations of axisymmetric thermal stress problems for homogeneous transversely isotropic bodies are the equilibrium equations (in the absence of body forces)

$$\sigma_{rr,r}+\sigma_{rz,z}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0 \qquad \sigma_{rz,r}+\sigma_{zz.z}+\frac{1}{r}\sigma_{rz}=0 \tag{2.1}$$

the strain-displacement relations

$$\begin{aligned} e_{rr}&=u_{r,r} & e_{\theta\theta}&=\frac{u_r}{r}\\ e_{zz}&=u_{z,z} & 2e_{rz}&=u_{r,z}+u_{z,r} \end{aligned} \tag{2.2}$$

the constitutive equations

$$\begin{aligned} \sigma_{rr}&=c_{11}e_{rr}+c_{12}e_{\theta\theta}+c_{13}e_{zz}-\beta_1T\\ \sigma_{\theta\theta}&=c_{12}e_{rr}+c_{11}e_{\theta\theta}+c_{13}e_{zz}-\beta_1T\\ \sigma_{zz}&=c_{13}e_{rr}+c_{13}e_{\theta\theta}+c_{33}e_{zz}-\beta_3T\\ \sigma_{rz}&=2c_{44}e_{rz} \end{aligned} \tag{2.3}$$

and the heat conduction equation (steady state without heat generation)

$$T_{,rr}+r^{-1}T_{,r}+s_0^{-2}T_{,zz}=0 \tag{2.4}$$

where partial differentiation is indicated by the comma followed by the variables, c_{ij} are the elastic constants of a transversely isotropic material, $\beta_1=(c_{11}+c_{12})\alpha_r+c_{13}\alpha_z$, $\beta_3=2c_{13}\alpha_r+c_{33}\alpha_z$ are the thermal stress coefficients, α_r and α_z are the coefficients of the linear thermal expansion in the radial and axial direction, $s_0^2=\lambda_r/\lambda_z$, λ_r and λ_z are the thermal conductivities in the radial and axial direction. By substituting Eq. (2.3) into equilibrium equations (2.1) and using relations (2.2), we obtain

$$\begin{aligned} &c_{11}\Big(u_{r,rr}+\frac{1}{r}u_{r,r}-\frac{1}{r^2}u_r\Big)+c_{44}u_{r,zz}+(c_{13}+c_{44})u_{z,rz}-\beta_1T_{,r}=0\\ &c_{44}\Big(u_{z,rr}+\frac{1}{r}u_{z,r}\Big)+c_{33}u_{z,zz}+(c_{13}+c_{44})\Big(u_{r,rz}+\frac{1}{r}u_{r,z}\Big)-\beta_3T_{,z}=0 \end{aligned} \tag{2.5}$$

To solve partial differential equations (2.4) and (2.5) we introduce potential functions which relate to the displacements as follows (Rogowski, 1978)

$$u_r = (k\varphi_1 + \varphi_2 + \varphi_0)_{,r} \qquad\qquad u_z = (\varphi_1 + k\varphi_2 + l\varphi_0)_{,z} \tag{2.6}$$

and the Hankel transforms defined as follows

$$u_r^* = \int\limits_0^\infty u_r r J_1(\xi r)\, dr \qquad\qquad u_z^* = \int\limits_0^\infty u_z r J_0(\xi r)\, dr \tag{2.7}$$

where ξ is the Hankel parameter and $J_\nu(\xi r)$ denotes the Bessel function of the first kind of order ν. The Hankel transform is its own inverse.

The potential functions must satisfy the following equations

$$\begin{aligned} &\varphi_{i,rr} + \frac{1}{r}\varphi_{i,r} + \frac{1}{s_i^2}\varphi_{i,zz} = 0 \qquad\qquad i = 0,1,2 \\ &\varphi_{0,zz} = M s_0^2 T \end{aligned} \tag{2.8}$$

where $s_0^2 = \lambda_r/\lambda_z$, s_i^2 $(i = 1,2)$ are the roots of the equation

$$c_{33}c_{44}s^4 - [c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})]s^2 + c_{11}c_{44} = 0 \tag{2.9}$$

and k, l, M are the material parameters defined as follows

$$\begin{aligned} k &= \frac{c_{33}s_1^2 - c_{44}}{c_{13} + c_{44}} \qquad\qquad l = \frac{\beta_1(c_{13} + c_{44}) - \beta_3(c_{11} - s_0^2 c_{44})}{\beta_1(c_{33}s_0^2 - c_{44}) - \beta_3 s_0^2(c_{13} + c_{44})} \\ M &= \frac{\beta_1(c_{33}s_0^2 - c_{44}) - \beta_3 s_0^2(c_{13} + c_{44})}{s_0^2(c_{13} + c_{44})^2 - (c_{11} - s_0^2 c_{44})(c_{33}s_0^2 - c_{44})} \end{aligned} \tag{2.10}$$

The thermal stresses components σ_{zz} and σ_{rz} are represented as follows

$$\begin{aligned} \sigma_{zz} &= G_z(k+1)(s_1^{-2}\varphi_1 + s_2^{-2}\varphi_2)_{,zz} + G_z M(1+l)T \\ \sigma_{rz} &= G_z(k+1)(\varphi_1 + \varphi_2)_{,rz} + G_z(1+l)\varphi_{0,rz} \end{aligned} \tag{2.11}$$

where $G_z = c_{44}$ is the shear modulus along the z-axis.

The stress components σ_{rr} and $\sigma_{\theta\theta}$ may be similarly expressed.

Consider an infinite transversely isotropic elastic solid containing a penny-shaped crack with its diameter $2a$ or an external crack covering the outside of a circle of the radius a, as shown in Fig. 1. Denote by (r,θ,z) the cylindrical co-ordinate system with its origin at the middle point of the penny-shaped crack face or of the bonding region, respectively. The thermal loading conditions (Fig. 2 and Fig. 3) may be decomposed into symmetrical (Fig. 4) and antisymmetrical (Fig. 5) with respect to the crack plane.

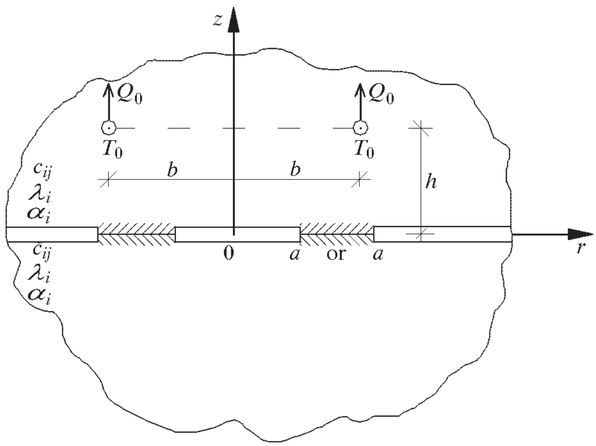


Fig. 1. Thermoelastic solid with a penny-shaped or external crack under thermal loadings

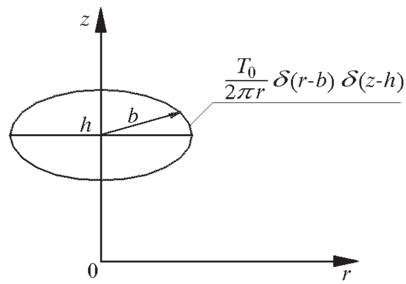


Fig. 2. Temperature loading acting along a circle

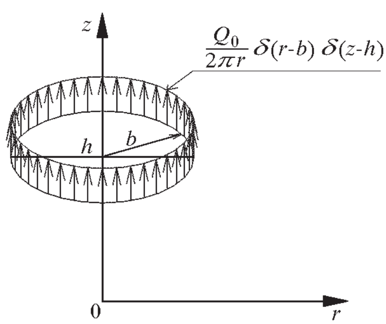


Fig. 3. Axial heat flux acting along a circle

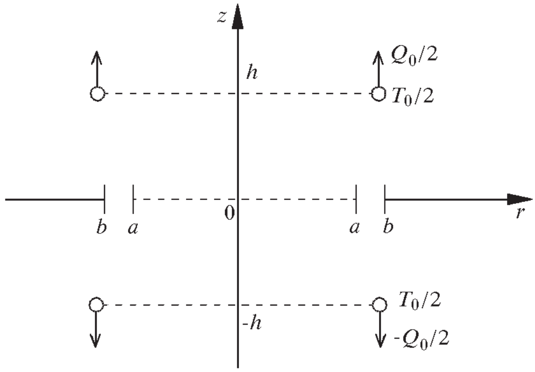


Fig. 4. Symmetric thermal loadings

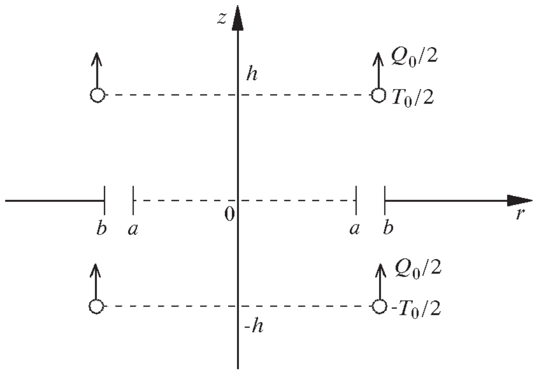


Fig. 5. Antisymmetric thermal loadings

3. Temperature field

For a uniform temperature and heat flux applied along the circumference $r = b$ on the plane $z = h$, the thermal loading conditions are

$$
\begin{aligned}
T(r, h+0) - T(r, h-0) &= \frac{T_0}{4\pi r}\delta(r-b) \\
T_{,z}(r, h+0) - T_{,z}(r, h-0) &= \frac{Q_0}{4\pi\lambda_z r}\delta(r-b)
\end{aligned}
\tag{3.1}
$$

where $\delta(r-b)$ is the Dirac delta function and T_0, Q_0 are the constant temperature and heat flux, respectively.

Applying the Hankel transforms to Eqs (2.4) and (3.1 a,b), we find the temperature as follows

$$
\begin{aligned}
T(r,z) &= \int\limits_0^\infty A_{ij}(\xi)\mathrm{e}^{-\xi s_0 z} J_0(\xi r)\, d\xi + \\
&+\frac{1}{2}\int\limits_0^\infty [\xi\nu_0 H_0(\xi s_0 z) - \nu_1 H_1(\xi s_0 z)] J_0(\xi b) J_0(\xi r)\, d\xi \qquad z \geqslant 0
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
H_0(\xi s_0 z) &= \operatorname{sgn}(z-h)\mathrm{e}^{-\xi s_0|z-h|} - (-1)^{i+j}\mathrm{e}^{-\xi s_0(z+h)} \\
H_1(\xi s_0 z) &= \mathrm{e}^{-\xi s_0|z-h|} + (-1)^{i+j}\mathrm{e}^{-\xi s_0(z+h)} \\
\operatorname{sgn}(z-h) &= \begin{cases} 1 & \text{for} \quad z > h \\ -1 & \text{for} \quad z < h \end{cases} \\
\nu_0 &= \frac{T_0}{4\pi} \qquad\qquad \nu_1 = \frac{Q_0}{4\pi\lambda_z s_0}
\end{aligned}
$$

for symmetric $(i = 1)$ and antisymmetric $(i = 2)$ thermal loading conditions, and where $A_{ij}(\xi)$ are unknown functions which may be determined by using the mixed thermal boundary conditions on the plane $z = 0$, where the penny-shaped crack $(j = 1)$ or the external crack $(j = 2)$ appear.

It is assumed that the crack faces remain insulated. The thermal conditions, therefore, are

$$
T_{,z} = 0 \qquad r \in A_c \qquad z = 0 \tag{3.3}
$$

and

$$
T_{,z} = 0 \qquad r \in \widetilde{A}_c \qquad z = 0 \tag{3.4}
$$

or

$$
T = 0 \qquad r \in \widetilde{A}_c \qquad z = 0 \tag{3.5}
$$

where A_c and $\widetilde{A}_c$ are the crack region and its complement, respectively.

Condition (3.4) corresponds to the symmetric problem, while condition (3.5) corresponds to the antisymmetric one.

Thermal conditions (3.3) and (3.4) or (3.5) yield:

(i) For the penny-shaped crack and symmetric temperature field

$$
A_{11}(\xi) = 0 \tag{3.6}
$$

(ii) For the penny-shaped crack and antisymmetric temperature field

$$
\begin{aligned}
&\int\limits_0^\infty \xi A_{21}(\xi) J_0(\xi r)\, d\xi = -\int\limits_0^\infty \xi(\nu_0\xi + \nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi && 0 \leqslant r < a \\
&\int\limits_0^\infty A_{21}(\xi) J_0(\xi r)\, d\xi = 0 && r > a
\end{aligned}
\tag{3.7}
$$

(iii) For the external crack and symmetric temperature field

$$
A_{12}(\xi) = -(\nu_0\xi + \nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b) \tag{3.8}
$$

(iv) For the external crack and antisymmetric temperature field

$$
\begin{aligned}
&\int\limits_0^\infty A_{22}(\xi) J_0(\xi r)\, d\xi = \int\limits_0^\infty (\nu_0\xi + \nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi && 0 \leqslant r < a \\
&\int\limits_0^\infty \xi A_{22}(\xi) J_0(\xi r)\, d\xi = 0 && r > a
\end{aligned}
\tag{3.9}
$$

Both solutions (3.6) and (3.8) give the temperature field

$$
\begin{aligned}
T(r,z) = \frac{1}{2}\int\limits_0^\infty & J_0(\xi b) J_0(\xi r)\,\cdot \\
&\cdot\Big[(\nu_0\xi\,\mathrm{sgn}\,(z-h) - \nu_1)\mathrm{e}^{-\xi s_0|z-h|} - (\nu_0\xi + \nu_1)\mathrm{e}^{-\xi s_0(z+h)}\Big]\, d\xi
\end{aligned}
\tag{3.10}
$$

related to the symmetric thermal loading conditions of the solid with the penny-shaped or an external crack.

Dual integral equations (3.7) are converted to the Abel integral equation by means of the following integral representation for $A_{21}(\xi)$ (Noble, 1963)

$$
A_{21} = \sqrt{\frac{2}{\pi}} \int\limits_0^a g_0(x)\sin(\xi x)\, dx \tag{3.11}
$$

on the assumption that $g_0(x) \to 0$ as $x \to 0^+$.

This representation of $A_{21}(\xi)$ identically satisfies Eq. $(3.7)_2$ (see Appendix, Eqs (A.1) and (A.9)).

Substituting $A_{21}(\xi)$ into Eq. $(3.7)_1$ leads to the following Abel integral equation in an auxiliary function $g_0(x)$

$$\sqrt{\frac{2}{\pi}}\int\limits_0^r \Big(\frac{dg_0(x)}{dx}\frac{1}{\sqrt{r^2-x^2}}\Big)\,dx = -\int\limits_0^\infty \xi(\nu_0\xi+\nu_1)J_0(\xi r)J_0(\xi b)\mathrm{e}^{-\xi s_0 h}\,d\xi \tag{3.12}$$

Applying the Abel solution method to invert the left hand side of Eq. (3.12) gives the formula for $g_0(x)$

$$g_0(x) = -\sqrt{\frac{2}{\pi}}\int\limits_0^\infty (\nu_0\xi+\nu_1)\sin(\xi x)J_0(\xi b)\mathrm{e}^{-\xi s_0 h}\,d\xi \tag{3.13}$$

The improper integrals appearing in Eq. (3.13) are calculated analytically (see Appendix, Eqs (A.1) and (A.2)). Consequently, the auxiliary function $g_0(x)$ is obtained in terms of the oblate spheroidal co-ordinates ζ_0 and η_0, defined in the Appendix, as

$$\begin{aligned} g_0(x) &= \sqrt{\frac{2}{\pi}}\Big[\nu_0\frac{d}{dx}\Big(\frac{\zeta_0}{D_0}\Big)-\nu_1\Big(\frac{\eta_0}{D_0}\Big)\Big] = \\ &= -\sqrt{\frac{2}{\pi}}\Big\{\nu_0\frac{\zeta_0}{D_0^2(\zeta_0^2+\eta_0^2)}[(1-\eta_0^2)(\eta_0^2-\zeta_0^2)+2\eta_0^2(1+\zeta_0^2)]+\nu_1\frac{\eta_0}{D_0}\Big\} \end{aligned} \tag{3.14}$$

where

$$D_0 = x(\zeta_0^2+\eta_0^2)$$

Finally, the temperature field is obtained as

$$\begin{aligned} T(r,z) &= \frac{2}{\pi}\int\limits_0^a \Big[\nu_0\frac{d}{dx}\Big(\frac{\zeta_0}{D_0}\Big)-\nu_1\Big(\frac{\eta_0}{D_0}\Big)\Big]\frac{\eta}{D}\,dx + \\ &+\frac{1}{2}\int\limits_0^\infty \Big\{[\nu_0\xi\,\mathrm{sgn}\,(z-h)-\nu_1]\mathrm{e}^{-\xi s_0|z-h|}+(\nu_0\xi+\nu_1)\mathrm{e}^{-\xi s_0(z+h)}\Big\}J_0(\xi b)J_0(\xi r)d\xi \end{aligned} \tag{3.15}$$

where

$$D = x(\zeta^2+\eta^2)$$

and where the oblate spheroidal co-ordinates ζ, η are associated with r, $s_0 z$ and x, while ζ_0, η_0 are associated with b, $s_0 h$ and x (see Appendix).

Dual integral equations (3.9) are converted to the Abel integral equations by means of the following integral representation of $A_{22}(\xi)$

$$A_{22}(\xi) = \sqrt{\frac{2}{\pi}} \int\limits_0^a f_0(x) \cos(\xi x)\, dx \tag{3.16}$$

In this representation the auxiliary function $f_0(x)$ is assumed to be continuous over the interval $[0, a]$. This representation of $A_{22}(\xi)$ identically satisfies Eq. $(3.9)_2$.

Substituting $A_{22}(\xi)$ into Eq. $(3.9)_1$ leads to the following Abel integral equation in an auxiliary function $f_0(x)$

$$\sqrt{\frac{2}{\pi}} \int\limits_0^r \frac{f_0(x)}{\sqrt{r^2 - x^2}}\, dx = \int\limits_0^\infty (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 h} J_0(\xi r) J_0(\xi b)\, d\xi \tag{3.17}$$

Applying the Abel solution method, give the formula for $f_0(x)$

$$f_0(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 h} \cos(\xi x) J_0(\xi b)\, d\xi \tag{3.18}$$

Substituting the integrals (A.1) and (A.2) (see Appendix), gives the final solution for $f_0(x)$

$$\begin{aligned} f_0(x) &= \sqrt{\frac{2}{\pi}} \Big[\nu_0 \frac{d}{dx} \Big(\frac{\eta_0}{D_0} \Big) + \nu_1 \frac{\zeta_0}{D_0} \Big] = \\ &= \sqrt{\frac{2}{\pi}} \Big\{ \nu_0 \frac{\eta_0}{D_0^2 (\zeta_0^2 + \eta_0^2)} [(1 + \zeta_0^2)(\zeta_0^2 - \eta_0^2) + 2\zeta_0^2 (1 - \eta_0^2)] + \nu_1 \frac{\zeta_0}{D_0} \Big\} \end{aligned} \tag{3.19}$$

For the external crack in the antisymmetric temperature field the temperature is obtained as

$$\begin{aligned} T(r,z) &= \frac{2}{\pi} \int\limits_0^a \Big[\nu_0 \frac{d}{dx} \Big(\frac{\eta_0}{D_0} \Big) + \nu_1 \frac{\zeta_0}{D_0} \Big] \frac{\zeta}{D}\, dx + \\ &+ \frac{1}{2} \int\limits_0^\infty \Big\{ [\nu_0 \xi \operatorname{sgn}(z - h) - \nu_1] \mathrm{e}^{-\xi s_0 |z-h|} - (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 (z+h)} \Big\} J_0(\xi r) J_0(\xi b) d\xi \end{aligned} \tag{3.20}$$

By using the superposition of two thermal fields (3.10) and (3.15) or (3.20), we obtain

$$
\begin{aligned}
T(r,z) &= \frac{2}{\pi}\int\limits_0^a \Big[\nu_0 \frac{d}{dx}\Big(\frac{\zeta_0}{D_0}\Big) - \nu_1 \frac{\eta_0}{D_0}\Big]\frac{\eta}{D}\,dx + \\
&+ \int\limits_0^\infty [\nu_0 \xi \,\mathrm{sgn}\,(z-h) - \nu_1]\mathrm{e}^{-\xi s_0|z-h|} J_0(\xi b) J_0(\xi r)\,d\xi \qquad z \geqslant 0
\end{aligned}
\tag{3.21}
$$

for the penny-shaped crack and

$$
\begin{aligned}
T(r,z) &= \frac{2}{\pi}\int\limits_0^a \Big[\nu_0 \frac{d}{dx}\Big(\frac{\eta_0}{D_0}\Big) + \nu_1 \frac{\zeta_0}{D_0}\Big]\frac{\zeta}{D}\,dx + \\
&+ \int\limits_0^\infty \Big\{[\nu_0 \xi \,\mathrm{sgn}\,(z-h) - \nu_1]\mathrm{e}^{-\xi s_0|z-h|} - (\nu_0\xi + \nu_1)\mathrm{e}^{-\xi s_0(z+h)}\Big\} J_0(\xi b) J_0(\xi r)\,d\xi
\end{aligned}
\tag{3.22}
$$

where $z \geqslant 0$, for the external crack.

4. Thermal stresses

Considering Eqs (2.8) and (3.2), we find the potential functions ($z \geqslant 0$)

$$
\begin{aligned}
\varphi_0(r,z) &= M\int\limits_0^\infty \xi^{-2}\Big\{A_{ij}(\xi)\mathrm{e}^{-\xi s_0 z} + \\
&\quad + \frac{1}{2}[\nu_0 \xi H_0(\xi s_0 z) - \nu_1 H_1(\xi s_0 z)] J_0(\xi b)\Big\} J_0(\xi r)\,d\xi \\
\varphi_1(r,z) &= \frac{s_2}{G_z(k+1)(s_1-s_2)}\int\limits_0^\infty \xi^{-1} B_{1j}(\xi)\mathrm{e}^{-\xi s_1 z} J_0(\xi r)\,d\xi \\
\varphi_2(r,z) &= -\frac{s_1}{G_z(k+1)(s_1-s_2)}\int\limits_0^\infty \xi^{-1} B_{2j}(\xi)\mathrm{e}^{-\xi s_2 z} J_0(\xi r)\,d\xi
\end{aligned}
\tag{4.1}
$$

Substituting Eqs (4.1) into Eqs (2.6) and (2.11), we obtain ($z \geqslant 0$)

$$
\begin{aligned}
u_r(r,z) = {} & -M\int\limits_0^\infty \xi^{-1}\Big\{A_{ij}(\xi)e^{-\xi s_0 z} + \\
& +\frac{1}{2}[\nu_0\xi H_0(\xi s_0 z) - \nu_1 H_1(\xi s_0 z)]J_0(\xi b)\Big\}J_1(\xi r)\,d\xi - \\
& -\frac{1}{G_z(k+1)(s_1-s_2)}\int\limits_0^\infty \Big[ks_2 B_{1j}(\xi)e^{-\xi s_1 z} - s_1 B_{2j}(\xi)e^{-\xi s_2 z}\Big]J_1(\xi r)\,d\xi
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
u_z(r,z) = {} & -Ms_0 l\int\limits_0^\infty \xi^{-1}\Big\{A_{ij}(\xi)e^{-\xi s_0 z} + \\
& +\frac{1}{2}[\nu_0\xi H_0'(\xi s_0 z) - \nu_1 H_1'(\xi s_0 z)]J_0(\xi b)\Big\}J_0(\xi r)\,d\xi - \\
& -\frac{s_1 s_2}{G_z(k+1)(s_1-s_2)}\int\limits_0^\infty \Big[B_{1j}(\xi)e^{-\xi s_1 z} - kB_{2j}(\xi)e^{-\xi s_2 z}\Big]J_0(\xi r)\,d\xi
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{zz}(r,z) = {} & G_z M(1+l)\int\limits_0^\infty \Big\{A_{ij}(\xi)e^{-\xi s_0 z} + \\
& +\frac{1}{2}[\nu_0\xi H_0(\xi s_0 z) - \nu_1 H_1(\xi s_0 z)]J_0(\xi b)\Big\}J_0(\xi r)\,d\xi + \\
& +\frac{1}{s_1-s_2}\int\limits_0^\infty \xi\Big[s_2 B_{1j}(\xi)e^{-\xi s_1 z} - s_1 B_{2j}(\xi)e^{-\xi s_2 z}\Big]J_0(\xi r)\,d\xi
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\sigma_{rz}(r,z) = {} & G_z M(1+l)s_0\int\limits_0^\infty \Big\{A_{ij}(\xi)e^{-\xi s_0 z} + \\
& +\frac{1}{2}[\nu_0\xi H_0'(\xi s_0 z) - \nu_1 H_0'(\xi s_0 z)]J_0(\xi b)\Big\}J_1(\xi r)\,d\xi + \\
& +\frac{s_1 s_2}{s_1-s_2}\int\limits_0^\infty \xi\Big[B_{1j}(\xi)e^{-\xi s_1 z} - B_{2j}(\xi)e^{-\xi s_2 z}\Big]J_1(\xi r)\,d\xi
\end{aligned}
$$

where

$$
\begin{aligned}
H_0'(\xi s_0 z) &= e^{-\xi s_0|z-h|} - (-1)^{i+j}e^{-\xi s_0(z+h)} \\
H_1'(\xi s_0 z) &= \operatorname{sgn}(z-h)e^{-\xi s_0|z-h|} + (-1)^{i+j}e^{-\xi s_0(z+h)}
\end{aligned}
\tag{4.4}
$$

The crack problem must be solved under the following conditions

$$\begin{aligned} &\sigma_{zr}(r,0) = 0 && r \geqslant 0 \\ &\sigma_{zz}(r,0) = 0 && r \in A_c \\ &u_z(r,0) = 0 && r \in \widetilde{A}_c \end{aligned} \tag{4.5}$$

for the symmetric thermal condition and

$$\begin{aligned} &\sigma_{zz}(r,0) = 0 && r \geqslant 0 \\ &\sigma_{zr}(r,0) = 0 && r \in A_c \\ &u_r(r,0) = 0 && r \in \widetilde{A}_c \end{aligned} \tag{4.6}$$

for the antisymmetric thermal condition.

Conditions $(4.5)_1$ and $(4.6)_1$ yield, respectively

$$\begin{aligned} B_{2j}(\xi) = B_{1j}(\xi) + G_zM(1+l)\Big(\frac{s_0}{s_2} - \frac{s_0}{s_1}\Big)\xi^{-1} \cdot \\ \cdot\Big\{A_{1j}(\xi) + \frac{1}{2}(\nu_0\xi + \nu_1)[1+(-1)^j]J_0(\xi b)\mathrm{e}^{-\xi s_0 h}\Big\} \end{aligned} \tag{4.7}$$

or

$$\begin{aligned} B_{2j}(\xi) = \frac{s_2}{s_1}B_{1j}(\xi) + G_zM(1+l)\Big(1 - \frac{s_2}{s_1}\Big)\xi^{-1} \cdot \\ \cdot\Big\{A_{2j}(\xi) - \frac{1}{2}(\nu_0\xi + \nu_1)[1+(-1)^j]J_0(\xi b)\mathrm{e}^{-\xi s_0 h}\Big\} \end{aligned} \tag{4.8}$$

The displacements and stresses meeting mixed boundary conditions $(4.5)_{2,3}$ and $(4.6)_{2,3}$ on the plane where the crack appears are

$$\begin{aligned} u_z(r,0) &= \frac{1}{G_zC}\int\limits_0^\infty B_{1j}(\xi)J_0(\xi r)\,d\xi + \frac{Ms_0(k-l)}{k+1} \cdot \\ &\cdot \int\limits_0^\infty \xi^{-1}\Big\{A_{1j}(\xi) + \frac{1}{2}(\nu_0\xi+\nu_1)[1+(-1)^j]\mathrm{e}^{-\xi s_0 h}J_0(\xi b)\Big\}J_0(\xi r)\,d\xi \\ \sigma_{zz}(r,0) &= -\int\limits_0^\infty \xi B_{1j}(\xi)J_0(\xi r)\,d\xi + G_zM(1+l)\int\limits_0^\infty\Big\{\Big(1-\frac{s_0}{s_2}\Big)A_{1j}(\xi) - \\ &-\frac{1}{2}(\nu_0\xi+\nu_1)\Big[1+\frac{s_0}{s_2} - \Big(1-\frac{s_0}{s_2}\Big)(-1)^j\Big]\mathrm{e}^{-\xi s_0 h}J_0(\xi b)\Big\}J_0(\xi r)\,d\xi \end{aligned} \tag{4.9}$$

$$u_r(r,z) = -\frac{1}{G_z C s_1}\int\limits_0^\infty B_{1j}(\xi)J_1(\xi r)\,d\xi -$$

$$-\frac{M(k-l)}{k+1}\int\limits_0^\infty \xi^{-1}\Big\{A_{2j}(\xi) - \frac{1}{2}(\nu_0\xi+\nu_1)[1+(-1)^j]\mathrm{e}^{-\xi s_0 h}J_0(\xi b)\Big\}J_1(\xi r)\,d\xi$$

$$\sigma_{rz}(r,z) = s_2\int\limits_0^\infty \xi B_{1j}(\xi)J_1(\xi r)\,d\xi + G_z M(1+l)\int\limits_0^\infty \Big\{(s_0-s_2)A_{2j} +$$

$$+\frac{1}{2}(\nu_0\xi+\nu_1)[s_0+s_2-(s_0-s_2)(-1)^j]\mathrm{e}^{-\xi s_0 h}J_0(\xi b)\Big\}J_1(\xi r)\,d\xi$$

where

$$C = \frac{(k+1)(s_1-s_2)}{(k-1)s_1 s_2} \tag{4.10}$$

5. Mode I loading

The Mode I crack problem corresponds to the symmetric thermal loading. The penny-shaped crack problem is obtained for $j=1$ and the external crack problem is obtained for $j=2$.

5.1. The penny-shaped crack problem

Substituting Eqs $(4.9)_{1,2}$ into boundary conditions $(4.5)_{2,3}$ and using that $A_{11}(\xi)=0$, the following dual integral equations are obtained

$$\begin{aligned}\int\limits_0^\infty \xi B_{11}(\xi)J_0(\xi r)\,d\xi &= \\ &= -G_z M(1+l)\int\limits_0^\infty (\nu_0\xi+\nu_1)\mathrm{e}^{-\xi s_0 h}J_0(\xi b)J_0(\xi r)\,d\xi\end{aligned} \qquad 0 \leqslant r < a \tag{5.1}$$

$$\int\limits_0^\infty B_{11}(\xi)J_0(\xi r)\,d\xi = 0 \qquad r > a \tag{5.2}$$

Dual integral equations (5.1), (5.2) are converted to the Abel integral equation by means of the following integral representation of $B_{11}(\xi)$

$$B_{11}(\xi) = \sqrt{\frac{2}{\pi}} \int\limits_0^a g(x) \sin(\xi x)\, dx \tag{5.3}$$

on the assumption that $g(x) \to 0$ as $x \to 0^+$.

This representation of $B_{11}(\xi)$ identically satisfies Eq. (5.2). Substituting $B_{11}(\xi)$ into Eq. (5.1) leads to the following Abel integral equation in an auxiliary function $g(x)$

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} \int\limits_0^r \Big(\frac{dg(x)}{dx} \frac{1}{\sqrt{r^2 - x^2}}\Big)\, dx = \\ &= -G_z M(1+l) \int\limits_0^\infty (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi \end{aligned} \tag{5.4}$$

Applying the Abel solution method to invert the left hand side of Eq. (5.4) gives the formula for $g(x)$

$$g(x) = -\sqrt{\frac{2}{\pi}} G_z M(1+l) \int\limits_0^\infty (\nu_0 + \nu_1 \xi^{-1}) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) \sin(\xi x)\, d\xi \tag{5.5}$$

The improper integrals appearing in Eq. (5.5) are calculated analytically (see Appendix, Eqs (A.1) and (A.3)). Consequently, the auxiliary function $g(x)$ is obtained explicitly in terms of the oblate spheroidal co-ordinates ζ_0 and η_0 (see Appendix) as

$$g(x) = -\sqrt{\frac{2}{\pi}} G_z M(1+l) \Big[\nu_0 \frac{\eta_0}{x(\zeta_0^2 + \eta_0^2)} + \nu_1 \Big(\frac{\pi}{2} - \tan^{-1} \zeta_0\Big)\Big] \tag{5.6}$$

The singular part of the axial stress is given by the formula

$$\sigma_{zz}(r,0) = \sqrt{\frac{2}{\pi}} \frac{g(a)}{\sqrt{r^2 - a^2}} \qquad \text{as} \qquad r \to a^+ \tag{5.7}$$

Defining the stress intensity factor of Mode I as

$$K_I = \lim_{r \to a^+} \sqrt{2(r-a)}\, \sigma_{zz}(r,0) \tag{5.8}$$

one obtains

$$K_I = -\frac{2}{\pi \sqrt{a}} G_z M(1+l) \Big[\nu_0 \frac{\overline{\eta}_0}{a(\overline{\zeta}_0^2 + \overline{\eta}_0^2)} + \nu_1 \Big(\frac{\pi}{2} - \tan^{-1} \zeta_0\Big)\Big] \tag{5.9}$$

where $\overline{\zeta}_0$, $\overline{\eta}_0$ are obtained from ζ_0, η_0 for $x = a$ (see Appendix).

Solution (5.9) contains three other problems as special cases, namely: (i) $h = 0$ and $b < a$, (ii) $h = 0$ and $b > a$, (iii) $b = 0$. We can deduce the results for these three cases from equations (5.9), (A.8) and (A.9) for $x = a$. The results are given in Table 1.

5.2. The external crack

The dual integral equations of the external crack problem are

$$\int\limits_0^\infty B_{12}(\xi) J_0(\xi r)\, d\xi = 0 \qquad 0 \leqslant r < a \tag{5.10}$$

$$\begin{aligned} &\int\limits_0^\infty \xi B_{12}(\xi) J_0(\xi r)\, d\xi = \\ &= -G_z M(1+l) \int\limits_0^\infty (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi \end{aligned} \qquad r > a \tag{5.11}$$

For the temperature loading we use the integral representation of $B_{12}(\xi)$

$$B_{12}(\xi) = \sqrt{\frac{2}{\pi}} \int\limits_0^a f(x) \cos(\xi x)\, dx - G_z M(1+l)\nu_0 \mathrm{e}^{-\xi s_0 h} J_0(\xi b) \tag{5.12}$$

and find the Abel integral equation in an auxiliary function $f(x)$

$$\sqrt{\frac{2}{\pi}} \int\limits_0^r \Big(\frac{f(x)}{\sqrt{r^2 - x^2}} \Big)\, dx = G_z M(1+l)\nu_0 \int\limits_0^\infty \mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi \tag{5.13}$$

The solution for this equation is

$$f(x) = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_0 \int\limits_0^\infty \mathrm{e}^{-\xi s_0 h} J_0(\xi b) \cos(\xi x)\, d\xi \tag{5.14}$$

Substituting the analytical expression for the improper integral (Eq. (A.2) in the Appendix), we get

$$f(x) = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_0 \frac{\zeta_0}{x(\zeta_0^2 + \eta_0^2)} \tag{5.15}$$

The stress transmitted through the neck is found to be

$$\sigma_{zz}(r,0) = -\sqrt{\frac{2}{\pi}}\frac{f(a)}{\sqrt{a^2-r^2}} + \int\limits_r^a \frac{df(x)}{dx}\frac{dx}{\sqrt{x^2-r^2}} \tag{5.16}$$

Defining the stress intensity factor of Mode I as

$$K_I = \lim_{r\to a^-} \sqrt{2(a-r)}\sigma_{zz}(r,0) \tag{5.17}$$

one obtains

$$K_I = -\frac{2}{\pi\sqrt{a}}G_z M(1+l)\nu_0 \frac{\overline{\zeta}_0}{a(\overline{\zeta}_0^2+\overline{\eta}_0^2)} \tag{5.18}$$

where $\overline{\zeta}_0$, $\overline{\eta}_0$ are the values of ζ_0, η_0 for $x = a$ (see Appendix).

For the heat flux problem we use the integral representation of $B_{12}(\xi)$

$$B_{12}(\xi) = \sqrt{\frac{2}{\pi}}\int\limits_0^a f_1(x)\Big[\frac{\sin(\xi x)}{\xi x} - \cos(\xi x)\Big]\, dx - G_z M(1+l)\nu_1 \xi^{-1}\mathrm{e}^{-\xi s_0 h} J_0(\xi b) \tag{5.19}$$

This representation identically satisfies Eq. (5.11) associated with the heat flux and converts Eq. (5.10) to the Abel integral equation

$$\begin{aligned} -\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{f_1(x)}{\sqrt{r^2-x^2}}\, dx + \sqrt{\frac{2}{\pi}}\int\limits_0^a \frac{f_1(u)}{u}\Big[\int\limits_0^\infty \frac{\sin(\xi u)}{\xi} J_0(\xi r)\, d\xi\Big]\, du = \\ = G_z M(1+l)\nu_1 \int\limits_0^\infty \xi^{-1}\mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_0(\xi r)\, d\xi \end{aligned} \tag{5.20}$$

Applying the Abel solution method we obtain

$$\begin{aligned} -f_1(x) + \frac{2}{\pi}\int\limits_0^a \frac{f_1(u)}{u}\Big[\int\limits_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi}\, d\xi\Big]\, du = \\ = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_1 \int\limits_0^\infty \xi^{-1}\mathrm{e}^{-\xi s_0 h} J_0(\xi b)\cos(\xi x)\, d\xi \end{aligned} \tag{5.21}$$

We use the integral

$$\int\limits_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi}\, d\xi = \frac{\pi}{2}H(u-x) \tag{5.22}$$

where $H(\cdot)$ is the Heaviside unit function.

Then we have

$$-f_1(x)+\int\limits_x^a \frac{f_1(u)}{u}\, du = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_1 \int\limits_0^\infty \xi^{-1} \mathrm{e}^{-\xi s_0 h} J_0(\xi b)\cos(\xi x)\, d\xi \quad (5.23)$$

It is seen that the integrand in the improper integral is unbounded as $\xi \to 0$. This improper behaviour at $\xi \to 0$ can be removed by adding to both sides of Eq. (5.23) the value of $f_1(0)$, obtained formally from this equation.

After adjusting the improper behaviour at $\xi \to 0$, Eq (5.23) becomes

$$\int\limits_0^x \frac{1}{x}\frac{d}{dx}[xf_1(x)]\, dx = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_1 \int\limits_0^\infty \frac{1-\cos(\xi x)}{\xi} \mathrm{e}^{-\xi s_0 h} J_0(\xi b)\, d\xi \quad (5.24)$$

The improper integral in Eq. (5.24) has an analytic expression given by Eqs (A.5) and (A.6) in the Appendix.

We use the following relationships

$$\frac{1-\cos(\xi x)}{\xi} = \int\limits_0^x \sin(\xi x)\, dx = \int\limits_0^x \frac{1}{x}\frac{d}{dx}\Big[\frac{x}{\xi}\Big(\frac{\sin(\xi x)}{\xi x} - \cos(\xi x)\Big)\Big]\, dx \quad (5.25)$$

and integral (A.4) from the Appendix.

Then, the solution to Eq (5.24) is obtained in the form

$$f_1(x) = \sqrt{\frac{2}{\pi}} G_z M(1+l)\nu_1 \eta_0 \Big[1 - \zeta_0\Big(\frac{\pi}{2} - \tan^{-1}\zeta_0\Big)\Big] \quad (5.26)$$

It is noted that $f_1(x)$ tends to zero as $x \to 0^+$.

The stress transmitted through the neck is found to be

$$\sigma_{zz}(r,0) = \sqrt{\frac{2}{\pi}}\Big[\frac{r^2 f_1(a)}{a^2\sqrt{a^2-r^2}} - r^2\int\limits_r^a \frac{d}{dx}\Big(\frac{f_1(x)}{x^2}\Big)\frac{dx}{\sqrt{x^2-r^2}} - 2\int\limits_r^a \frac{f_1(x)}{x}\frac{dx}{\sqrt{x^2-r^2}}\Big] \quad (5.27)$$

The stress intensity factor of Mode I is given by

$$K_I = \frac{2}{\pi\sqrt{a}} G_z M(1+l)\nu_1 \overline{\eta}_0 \Big[1 - \overline{\zeta}_0\Big(\frac{\pi}{2} - \tan^{-1}\overline{\zeta}_0\Big)\Big] \quad (5.28)$$

where $\overline{\zeta}_0$, $\overline{\eta}_0$ are the values of ζ_0, η_0 for $x = a$ (see Appendix).

In special cases, K_I takes the values which are shown in Table 1.

6. Mode II loading

6.1. The penny-shaped crack

The dual integral equations are:

— for $0 \leqslant r < a$

$$\int_0^\infty \xi B_{11}(\xi) J_1(\xi r)\, d\xi = \tag{6.1}$$
$$= -G_z M(1+l) \int_0^\infty \Big[\Big(\frac{s_0}{s_2} - 1\Big) A_{21}(\xi) + (\nu_0 \xi + \nu_1) \frac{s_0}{s_2} \mathrm{e}^{-\xi s_0 h} J_0(\xi b)\Big] J_1(\xi r)\, d\xi$$

— for $r > a$

$$\int_0^\infty B_{11}(\xi) J_1(\xi r)\, d\xi = -\frac{G_z M(k-l)}{k-1} \Big(\frac{s_1}{s_2} - 1\Big) \int_0^\infty \xi^{-1} A_{21}(\xi) J_1(\xi r)\, d\xi \tag{6.2}$$

The integral representation of $B_{11}(\xi)$

$$B_{11}(\xi) = \sqrt{\xi} \int_0^a \sqrt{x} h(x) J_{3/2}(\xi x)\, dx - \frac{G_z M(k-l)}{k-1} \Big(\frac{s_1}{s_2} - 1\Big) \xi^{-1} A_{21}(\xi) \tag{6.3}$$

on the assumption that $\sqrt{x} h(x) \to 0$ as $x \to 0^+$, satisfies identically Eq. (6.2), while Eq. (6.1) is converted to the Abel integral equation

$$\sqrt{\frac{2}{\pi}} \int_0^r \Big(\frac{d[xh(x)]}{dx} \frac{1}{\sqrt{r^2 - x^2}}\Big)\, dx =$$
$$= G_z M r \Big\{\Big[\frac{k-l}{k-1}\Big(\frac{s_1}{s_2} - 1\Big) - (1+l)\Big(\frac{s_0}{s_2} - 1\Big)\Big] \int_0^\infty A_{21}(\xi) J_1(\xi r)\, d\xi - \tag{6.4}$$
$$-(1+l)\frac{s_0}{s_2} \int_0^\infty (\nu_0 \xi + \nu_1) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_1(\xi r)\, d\xi \Big\}$$

The solution to this equation is

$$\begin{aligned} h(x) = \sqrt{\frac{2}{\pi}} G_z M \Big\{ \Big[\frac{k-l}{k-1} \Big(\frac{s_1}{s_2} - 1 \Big) - (1+l) \Big(\frac{s_0}{s_2} - 1 \Big) \Big] \cdot \\ \cdot \int\limits_0^\infty A_{21}(\xi) \frac{d}{d\xi} \Big(\frac{-\sin(\xi x)}{\xi x} \Big) \; d\xi - \\ -(1+l) \frac{s_0}{s_2} \int\limits_0^\infty (\nu_0 + \nu_1 \xi^{-1}) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) \Big[\frac{\sin(\xi x)}{\xi x} - \cos(\xi x) \Big] \; d\xi \Big\} \end{aligned} \tag{6.5}$$

Integrating the first integral in Eq. (6.5) by parts, substituting $A_{21}(\xi)$ and $g_0(x)$ from Eqs (3.11) and (3.13) and substituting for the second integral the analytical formula (see Appendix, Eqs (A.2), (A.3) and (A.4)), lead to the following exact formula for $h(x)$

$$\begin{aligned} h(x) = \sqrt{\frac{2}{\pi}} G_z M (1+l) \frac{\kappa}{s_2} \cdot \\ \cdot \Big\{ \frac{\nu_0}{x} \Big(\frac{\pi}{2} - \tan^{-1} \zeta_0 - \frac{\zeta_0}{\zeta_0^2 + \eta_0^2} \Big) + \nu_1 \eta_0 \Big[\zeta_0 \Big(\frac{\pi}{2} - \tan^{-1} \zeta_0 \Big) - 1 \Big] \Big\} \end{aligned} \tag{6.6}$$

where

$$\kappa = s_2 + \frac{k-l}{k-1} \frac{s_1 - s_2}{1+l} \tag{6.7}$$

The singular part of the shear stress is given by

$$\sigma_{rz}(r,0) = -\sqrt{\frac{2}{\pi}} \frac{s_2 a h(a)}{r\sqrt{r^2 - a^2}} \qquad \text{as} \qquad r \to a^+ \tag{6.8}$$

The stress intensity factor of Mode II is obtained as follows

$$\begin{aligned} K_{II} = -\frac{2}{\pi\sqrt{a}} G_z M (1+l) \kappa \cdot \\ \cdot \Big\{ \frac{\nu_0}{a} \Big(\frac{\pi}{2} - \tan^{-1} \overline{\zeta}_0 - \frac{\overline{\zeta}_0}{\overline{\zeta}_0^2 + \overline{\eta}_0^2} \Big) + \nu_1 \overline{\eta}_0 \Big[\overline{\zeta}_0 \Big(\frac{\pi}{2} - \tan^{-1} \overline{\zeta}_0 \Big) - 1 \Big] \Big\} \end{aligned} \tag{6.9}$$

In special cases K_{II}, takes the values which are shown in Table 1.

6.2. The external crack

The dual integral equations are: — for $0 \leqslant r < a$

$$\int\limits_0^\infty B_{12}(\xi) J_1(\xi r)\, d\xi = \tag{6.10}$$
$$= -\frac{G_z M(k-l)}{k-1}\Big(\frac{s_1}{s_2}-1\Big)\int\limits_0^\infty \xi^{-1}\Big[A_{22}(\xi) - (\nu_0\xi+\nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b)\Big] J_1(\xi r)\, d\xi$$

— for $r > a$

$$\int\limits_0^\infty \xi B_{12}(\xi) J_1(\xi r)\, d\xi = \tag{6.11}$$
$$= -G_z M(1+l)\int\limits_0^\infty \Big[\Big(\frac{s_0}{s_2}-1\Big)A_{22}(\xi) + (\nu_0\xi+\nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b)\Big] J_1(\xi r)\, d\xi$$

The integral representation of $B_{12}(\xi)$

$$B_{12}(\xi) = \sqrt{\frac{2}{\pi}}\int\limits_0^a t(x)\sin(\xi x)\, dx - \tag{6.12}$$
$$-G_z M(1+l)\xi^{-1}\Big[\Big(\frac{s_0}{s_2}-1\Big)A_{22}(\xi) + (\nu_0\xi+\nu_1)\mathrm{e}^{-\xi s_0 h} J_0(\xi b)\Big]$$

gives the Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{x t(x)}{\sqrt{r^2-x^2}}\, dx = -G_z M r \cdot$$
$$\cdot\Big\{\Big[\frac{k-l}{k-1}\Big(\frac{s_1}{s_2}-1\Big) - (1+l)\Big(\frac{s_0}{s_2}-1\Big)\Big]\int\limits_0^\infty \xi^{-1}A_{22}(\xi) J_1(\xi r)\, d\xi - \tag{6.13}$$
$$-\Big[\frac{k-l}{k-1}\Big(\frac{s_1}{s_2}-1\Big) + 1 + l\Big]\int\limits_0^\infty (\nu_0+\nu_1\xi^{-1})\mathrm{e}^{-\xi s_0 h} J_0(\xi b) J_1(\xi r)\, d\xi\Big\}$$

The solution to this equation is

$$
\begin{aligned}
t(x) = & -\sqrt{\frac{2}{\pi}} G_z M \cdot \\
& \cdot \Big\{ \Big[\frac{k-l}{k-1} \Big(\frac{s_1}{s_2} - 1 \Big) - (1+l) \Big(\frac{s_0}{s_2} - 1 \Big) \Big] \int\limits_0^\infty \xi^{-1} A_{22}(\xi) \sin(\xi x) \, d\xi - \qquad (6.14) \\
& - \Big[\frac{k-l}{k-1} \Big(\frac{s_1}{s_2} - 1 \Big) + 1 + l \Big] \int\limits_0^\infty (\nu_0 + \nu_1 \xi^{-1}) \mathrm{e}^{-\xi s_0 h} J_0(\xi b) \sin(\xi x) \, d\xi \Big\}
\end{aligned}
$$

Substituting $A_{22}(\xi)$ from Eq. (3.16) and $f_0(x)$ from Eq. (3.19), integrating and using Eqs (A.1) and (A.3) from the Appendix, we obtain

$$
t(x) = \sqrt{\frac{2}{\pi}} G_z M (1+l) \frac{s_0}{s_2} \Big[\nu_0 \frac{\eta_0}{x(\zeta_0^2 + \eta_0^2)} + \nu_1 \Big(\frac{\pi}{2} - \tan^{-1} \zeta_0 \Big) \Big] \qquad (6.15)
$$

The singular part of the shear stress is

$$
\sigma_{zr}(r,0) = \sqrt{\frac{2}{\pi}} \frac{r t(a) s_2}{a\sqrt{a^2 - r^2}} \qquad \text{as} \qquad r \to a^- \qquad (6.16)
$$

The stress intensity factor of Mode II is obtained in the form

$$
K_{II} = \frac{2}{\pi\sqrt{a}} G_z M (1+l) s_0 \Big[\nu_0 \frac{\overline{\eta}_0}{a(\overline{\zeta}_0^2 + \overline{\eta}_0^2)} + \nu_1 \Big(\frac{\pi}{2} - \tan^{-1} \overline{\zeta}_0 \Big) \Big] \qquad (6.17)
$$

where the oblate spheroidal co-ordinates $\overline{\zeta}_0$, $\overline{\eta}_0$ are calculated for $x = a$.

In special cases K_{II}, takes the values presented in Table 1.

7. Applications

The exact solutions have been presented for the stress intensity factors of Mode I and II at the tips of the penny-shaped crack and external crack under thermal loadings. These solutions are obtained explicitly in terms of elementary functions. For any axisymmetrical distribution of thermal loadings of the medium with internal or external cracks the integration and/or simple superposition of the obtained results can yield the stress intensity factors.

When the cracked solid is subjected to temperature $T(r,z) = T_0 t(r,z)$ and/or heat flux $Q(r,z) = Q_0 q(r,z)$, then the components K_i $(i = I, II)$ of the stress intensity factor may be calculated as follows

$$K_i = \int\limits_V [t(r,z)K_{i0}(r,z) + q(r,z)K_{i1}(r,z)]\, dV \tag{7.1}$$

where V denotes the domain volume of the thermally loaded region and $K_{i0}(r,z)$, $K_{i1}(r,z)$ denote the stress intensity factors when the temperature and heat flux ring loading (index 0 or 1, respectively) act along a circle (r,z) of the radius r on the plane z (the co-ordinates b, h should be replaced by r, z in the obtained results).

We now proceed to consider some specific cases of thermal loadings, when the temperature $T_0/2$ and the heat flux $Q_0/2$ are applied on the planes $z = \pm h$ in an annular region $b \leqslant r \leqslant c$ symmetrically or asymmetrically with respect to $z = 0$ plane.

Then, equation (7.1) yields

$$K_i = 2\pi \int\limits_b^c [K_{i0}(r,h) + K_{i1}(r,h)]r\, dr \tag{7.2}$$

where

$$r\, dr = a^2(\zeta^2 + \eta^2)\frac{d\zeta}{\zeta} \qquad\qquad \frac{d\zeta}{\zeta} = -\frac{d\eta}{\eta}$$

in the oblate spheroidal co-ordinates $r^2 = a^2(1+\zeta^2)(1-\eta^2)$, $s_0 h = a\zeta\eta$ and $K_{i0}(r,h)$, $K_{i1}(r,h)$ are presented in those co-ordinates.

Example 1: Consider the case of the temperature loading $T_0/2$ on the planes $z = \pm h$ in the annular region $b \leqslant r \leqslant c$.

From equation (5.8) and (5.18) we obtain:

— for the penny-shaped crack $(0 \leqslant r \leqslant a)$

$$\begin{aligned} K_I &= -\frac{T_0\sqrt{a}}{\pi} G_z M(1+l) \int\limits_{\eta(b)}^{\eta(c)} \frac{\eta}{\zeta^2+\eta^2}\Big(-\frac{\zeta^2+\eta^2}{\eta}\Big)\, d\eta = \\ &= -\frac{T_0\sqrt{a}}{\pi} G_z M(1+l)[\eta(b) - \eta(c)] \end{aligned} \tag{7.3}$$

where

$$\eta(r) = \frac{1}{a\sqrt{2}}\sqrt{\sqrt{(r^2 + s_0^2h^2 - a^2)^2 + 4a^2s_0^2h^2} - (r^2 + s_0^2h^2 - a^2)} \tag{7.4}$$

— for the external crack $(r \geqslant a)$

$$\begin{aligned} K_I &= -\frac{T_0\sqrt{a}}{\pi}G_zM(1+l)\int\limits_{\xi(b)}^{\xi(c)} \frac{\zeta}{\zeta^2+\eta^2}\frac{\zeta^2+\eta^2}{\zeta}\,d\zeta = \\ &= -\frac{T_0\sqrt{a}}{\pi}G_zM(1+l)[\zeta(c) - \zeta(b)] \end{aligned} \tag{7.5}$$

where

$$\zeta(r) = \frac{1}{a\sqrt{2}}\sqrt{\sqrt{(r^2 + s_0^2h^2 - a^2)^2 + 4a^2s_0^2h^2} + r^2 + s_0^2h^2 - a^2} \tag{7.6}$$

Since for real materials $G_zM(1+l) < 0$, the cracks open if $T_0 > 0$.

In special cases, K_I assumes the values:

— for the penny-shaped crack

$$K_I = \mathcal{A}\begin{cases} 1 & \text{for} \quad b=0 \quad c \to \infty \\ 1 - \eta(c) & \text{for} \quad b=0 \quad c \text{ finite} \\ 1 & \text{for} \quad h=0 \quad b=0 \qquad c=a \\ \dfrac{\sqrt{a^2-b^2} - \sqrt{a^2-c^2}}{a} & \text{for} \quad h=0 \quad b<c\leqslant a \\ 0 & \text{for} \quad h=0 \quad b \geqslant a \qquad c>a \end{cases} \tag{7.7}$$

— for the external crack

$$K_I = \mathcal{A}\begin{cases} \zeta(c) - \dfrac{s_0h}{a} & \text{for} \quad b=0 \quad c \text{ finite} \\ \dfrac{\sqrt{c^2-a^2} - \sqrt{b^2-a^2}}{a} & \text{for} \quad h=0 \quad a\leqslant b<c \\ 0 & \text{for} \quad h=0 \quad b<a \qquad c\leqslant a \end{cases} \tag{7.8}$$

where

$$\mathcal{A} = -\frac{T_0\sqrt{a}}{\pi}G_zM(1+l)$$

When the temperature change takes place in the plane of the crack but outside of the crack surface, then K_I are zero. For the penny-shaped crack and temperature change over the crack surface $0 \leqslant r \leqslant a$, $h = 0$ or on the plane $z = h$, $r \geqslant 0$, the stress intensity factors are equal. Note that, if the temperature is applied in an infinite region $r \geqslant 0$ on the planes $z = \pm h$, the K_I is independent on h for the penny-shaped crack problem.

Example 2: Consider the case where the heat flux of the intensity $Q_0/2$ is applied on the plane $z = h$ in the annular region $b \leqslant r \leqslant c$, and the opposite heat flux $(-Q_0/2)$ acts on the plane $z = -h$.

From equations (5.8) or (5.18) we obtain:

— for the penny-shaped crack $(0 \leqslant r \leqslant a)$

$$\begin{aligned} K_I &= -\frac{Q_0}{\pi\sqrt{a}} \frac{G_z M(1+l)}{\lambda_z s_0} \int\limits_b^c \Big(\frac{\pi}{2} - \tan^{-1}\zeta\Big) r\,dr = \\ &= -\frac{Q_0 a\sqrt{a}}{\pi} \frac{G_z M(1+l)}{\lambda_z s_0} [f(c) - f(b)] \end{aligned} \tag{7.9}$$

where

$$f(r) = \frac{r^2}{2a^2}\Big(\frac{\pi}{2} - \tan^{-1}\zeta + \frac{\zeta}{1+\zeta^2}\frac{1-\eta}{1+\eta}\Big) \tag{7.10}$$

and ζ, η are defined by equations (7.4) and (7.6), respectively

— for the external crack $(r \geqslant a)$

$$\begin{aligned} K_I &= -\frac{Q_0}{\pi\sqrt{a}} \frac{G_z M(1+l)}{\lambda_z s_0} \int\limits_b^c \Big[\frac{s_0 h}{a}\Big(\frac{\pi}{2} - \tan^{-1}\zeta\Big) - \eta\Big] r\,dr = \\ &= -\frac{Q_0 a\sqrt{a}}{\pi} \frac{G_z M(1+l)}{\lambda_z s_0} \Big\{\frac{s_0 h}{a}[f(c) - f(b) + \zeta(c) - \zeta(b)] + \frac{1}{3}[\eta^3(c) - \eta^3(b)]\Big\} \end{aligned} \tag{7.11}$$

where $f(r)$, $\eta(r)$, $\zeta(r)$ are defined by equations (7.10), (7.4) and (7.6), respectively.

Example 3: Consider the case of the temperature loading $T_0/2$ on the plane $z = h$ and $(-T_0/2)$ on the plane $z = -h$ applied in the annular region $b \leqslant r \leqslant c$.

From equations (6.9) or (6.17) we obtain:

— for the penny-shaped crack $(0 \leqslant r \leqslant a)$

$$K_{II} = -\frac{T_0\sqrt{a}}{\pi} G_z M(1+l)\kappa[f(c) - f(b) - \zeta(c) + \zeta(b)] \tag{7.12}$$

where $f(r)$ is defined by equation (7.10) and $\zeta(r)$ by eqation (7.6)

— for the external crack $(r \geqslant a)$

$$K_{II} = \frac{T_0\sqrt{a}}{\pi} G_z M(1+l)s_0[\eta(b) - \eta(c)] \tag{7.13}$$

where $\eta(r)$ is defined by equation (7.4).

Example 4: Consider the case where the heat flux of the intensity $Q_0/2$ is applied on the planes $z = \pm h$ in the z-direction over the annular region $b \leqslant r \leqslant c$.

From equation (6.9) and (6.17) we obtain:

— for the penny-shaped crack $(0 \leqslant r \leqslant a)$

$$K_{II} = -\frac{Q_0 a\sqrt{a}}{\pi} G_z M(1+l)\kappa\Big\{\frac{s_0 h}{a}[f(c) - f(b) + \zeta(c) - \zeta(b)] + \frac{1}{3}[\eta^3(c) - \eta^3(b)]\Big\} \tag{7.14}$$

— for the external crack $(r \geqslant a)$

$$K_{II} = \frac{Q_0 a\sqrt{a}}{\pi} G_z M(1+l)s_0[f(c) - f(b)] \tag{7.15}$$

where $f(r)$, $\eta(r)$, $\zeta(r)$ are defined by equations (7.10), (7.4) and (7.6), respectively.

In above examples, the loading was either symmetric or asymmetric with respect to the crack plane.

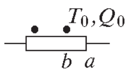
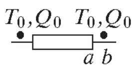
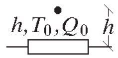
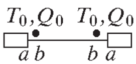
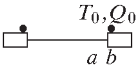
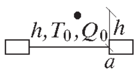
Thus, one can superpose solutions to obtain the solution for a thermal loading on one half-space only.

Defining the stress intensity factors as follows

$$K^*_{I,II} = K_{I,II}\frac{\pi}{2G_z M(1+l)\sqrt{a}} \tag{7.16}$$

the formulae for the special cases of thermal ring loadings are summarized in Table 1.

Table 1. Values of the stress intensity factor in [K/m^2] (κ is defined by Eq. (6.7))

Case of loading	K_I^*		K_{II}^*	
	$\frac{T_0}{4\pi a^2}\times$	$\frac{Q_0}{4\pi\lambda_z s_0 a}\times$	$\frac{T_0}{4\pi a^2}\times$	$\frac{Q_0}{4\pi\lambda_z s_0 a}\times$
	$-\frac{a}{\sqrt{\mathcal{A}_1}}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}\kappa$	$\frac{\sqrt{\mathcal{A}_1}}{a}\kappa$
	0	$-\mathcal{A}_2$	$\kappa\left(\frac{a}{\sqrt{-\mathcal{A}_1}}-\mathcal{A}_2\right)$	0
	$-\frac{a^2}{\mathcal{A}_3}$	$-\mathcal{A}_4$	$\kappa\left(\frac{s_0 ha}{\mathcal{A}_3}-\mathcal{A}_4\right)$	$\kappa\left(1-\frac{s_0 h}{a}\mathcal{A}_4\right)$
	0	$\frac{\sqrt{\mathcal{A}_1}}{a}$	$\frac{as_0}{\sqrt{\mathcal{A}_1}}$	$\frac{\pi}{2}s_0$
	$-\frac{a}{\sqrt{-\mathcal{A}_1}}$	0	0	$s_0\mathcal{A}_2$
	$-\frac{s_0 ha}{\mathcal{A}_3}$	$1-\frac{s_0 h}{a}\mathcal{A}_4$	$\frac{s_0 a^2}{\mathcal{A}_3}$	$s_0\mathcal{A}_4$

where

$$\mathcal{A}_1 = a^2 - b^2 \qquad \mathcal{A}_3 = a^2 + s_0^2 h^2$$

$$\mathcal{A}_2 = \sin^{-1}\frac{a}{b} \qquad \mathcal{A}_4 = \tan^{-1}\frac{a}{s_0 h}$$

All of the results obtained before are valid for isotropic solids, provided that we take

$$\begin{aligned} & s_0 = s_1 = s_2 = k = 1 && \alpha_r = \alpha_z = \alpha \\ & \beta_1 = \beta_3 = \frac{E\alpha}{1-2\nu} && l = -3 + 4\nu \\ & M = \frac{(1+\nu)\alpha}{2(1-\nu)} && \kappa = -\frac{2\nu}{1-2\nu} \\ & G_z M(1+l) = -\frac{1-2\nu}{2(1-\nu)}E\alpha \end{aligned} \tag{7.17}$$

where E is the elastic modulus, and ν is Poisson's ratio.

The limits were computed according to de L'Hospital's rule.

A. Appendix

The following integrals are used to evaluate the auxiliary functions appearing in this paper

$$\int_0^\infty J_0(\xi b)\sin(\xi x)\mathrm{e}^{-\xi s_0 h}\,d\xi = \frac{\eta_0}{x(\zeta_0^2+\eta_0^2)} \tag{A.1}$$

$$\int_0^\infty J_0(\xi b)\cos(\xi x)\mathrm{e}^{-\xi s_0 h}\,d\xi = \frac{\zeta_0}{x(\zeta_0^2+\eta_0^2)} \tag{A.2}$$

$$\int_0^\infty \frac{1}{\xi}J_0(\xi b)\sin(\xi x)\mathrm{e}^{-\xi s_0 h}\,d\xi = \frac{\pi}{2} - \tan^{-1}\zeta_0 \tag{A.3}$$

$$\int_0^\infty \frac{1}{\xi}J_0(\xi b)\Big(\frac{\sin(\xi x)}{\xi x} - \cos(\xi x)\Big)\mathrm{e}^{-\xi s_0 h}\,d\xi = \eta_0\Big[1-\zeta_0\Big(\frac{\pi}{2} - \tan^{-1}\zeta_0\Big)\Big] \tag{A.4}$$

$$\int_0^\infty \frac{1-\cos(\xi x)}{\xi}\mathrm{e}^{-\xi s_0 h}J_0(\xi b)\,d\xi = \frac{1}{2}\ln\Big(\frac{1+\eta_0}{1-\eta_0}\frac{1-\eta_0'}{1+\eta_0'}\Big) \tag{A.5}$$

$$\eta_0' = \frac{s_0 h}{\sqrt{s_0^2h^2+b^2}} \tag{A.6}$$

The oblate spheroidal co-ordinates ζ_0, η_0 are related to b, s_0, h, x by the equations

$$b^2 = x^2(1+\zeta_0^2)(1-\eta_0^2) \qquad\qquad s_0 h = x\zeta_0\eta_0 \tag{A.7}$$

where $-1 \leqslant \eta_0 \leqslant 1$ and $\zeta_0 \geqslant 0$.

The surfaces $\zeta_0 = 0$ and $\eta_0 = 0$ are the interior and exterior of the circle $b = x$, $h = 0$, respectively; here therefore

$$\zeta_0 = \begin{cases} 0 & \text{for} \quad h=0 \quad b<x \\ \sqrt{\dfrac{b^2}{x^2}-1} & \text{for} \quad h=0 \quad b>x \\ \dfrac{s_0 h}{x} & \text{for} \quad b=0 \end{cases} \tag{A.8}$$

$$\eta_0 = \begin{cases} \sqrt{1-\dfrac{b^2}{x^2}} & \text{for} \quad h=0 \quad b<x \\ 0 & \text{for} \quad h=0 \quad b>x \\ 1 & \text{for} \quad b=0 \end{cases} \tag{A.9}$$

The co-ordinates ζ_0, η_0 for $x = a$ are denoted by $\overline{\zeta}_0$, $\overline{\eta}_0$. The co-ordinates for $b = r$, $h = z$ are denoted by ζ, η and those for $x = a$ by $\overline{\zeta}$, $\overline{\eta}$.

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Rozwiązania podstawowe dla termicznych współczynników intensywności naprężenia typów I i II. Zagadnienie osiowo symetryczne

Streszczenie

W pracy rozpatrzono zagadnienia szczeliny dla nieograniczonego termosprężystego ciała stałego poddanego działaniu ustalonej temperatury lub strumienia ciepła. Założono, że powierzchnie szczeliny są termicznie izolowane. Otrzymano funkcje Greena dla współczynników intensywności naprężenia typów I i II. Funkcje Greena zde-

finiowano jako rozwiązanie zagadnienia termosprężystego, poprzecznie izotropowego ciała z kołową lub zewnętrzną szczeliną, gdy na płaszczyźnie równoległej do płaszczyzny szczeliny działają dowolne osiowo symetryczne termiczne obciążenia w postaci ustalonej temperatury lub strumienia ciepła, rozłożonych na okręgu.

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