ON ELASTODYNAMICS OF BIPERIODIC COMPOSITE MEDIA[1]

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The aim of this paper is twofold. First, we formulate a mathematical model for the analysis of waves propagating in a linear-elastic composite, which in every plane normal to a certain straight line has an identical periodic structure. Second, we apply the derived model equations to the investigations of waves propagating across a laminated medium with periodically folded laminae. Lower and higher propagation speeds for the longitudinal and transversal waves are calculated and for the long waves represented in the form of simple asymptotic formulas.

Key words: modelling, composites, waves, dispersion

1. Introduction

By a biperiodic composite we understand a heterogeneous solid which has a periodic structure in a certain plane, and the properties of which are constant in the direction normal to this plane. A formulation of an approximate theory describing the dynamic response of a biperiodic composite solid was detailed in a book by Woźniak and Wierzbicki (2000), and applied to the analysis of some initial-boundary-value problems in papers by Wierzbicki *et al.* (2001), and Woźniak *et al.* (2002). This formulation was based on what is called the tolerance averaging of differential equations with periodic coefficients.

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Various applications of the tolerance averaging technique to the investigation of selected dynamic problems for composite solids and structures can be found in a series of papers by Baron and Woźniak (1995), Dell' Isola *et al.* (1998), Ignaczak (1998), Jędrysiak (1999, 2000), Michalak (1998, 2000), Mazur-Śniady (2001), Woźniak (1999), Woźniak and Wierzbicki (2002), Woźniak (1996) and others.

The approximate theory of biperiodic composites, based on the tolerance averaging of the equations of elastodynamics, which has been formulated in Woźniak and Wierzbicki (2000), cannot be applied to the analysis of waves propagating in an arbitrary direction. The aim of this contribution is to provide the reader with a certain modified version of this theory which is free from the above drawback. The proposed version makes it possible to investigate some dispersion phenomena related to the propagation of waves in an arbitrary direction. The obtained equations are applied to the analysis of harmonic waves in a laminated medium with periodically folded laminae. The considerations are restricted to biperiodic composites made of perfectly bonded linear-elastic constituents.

To make this paper self-consistent, in the subsequent section the fundamental ideas and assumptions of the tolerance averaging technique are outlined; for details the reader is referred to Woźniak and Wierzbicki (2000).

Denotations. Considerations are carried out in the orthogonal Cartesian coordinate system $0x_1x_2x_3$. Partial derivatives with respect to x_1, x_2, x_3 are denoted by ∂_1, ∂_2, ∂_3, respectively, and the time derivative is denoted by the overdot. The gradient operators are introduced in the form $\nabla = (\partial_1, \partial_2, \partial_3)$, $\overline{\nabla} = (\partial_1, \partial_2, 0)$ and $\partial = (0, 0, \partial_3)$. We also denote $\boldsymbol{x} = (x_1, x_2, x_3)$, $\overline{\boldsymbol{x}} = (x_1, x_2)$; hence $\boldsymbol{x} = (\overline{\boldsymbol{x}}, x_3)$. Superscripts A, B run over $1, ..., N$, summation convention holds. We also use the index notation; subscripts $k, l, ...$ run over $1, 2, 3$, subscripts α, β run over $1, 2$ and partial derivatives are indicated by a comma.

2.   Modelling technique

In this section we shall assume that the biperiodic composite solid under consideration occupies in the reference configuration a region $\Omega = \Pi \times (0, L)$, where Π is a region on the $0x_1x_2$-plane. Define $\Delta = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times \{0\}$ as a cell on the $0x_1x_2$-plane, where l_1 and l_2 are the periods of inhomogeneity in directions of the x_1- and x_2-axes, respectively.

We also assume that the smallest characteristic length dimension of the region Π is sufficiently large when compared to the periods l_1, l_2. For every position vector $\boldsymbol{x} = (x_1, x_2, x_3)$ we define $\Delta(\boldsymbol{x}) := \boldsymbol{x} + \Delta$ and

$$\Omega_0 := \{\boldsymbol{x} \in \Omega : \quad \Delta(\boldsymbol{x}) \subset \Omega\}$$

For every $\boldsymbol{x} \in \Omega_0$ and for an arbitrary integrable function f defined in Ω we introduce the averaging operator given by

$$\langle f \rangle(\boldsymbol{x}) = \frac{1}{l_1 l_2} \int_{\Delta(\boldsymbol{x})} f(\overline{\boldsymbol{y}}, x_3) \, d\overline{\boldsymbol{y}} \qquad \boldsymbol{x} \in \Omega_0 \qquad (2.1)$$

Subsequently, the function f can also depend on the time coordinate t.

The philosophy of the proposed modelling approach is based on the supposition that to every physical field Φ can be assigned a positive number ε_Φ called the tolerance parameter such that every two values Φ_1, Φ_2 of this field satisfying the condition $|\Phi_1 - \Phi_2| < \varepsilon_\Phi$ can be treated as indiscernible. Following Fichera (1992) we say that the values of Φ which do not exceed ε_Φ cannot be detected by instruments. Hence, ε_Φ represents a certain degree of accuracy in performing the measurement or calculations. The above philosophy has been applied in Fichera (1992), where ε_Φ was referred to as an upper bound for negligibles.

Setting $l = \sqrt{l_1^2 + l_2^2}$, denoting by T a set of all tolerance parameters regarded in the modelling procedure and by $\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|$ the distance between the points $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{y}}$ we shall introduce two important definitions.

Definition 1. The function F defined on Π will be called *slowly-varying*, $F \in SV_l(T)$, if for every $\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}} \in \Pi$ the condition $\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\| \leqslant l$ implies $|F(\overline{\boldsymbol{x}}) - F(\overline{\boldsymbol{y}})| \leqslant \varepsilon_F$.

Definition 2. The function φ defined on Π will be called *periodic-like*, $\varphi \in PL_l(T)$, if for every $\overline{\boldsymbol{x}} \in \Pi$ there exists a Δ-periodic function $\varphi_{\overline{\boldsymbol{x}}}$ such that for every $\overline{\boldsymbol{y}} \in \Pi$ the condition $\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\| \leqslant l$ implies $|\varphi(\overline{\boldsymbol{y}}) - \varphi_{\overline{\boldsymbol{x}}}(\overline{\boldsymbol{y}})| \leqslant \varepsilon_\varphi$.

Function $\varphi_{\overline{\boldsymbol{x}}}$ will be referred to as *the Δ-periodic approximation* of φ in the vicinity of the point $\overline{\boldsymbol{x}}$. It can be shown that if $\varphi \in PL_l(T)$ then $\langle \varphi \rangle$ is a slowly-varying function.

Remark. Subsequently, the functions F, φ will also depend on $x_3 \in (0, L)$ (and on time t); that is why instead of $\varphi_{\overline{\boldsymbol{x}}}$ we shall write $\varphi_{\boldsymbol{x}}$.

The tolerance averaging technique of differential equations with periodic coefficients is based on two modelling assumptions. The first is strictly related to the concepts of slowly-varying and periodic-like functions.

Tolerance Averaging Approximation (TAA). *For every Δ-periodic integrable function f and every $F \in SV_l(T)$, $\varphi \in PL_l(T)$, the following approximations are assumed to hold*

$$\langle fF\rangle(\boldsymbol{x}) \approx \langle f\rangle F(\boldsymbol{x}) \qquad\qquad \langle f\varphi\rangle(\boldsymbol{x}) \approx \langle f\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x}) \qquad\qquad \boldsymbol{x} \in \Omega_0 \qquad (2.2)$$

From $(2.2)_1$ it follows that in the course of averaging the increments $F(\overline{\boldsymbol{y}}) - F(\overline{\boldsymbol{x}})$, $\overline{\boldsymbol{y}} \in \Delta(\boldsymbol{x})$, of the slowly varying function $F(\cdot)$ can be neglected.

Before formulating the second modelling assumption, let us recall the well known equation of the linear elastodynamics

$$\nabla \cdot (\mathbf{C} : \nabla \boldsymbol{u}) - \rho\ddot{\boldsymbol{u}} + \rho\boldsymbol{f} = \boldsymbol{0} \qquad\qquad (2.3)$$

where $\boldsymbol{u}$ is a displacement field, $\boldsymbol{f}$ is a body force, and where the elasticity tensor field $\mathbf{C}$ as well as the mass density scalar field ρ are always assumed to be the known Δ-periodic functions independent of the x_3-coordinate. The above equation has to be satisfied for every time t in the region $\Omega = \Pi \times (0, L)$, and holds together with the known continuity conditions on the interfaces between the constituents of the composite, and with the prescribed boundary and initial conditions.

The second modelling assumption is based on heuristic premises and restricts the class of elastodynamic problems under consideration to those in which a typical wavelength of what is called a macroscopic deformation pattern is sufficiently large when compared to the diameter l of cell Δ.

Conformability Assumption (CA). *The displacement field $\boldsymbol{u} = \boldsymbol{u}(\overline{\boldsymbol{x}}, x_3, t)$, $\overline{\boldsymbol{x}} \in \Pi$, $x_3 \in (0, L)$, away from the boundary $\partial\Pi$ of Π, conforms to the Δ-periodic heterogeneous structure of the composite, i.e., $\boldsymbol{u} = \boldsymbol{u}(\cdot, x_3, t)$ is for every $x_3 \in (0, L)$ and for every time t a periodic-like function.*

The tolerance averaging procedure related to equation (2.3) will be realized in five steps.

1. Setting $\boldsymbol{w} = \langle\rho\rangle^{-1}\langle\rho\boldsymbol{u}\rangle$ and defining $\boldsymbol{r} = \boldsymbol{u} - \boldsymbol{w}$, we introduce the decomposition of the displacement field

$$\boldsymbol{u}(\overline{\boldsymbol{x}}, x_3, t) = \boldsymbol{w}(\overline{\boldsymbol{x}}, x_3, t) + \boldsymbol{r}(\overline{\boldsymbol{x}}, x_3, t) \qquad\qquad (\overline{\boldsymbol{x}}, x_3) \in \Omega_0 \qquad (2.4)$$

where by means of (CA) we conclude that $\boldsymbol{w}(\cdot, x_3, t)$ is a slowly-varying function and $\langle\rho\boldsymbol{r}\rangle(\boldsymbol{x}, t) = \boldsymbol{0}$. Hence $\boldsymbol{w}$ and $\boldsymbol{r}$ represent the averaged

and oscillating (residual) parts of $\boldsymbol{u}$, respectively. At the same time the values of $\boldsymbol{r}$ have to be quantities of an order l, $\boldsymbol{r}(\boldsymbol{x},t) \in \mathcal{O}(l)$.

2. Substituting (2.4) into (2.3), averaging the resulting equation over $\Delta(\boldsymbol{x})$ and using (2.2), we obtain a variational Δ-periodic cell problem for the Δ-periodic approximation $\boldsymbol{r_x}$ of $\boldsymbol{r}$ in $\Delta(\boldsymbol{x})$. This problem is governed by

$$\langle \rho \overline{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}_x \rangle(\boldsymbol{x},t) + \langle \overline{\nabla \boldsymbol{r}} : \mathbf{C} : \nabla \boldsymbol{r_x} \rangle(\boldsymbol{x},t) - \partial \cdot \langle \overline{\boldsymbol{r}} \cdot \mathbf{C} : \nabla \boldsymbol{r_x} \rangle(\boldsymbol{x},t) =$$
$$= -\langle \overline{\nabla \boldsymbol{r}} : \mathbf{C} \rangle : \nabla \boldsymbol{w}(\boldsymbol{x},t) + \partial \cdot [\langle \overline{\boldsymbol{r}} \cdot \mathbf{C} \rangle : \nabla \boldsymbol{w}(\boldsymbol{x},t)] + \langle \rho \overline{\boldsymbol{r}} \cdot \boldsymbol{f} \rangle(\boldsymbol{x},t)$$
$$\text{(2.5)}$$

$$\langle \rho \boldsymbol{r_x} \rangle(\boldsymbol{x},t) = 0 \qquad\qquad \boldsymbol{x} \in \Omega_0$$

where equation $(2.5)_1$ has to hold for every integrable Δ-periodic test function $\overline{\boldsymbol{r}}$ of $\overline{\boldsymbol{y}} = (y_1, y_2)$ satisfying the conditions $\langle \rho \overline{\boldsymbol{r}} \rangle = \boldsymbol{0}$ and $\overline{\boldsymbol{r}}(x_1, x_2) \in \mathcal{O}(l)$.

3. We look for an approximate solution to (2.5) in the form

$$\boldsymbol{r_x}(\overline{\boldsymbol{y}}, x_3, t) = h^A(\overline{\boldsymbol{y}}) \boldsymbol{v}^A(\overline{\boldsymbol{x}}, x_3, t) \qquad (\overline{\boldsymbol{y}}, x_3) \in \Delta(\boldsymbol{x}) \qquad \text{(2.6)}$$

where $h^A(\cdot)$ are certain postulated *a priori* linear-independent continuous Δ-periodic functions satisfying the conditions $\langle \rho h^A \rangle = 0$ and $h^A(\overline{\boldsymbol{y}}) \in \mathcal{O}(l)$. Moreover, $\boldsymbol{v}^A(\cdot, x_3, t)$ are assumed to be slowly-varying functions. The functions $\boldsymbol{v}^A$ represent new unknowns which will be termed fluctuation variables. The functions $h^A(\cdot)$ can be assumed as the interpolation functions related to the periodic FEM discretization of the cell Δ, Żmijewski (1987), Augustowska and Wierzbicki (2002).

4. Substituting (2.6) into $(2.5)_1$ and assuming $\overline{\boldsymbol{r}} = h^A(\overline{\boldsymbol{y}}) \boldsymbol{c}^A$, $\overline{\boldsymbol{y}} \in \Delta(\boldsymbol{x})$ where $\boldsymbol{c}^A$ are arbitrary constant vectors, we obtain a system of N vector equations

$$\langle h^A h^B \rho \rangle \ddot{\boldsymbol{v}}^B + \langle \overline{\nabla} h^A \cdot \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B + \underline{\langle h^B \mathbf{C} \cdot \overline{\nabla} h^A - h^A \mathbf{C} \cdot \overline{\nabla} h^B \rangle : \partial \boldsymbol{v}^B} -$$
$$\text{(2.7)}$$
$$-\underline{\partial \cdot (\langle h^A h^B \mathbf{C} \rangle : \nabla \boldsymbol{v}^B)} = -\langle \overline{\nabla} h^A \cdot \mathbf{C} \rangle : \nabla \boldsymbol{w} + \underline{\partial \cdot (\langle h^A \cdot \mathbf{C} \rangle : \nabla \boldsymbol{w})} + \langle \boldsymbol{f} h^A \rho \rangle$$

which together with (2.6) represent a certain approximation to periodic cell problem (2.5).

5. Substituting (2.4) into (2.3) and averaging the resulting equation over $\Delta(\boldsymbol{x})$, after using (2.2) and (2.6), we obtain the vector equation

$$\nabla \cdot (\langle \mathbf{C} \rangle : \nabla \boldsymbol{w} + \langle \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B + \underline{\langle h^B \mathbf{C} \rangle : \partial \boldsymbol{v}^B}) - \langle \rho \rangle \ddot{\boldsymbol{w}} + \langle \boldsymbol{f} \rho \rangle = \boldsymbol{0} \quad \text{(2.8)}$$

Equations (2.7) and (2.8) for the unknowns $\boldsymbol{w}$, $\boldsymbol{v}^A$, $A = 1, ..., N$, have constant coefficients and hence they represent *the averaged model of a biperiodic composite* for the analysis of dynamic problems restricted by the heuristic hypothesis (CA).

It has to be emphasized that the solutions $\boldsymbol{w}$, $\boldsymbol{v}^A$ to equations (2.7), (2.8) have a physical sense only if the functions $\boldsymbol{w}(\cdot, x_3, t)$, $\boldsymbol{v}^A(\cdot, x_3, t)$ together with their derivatives are slowly-varying (possibly except for the vicinity of the boundary $\partial\Pi$). The above requirement can be used as a certain *a posteriori* condition for the evaluation of tolerance parameters, Woźniak and Wierzbicki (2000).

We have stated above that the functions h^A can be derived by the periodic discretization of Δ, and hence they are periodic interpolation functions satisfying extra conditions $\langle\rho h^A\rangle = 0$. In most cases, the number N of these functions has to be large and hence in model equations (2.7), (2.8) we deal with a large number N of the unknown fluctuation variables $\boldsymbol{v}^A$. To eliminate this drawback we shall introduce into the modelling technique the problem of free periodic vibrations of cell Δ. This is an eigenvalue problem of finding a continuous function $\boldsymbol{h}(\overline{\boldsymbol{y}})$, $\overline{\boldsymbol{y}} \in \Delta$, which is Δ-periodic and satisfies the condition $\langle\rho\boldsymbol{h}\rangle = \boldsymbol{0}$ as well as the variational condition

$$\langle\overline{\nabla\boldsymbol{h}} : \mathbf{C} : \overline{\nabla}\boldsymbol{h}\rangle - \lambda\langle\rho\overline{\boldsymbol{h}} \cdot \boldsymbol{h}\rangle = 0 \qquad (2.9)$$

which holds for every Δ-periodic test function $\overline{\boldsymbol{h}} = \overline{\boldsymbol{h}}(\boldsymbol{y})$ such that $\langle\overline{\boldsymbol{h}}\rho\rangle = \boldsymbol{0}$. The eigenvalues λ of (2.9) represent here the squares of the free periodic vibration frequencies of the cell Δ. We shall look for an approximate solution to this eigenvalue problem in the form

$$\boldsymbol{h}(\overline{\boldsymbol{y}}) = h^A(\overline{\boldsymbol{y}})\boldsymbol{a}^A \qquad\qquad \overline{\boldsymbol{y}} \in \overline{\Delta} \qquad (2.10)$$

where h^A have the same meaning as before (being derived by the periodic FEM discretization of Δ) and $\boldsymbol{a}^A$ are arbitrary constant vectors. Combining (2.10) and (2.9), we obtain a new eigenvalue problem

$$(\langle\overline{\nabla}h^A \cdot \mathbf{C} \cdot \overline{\nabla}h^B\rangle - \mathbf{I}\lambda\langle\rho h^A h^B\rangle) \cdot \boldsymbol{a}^B = \boldsymbol{0} \qquad (2.11)$$

where $\mathbf{I}$ stands for a unit tensor in R^2. Let $(\boldsymbol{a}_a^1, ..., \boldsymbol{a}_a^N)$, $a = 1, ..., n$, $n < N$, be the first n eigenvectors related to problem (2.11). In this case, from (2.10) we obtain

$$\boldsymbol{h}_a(\overline{\boldsymbol{y}}) = h^A(\overline{\boldsymbol{y}})\boldsymbol{a}_a^A \qquad\qquad a = 1, ..., n \qquad (2.12)$$

and instead of (2.6) we shall look for an approximate solution to (2.5) in the form

$$\boldsymbol{r}_{\boldsymbol{x}}(\overline{\boldsymbol{y}}, x_3, t) = \boldsymbol{h}_b(\overline{\boldsymbol{y}})v_b(\overline{\boldsymbol{x}}, x_3, t) \qquad\qquad (\overline{\boldsymbol{y}}, x_3) \in \Delta(\boldsymbol{x}) \qquad (2.13)$$

where here and hereafter the summation convention over $b = 1, ..., n$ holds. The functions $v_a(\cdot)$, $a = 1, ..., n$, are new unknowns which are assumed to be slowly-varying functions of $\overline{\boldsymbol{x}} = (x_1, x_2)$. Using approximation (2.13), instead of (2.7), (2.8), we obtain the following system of equations for $\boldsymbol{w}$ and v_a

$$\langle \boldsymbol{h}_a \cdot \boldsymbol{h}_b \rho \rangle \ddot{v}_b + \langle \nabla \boldsymbol{h}_a : \mathsf{C} : \nabla \boldsymbol{h}_b \rangle v_b + \langle \boldsymbol{h}_b \cdot \mathsf{C} : \overline{\nabla} \boldsymbol{h}_a - \boldsymbol{h}_a \cdot \mathsf{C} : \overline{\nabla} \boldsymbol{h}_b \rangle \cdot \partial v_b -$$
$$- \partial \cdot [\langle (\boldsymbol{h}_a \otimes \boldsymbol{h}_b) : \mathsf{C} \rangle \cdot \nabla v_b] = -\langle \overline{\nabla} \boldsymbol{h}_a : \mathsf{C} \rangle : \nabla \boldsymbol{w} + \partial \cdot (\langle \boldsymbol{h}_a \cdot \mathsf{C} \rangle : \nabla \boldsymbol{w}) +$$
$$+ \langle \boldsymbol{f} \cdot \boldsymbol{h}_a \rangle \qquad (2.14)$$
$$\nabla \cdot (\langle \mathsf{C} \rangle : \nabla \boldsymbol{w} + \langle \mathsf{C} : \overline{\nabla} \boldsymbol{h}_b \rangle v_b + \langle \boldsymbol{h}_b \cdot \mathsf{C} \rangle : \partial v_b) - \langle \rho \rangle \ddot{\boldsymbol{w}} + \langle \boldsymbol{f} \rangle = \boldsymbol{0}$$

The number n of the unknowns v_a can be small and that is why equations (2.14) represent what will be called *the reduced order averaged model of a biperiodic composite.*

In many special cases the form of the functions $\boldsymbol{h}_a$ as well as h^A can be also based on a heuristic assumption that $\boldsymbol{h}_a$ and $h^A \boldsymbol{e}_\alpha$ (where $\boldsymbol{e}_\alpha$, $\alpha = 1, 2$, constitute a vector base in E^2) approximate the expected form of the free periodic vibrations of the cell Δ and satisfy the conditions $\langle \rho \boldsymbol{h}_a \rangle = \boldsymbol{0}$ and $\langle \rho h^A \rangle = 0$, respectively.

In the subsequent part of this contribution, the considerations will be based on equations (2.7), (2.8), but the application of the reduced order model leads to similar conclusions.

3. Averaged wave-type model

It can be observed that the averaged model of biperiodic composites, represented by equations (2.7), (2.8), in the general case cannot be applied to the analysis of waves propagating along the $0x_3$-axis. In order to obtain the wave-type averaged equations we introduce an extra assumption that the fields $\boldsymbol{w}$, $\boldsymbol{v}^A$ together with their derivatives are slowly-varying functions not only with respect to $\overline{\boldsymbol{x}} = (x_1, x_2)$ but also with respect to the x_3-coordinate. It means that for every x_3, y_3 the condition $|x_3 - y_3| \leqslant l$ implies $\|\boldsymbol{w}(\overline{\boldsymbol{x}}, x_3, t) - \boldsymbol{w}(\overline{\boldsymbol{x}}, y_3, t)\| \leqslant \varepsilon_u$ and $\|\boldsymbol{v}^A(\overline{\boldsymbol{x}}, x_3, t) - \boldsymbol{v}^A(\overline{\boldsymbol{x}}, y_3, t)\| \leqslant \varepsilon_u$, where ε_u is a tolerance parameter assigned to the evaluation of displacements; similar conditions hold also for the derivatives of $\boldsymbol{w}$ and $\boldsymbol{v}^A$. For the sake of simplicity the set of these slowly-varying functions will be also denoted by $SV_l(T)$. Hence, averaging the aforementioned slowly-varying functions over the three dimensional cell $V(\boldsymbol{x}) \equiv \Delta(\overline{\boldsymbol{x}}, x_3) \times (x_3 - l/2, x_3 + l/2)$, we shall neglect in $V(\boldsymbol{x})$ increments of $\boldsymbol{w}$, $\boldsymbol{v}^A$ also in the $0x_3$-axis direction. In this

case, all underlined terms in (2.7), (2.8) can be neglected, and in the absense of body forces we obtain

$$\nabla \cdot (\langle \mathbf{C} \rangle : \nabla \boldsymbol{w} + \langle \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B) - \langle \rho \rangle \ddot{\boldsymbol{w}} = \mathbf{0}$$

$$\langle h^A h^B \rho \rangle \ddot{\boldsymbol{v}}^B + \langle \overline{\nabla} h^A \cdot \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B + \langle \overline{\nabla} h^A \cdot \mathbf{C} \rangle : \nabla \boldsymbol{w} = \mathbf{0}$$

$$(3.1)$$

Equations (3.1) represent *the wave-type averaged model of a biperiodic composite*. It has to be remembered that solutions to (3.1) have a physical sense only if the following conditions

$$\boldsymbol{w}(\cdot, t) \in SV_l(T) \qquad\qquad \boldsymbol{v}^A(\cdot, t) \in SV_l(T) \tag{3.2}$$

as well as similar conditions for the derivatives of $\boldsymbol{w}$, $\boldsymbol{v}^A$ occurring in (3.1) hold.

The final conclusion is that in a biperiodic medium waves can propagate also in the $0x_3$-axis direction provided that the pertinent averaged parts of the displacements $\boldsymbol{w}$ and the fluctuation variables $\boldsymbol{v}^A$ are slowly-varying with respect to all spatial coordinates.

Independently of this general statement, we can also deal with some special situations in which wave equations (3.1) can be obtained directly from equations (2.7), (2.8). They are situations where the functions h^A can be assumed in the form satisfying identically the extra conditions

$$\langle h^A \mathbf{C} \rangle = \mathbf{0} \qquad\qquad \langle h^A \mathbf{C} \cdot \nabla h^B \rangle = \mathbf{0}$$

Hence, after neglecting the body forces, equations (2.7), (2.8) yield

$$\nabla \cdot (\langle \mathbf{C} \rangle : \boldsymbol{w} + \langle \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B) - \langle \rho \rangle \ddot{\boldsymbol{w}} = \mathbf{0}$$

$$\langle h^A h^B \rho \rangle \ddot{\boldsymbol{v}}^B + \langle \overline{\nabla} h^A \cdot \mathbf{C} \cdot \overline{\nabla} h^B \rangle \cdot \boldsymbol{v}^B - \partial \cdot (\langle h^A h^B \mathbf{C} \rangle : \nabla \boldsymbol{v}^B) +$$

$$+ \langle \nabla h^A \cdot \mathbf{C} \rangle : \nabla \boldsymbol{w} = \mathbf{0}$$

$$(3.3)$$

In this case, instead of conditions (3.2) we shall deal with weaker conditions of the form

$$\boldsymbol{w}(\cdot, x_3, t) \in SV_l(T) \qquad\qquad \boldsymbol{v}^A(\cdot, x_3, t) \in SV_l(T) \tag{3.4}$$

It means that the solutions $\boldsymbol{w}$, $\boldsymbol{v}^A$ to (3.3) have to be slowly-varying functions only with respect to the x_1- and the x_2-coordinates.

Subsequently, we shall take into account the averaged wave equations in the form (3.3) bearing in mind that in the case described by equations (3.1)

the terms depending on the coefficients $\langle h^A h^B \mathbf{C}\rangle$ have to be neglected, and the fields $\boldsymbol{w}$, $\boldsymbol{v}^A$ are slowly-varying functions with respect to all spatial coordinates. We shall also assume that the constituents of a composite are isotropic with the moduli λ, μ and mass density ρ as Δ-periodic functions of x_1, x_2. In this case, equations (3.3) in the index notation take the form

$$\langle\lambda+\mu\rangle w_{k,k\alpha} + \langle\mu\rangle w_{\alpha,kk} + \langle\lambda h^A{}_{,\gamma}\rangle v^A_{\gamma}{}_{,\alpha} + \langle\mu h^A{}_{,\gamma}\rangle v^A_{\alpha}{}_{,\gamma} +$$

$$+\langle\mu h^A{}_{,\alpha}\rangle v^A_{k}{}_{,k} - \langle\rho\rangle\ddot{w}_{\alpha} = 0$$

$$\langle\lambda+\mu\rangle w_{k,k3} + \langle\mu\rangle w_{3,kk} + \langle\lambda h^A{}_{,\gamma}\rangle v^A_{\gamma}{}_{,3} + \langle\mu h^A{}_{,\gamma}\rangle v^A_{3}{}_{,\gamma} - \langle\rho\rangle\ddot{w}_3 = 0$$

$$\langle\rho h^A h^B\rangle\ddot{v}^B_{\alpha} + \langle\lambda h^A{}_{,\alpha}h^B{}_{,\beta}\rangle v^B_{\beta} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\gamma}\rangle v^B_{\alpha} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\alpha}\rangle v^B_{\gamma} - \quad (3.5)$$

$$-\langle\mu h^A h^B\rangle v^B_{\alpha}{}_{,3\alpha} = -\langle\lambda h^A{}_{,\alpha}\rangle w_{k,k} - \langle\mu h^A{}_{,\gamma}\rangle(w_{\alpha,\gamma}+w_{\gamma,\alpha})$$

$$\langle\rho h^A h^B\rangle\ddot{v}^B_{3} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\gamma}\rangle v^B_{3} - \langle(\lambda+2\mu)h^A h^B\rangle v^B_{3}{}_{,33} - \langle\lambda h^A h^B\rangle v^B_{\alpha,\alpha3} =$$

$$= -\langle\mu h^A{}_{,\alpha}\rangle(w_{3,\alpha}+w_{\alpha,3})$$

Let us investigate the plane wave propagating along the x_3-axis. Setting $w_k = w_k(x_3,t)$, $v^A_k = v^A_k(x_3,t)$ we obtain the longitudinal wave equations

$$\langle\lambda+2\mu\rangle w_{3,33} + \langle\lambda h^A{}_{,\gamma}\rangle v^A_{\gamma}{}_{,3} - \langle\rho\rangle\ddot{w}_3 = 0$$

$$\langle\rho h^A h^B\rangle\ddot{v}^B_{\alpha} + \langle\lambda h^A{}_{,\alpha}h^B{}_{,\beta}\rangle v^B_{\beta} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\gamma}\rangle v^B_{\alpha} - \langle\mu h^A h^B\rangle v^B_{\alpha}{}_{,33} = \quad (3.6)$$

$$= -\langle\lambda h^A{}_{,\alpha}\rangle w_{3,3}$$

for w_3, v^A_α and the transversal wave equations

$$\langle\mu\rangle w_{\alpha,33} + \langle\mu h^A{}_{,\alpha}\rangle v^A_{3}{}_{,3} - \langle\rho\rangle\ddot{w}_{\alpha} = 0$$

$$\langle\rho h^A h^B\rangle\ddot{v}^B_{3} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\gamma}\rangle v^B_{3} - \langle(\lambda+2\mu)h^A h^B\rangle v^B_{3}{}_{,33} = -\langle\mu h^A{}_{,\alpha}\rangle w_{\alpha,3} \quad (3.7)$$

for w_α, v^A_3. Let us observe that to the averaged displacements w_3 and w_α there are assigned the displacement fluctuations $h^A v^A_\alpha$ and $h^A v^A_3$, respectively, in the directions normal to the pertinent averaged displacements.

For a cylindrical wave propagating in the direction normal to the x_3-axis, by setting $w_k = w_k(x_1,x_2,t)$, $v^A_k = v^A_k(x_1,x_2,t)$, we obtain from (3.5) the independent equations for w_3, v^A_3 representing transversal waves in the form

$$\langle\mu\rangle w_{3,\alpha\alpha} + \langle\mu h^A{}_{,\gamma}\rangle v^A_{3}{}_{,\gamma} - \langle\rho\rangle\ddot{w}_3 = 0$$

$$\langle\rho h^A h^B\rangle\ddot{v}^B_{3} + \langle\mu h^A{}_{,\gamma}h^B{}_{,\gamma}\rangle v^B_{3} = -\langle\mu h^A{}_{,\alpha}\rangle w_{3,\alpha} \quad (3.8)$$

For an arbitrary biperiodic composite with isotropic constituents, in the directions normal to the x_3-axis, the longitudinal wave as well as the transversal wave in the $0x_1x_2$ plane cannot propagate.

4.   Example of application

The objective of the analysis will be now a biperiodic two component unbounded laminated medium in which the laminae are isotropic and periodically slightly folded in the direction of the x_2-axis. A fragment of a cross section $x_3 = $ const of this laminate is shown in Fig. 1.
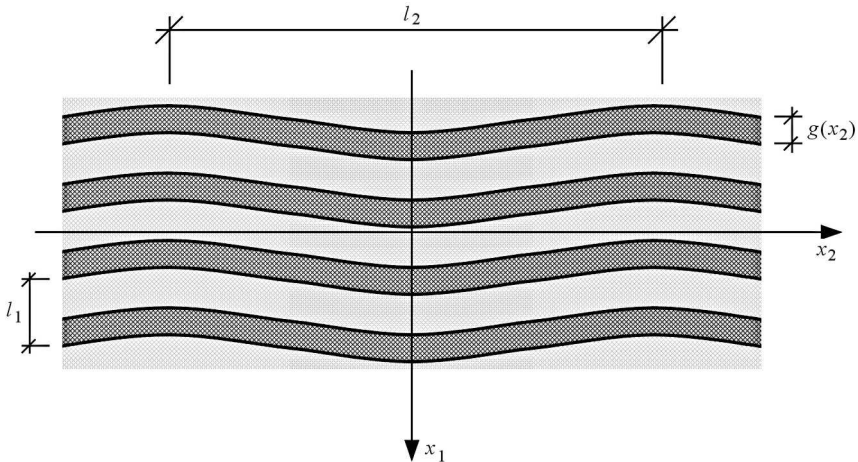
Fig. 1. Cross-section of a biperiodic laminated medium

In order to describe the biperiodic material structure under consideration we denote by $x_1 = \alpha(x_2)$ the periodic function with the period l_2, the mean value of which in $(0, l)$ is equal to zero and its amplitude A satisfies the condition $A/l_2 \ll 1$. We also assume that $\alpha(0) = 0$. The interfaces between components are assumed to be cylindrical and given by

$$x_1 = \alpha(x_2) + nl_1 \pm \frac{g(x_2)}{2} \qquad\qquad n = 0, \pm 1, \pm 2, ...$$

where $g(x_2)$, $l_2 - g(x_2)$ are the thicknesses of laminae measured along the x_1-axis. By means of the condition $A/l_2 \ll 1$ we shall assume that $g \approx g_0$, where now g_0, and $l_1 - g_0$ are the mean thicknesses of the laminae. We define

$\nu' = g_0/l_1$, $\nu'' = (l_1 - g_0)/l_1$ and denote by ρ', λ', μ' and ρ'', λ'', μ'' the mass densities and Lame's moduli in the laminae with the mean thickness g_0 and $l_1 - g_0$, respectively. Moreover, let $\varphi = \varphi(x_1)$ be a periodic saw-like function the diagram of which is shown in Fig. 2.
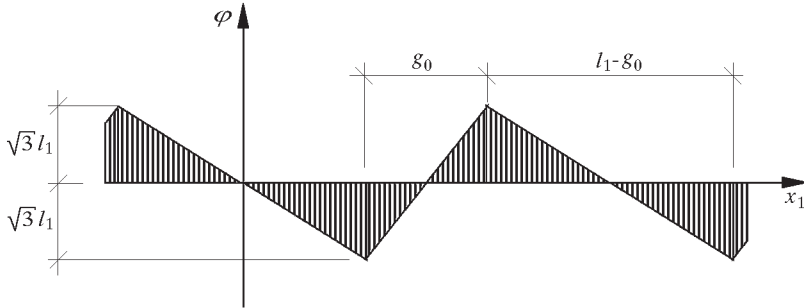

Fig. 2. Diagram of function $\varphi = \varphi(x_1)$

We shall introduce an approximated mathematical model of the biperiodic laminate under consideration by assuming $N = 1$ and defining a function $h = h^1(x_1, x_2)$, $(x_1, x_2) \in R^2$, in the form

$$h(x_1, x_2) = \varphi(x_1 - \alpha(x_2))$$

We deal here with a certain generalization of the known approach to the modelling of a layered medium which was applied in papers by Ignaczak (1998), Matysiak and Nagórko (1995), Wierzbicki *et al.* (2001), Woźniak (1996), Woźniak *et al.* (2002) and many others. Denoting by η an arbitrary from the moduli λ, μ, we obtain the following values of coefficients in equations (3.5)-(3.8)

$$\langle \eta \rangle = \eta'\nu' + \eta''nu'' \qquad \langle \eta h_{,1} \rangle = 2\sqrt{3}(\eta'' - \eta')$$

$$\langle \eta h_{,2} \rangle = 0 \qquad \langle \eta(h_{,1})^2 \rangle = 12\Big(\frac{\eta'}{\nu'} + \frac{\eta''}{\nu''}\Big)$$

$$\langle \eta h_{,1} h_{,2} \rangle = 0 \qquad \langle \eta(h_{,2})^2 \rangle = 12\xi\Big(\frac{\eta'}{\nu'} + \frac{\eta''}{\nu''}\Big) \qquad (4.1)$$

$$\langle \rho \rangle = \nu'\rho' + \nu''\rho'' \qquad \langle \rho(h)^2 \rangle = (l_1)^2\langle\rho\rangle$$

$$\langle \eta(h)^2 \rangle = (l_1)^2\langle\eta\rangle$$

From equations (3.5) under the extra denotation

$$\xi = \int\limits_0^{l_2} \alpha'(x_2)\, dx_2$$

we obtain

$$\langle \lambda + 2\mu \rangle w_{3,33} + \langle \lambda h_{,1} \rangle v_{1,3} - \langle \rho \rangle \ddot{w}_3 = 0$$

$$(l_1)^2 \langle \rho \rangle \ddot{v}_1 + \langle (\lambda + \mu + \xi\mu)(h_{,1})^2 \rangle v_1 - (l_1)^2 \langle \mu \rangle v_{1,33} = -\langle \lambda h_{,1} \rangle w_{3,3}$$

(4.2)

and without the loss of generality we can assume $v_2 = 0$. From (3.6) we get

$$\langle \mu \rangle w_{1,33} + \langle \mu h_{,1} \rangle v_{3,3} - \langle \rho \rangle \ddot{w}_1 = 0$$

$$(l_1)^2 \langle \rho \rangle \ddot{v}_3 + \langle \mu(1 + \xi)(h_{,1})^2 \rangle v_3 - (l_1)^2 \langle (\lambda + 2\mu) \rangle v_{3,33} = -\langle \mu h_{,1} \rangle w_{1,3}$$

(4.3)

and

$$\langle \mu \rangle w_{2,33} - \langle \rho \rangle \ddot{w}_2 = 0 \tag{4.4}$$

It means that in the laminated biperiodic medium under consideration three kinds of waves can propagate in the x_3-axis direction: the longitudinal wave described by (4.2) and two transversal waves governed by (4.3) and (4.4), for which free vibrations take place in the directions of the $0x_1$- and $0x_2$-axis, respectively. The wave described by (4.4) is a nondispersive wave. In order to investigate these waves we shall write equations (4.2) and (4.3) in the following form valid for both of them

$$aw_{,33} + bv_{,3} - \langle \rho \rangle \ddot{w} = 0$$

$$l_2 \langle \rho \rangle \ddot{v} + ev - l^2 d v_{,33} = -bw_{,3}$$

(4.5)

where $w = w(x_3, t)$, $v = v(x_3, t)$, $l = l_1$ and a, b, e, d are the pertinent constant coefficients occurring either in (4.2) or in (4.3), satisfying the conditions $a > 0$, $e > 0$, $d > 0$ and $ae - b^2 > 0$.

Let us investigate the propagation of harmonic waves by substituting to (4.5) the right-hand sides of the formulae

$$w = A_w \exp[ik(x_3 - ct)] \qquad v = A_v \exp[ik(x_3 - ct)] \tag{4.6}$$

where $k = 2\pi/L$ is the wave number (here L is the wavelength) and A_w, A_v are the amplitudes. Substituting the right-hand sides of (4.6) into (4.5), we obtain the following dispersion relation

$$l^2 \langle \rho \rangle^2 k^2 c^4 - \langle \rho \rangle (e + k^2 l^2 d + k^2 l^2 a) c^2 + ae - b^2 + k^2 l^2 da = 0 \tag{4.7}$$

which yields two propagation speeds c_1 and c_2.

Let us observe that in the problem under consideration we deal with the situation described by equations (3.3) and conditions (3.4). Hence, the functions in (4.2)-(4.6) may not be slowly varying with respect to the x_3-coordinate, and dispersion relation (4.7) has a physical sense for an arbitrary wave number k. However, if the wavelength L is large when compared to the period $l = l_1$ (i.e., if functions (3.4) are slowly varying) then the nondimensional wave number $q = kl = 2\pi l/L$ is small when compared to 1. Transforming equation (4.7) to the form

$$\langle\rho\rangle q^2 c^4 - \langle\rho\rangle[e + (a + d)q^2]c^2 + ae - b^2 + adq^2 = 0$$

and restricting considerations to the long waves (when compared to the period l), after denotation

$$\widetilde{a} = a - \frac{b^2}{e} + \frac{ad}{e}$$

we obtain the solutions c_1, c_2 to dispersion relation (4.7) in the asymptotic form

$$(c_1)^2 = \frac{\widetilde{a}}{\langle\rho\rangle} + \mathcal{O}(q^2)$$

$$(c_2)^2 = \frac{e}{\langle\rho\rangle}\frac{1}{q^2} - \frac{ad - b^2 - ed}{\langle\rho\rangle e} + \mathcal{O}(q^2)$$

Thus, we conclude that in the biperiodic laminated medium under consideration the following kinds of waves can propagate along the x_3-axis:

- the longitudinal and transversal wave (with vibrations in the $0x_1x_3$-plane) propagating with two different speeds c_1, c_2 determined by dispersion relation (4.7),

- the transversal nondispersive wave described by (4.4).

As we have stated above, if the biperiodic medium is modelled by equations (3.3), then the waves propagating in the x_3-axis direction can have arbitrary lengths; this situation takes place in the above problem.

5.   Conclusions

The results obtained in this contribution can be summarized by the following conclusions.

- Dynamic problems of biperiodic linear-elastic composites, in which displacement fields are represented by periodic-like functions with respect to the x_1- and x_2-coordinates, can be investigated in the framework of the averaged model governed by equations (2.7), (2.8). Using this model we can satisfy, with a required accuracy, the initial conditions for every constituent of the biperiodic composite as well as the boundary conditions for this constituent on the boundaries $x_3 = 0, L$. On the remaining part $\partial\Pi \times (0, L)$ of the solid boundary, the displacement conditions can be imposed only on the averaged displacement field. Some remarks on this subject can also be found in Woźniak and Wierzbicki, (2000).

- In the general case the waves of an arbitrary length, represented by the periodic-like functions of x_1, x_2, cannot propagate across the biperiodic medium in the direction of the $0x_3$-axis. The wave propagation in this direction is possible only in special situations which are described by equations (3.3).

- In the biperiodic linear-elastic medium only the waves can propagate which are represented by functions being periodic-like not only with respect to the x_1- and x_2-coordinates but also with respect to the x_3-coordinate. In this case the propagation of waves is described by equations (3.1).

- The main difficulty in the formulation of the proposed models lies in finding proper approximation (2.6) to periodic cell problem (2.5). An approximate solution to this problem can be found on the basis of a certain heuristic hypothesis as it was done in Section 4, where only one Δ-periodic function $h(x_1, x_2)$ described the form of displacement fluctuations. In general, the functions $h^A(x_1, x_2)$ can be derived from a periodic discretizations of the cell Δ, cf. Augustowska and Wierzbicki (2002), but the modelling approach can lead to a large number of the unknowns $\boldsymbol{v}^A$ in the model equations. In these situations we have to apply the reduced order models represented by equations (2.14).

- By a formal limit passage $l \to 0$, differential equations (2.7) are reduced to a system of linear algebraic equations for $\boldsymbol{v}^A$; this passage is due to the fact that $h^A \to 0$ together with $l \to 0$ but $\overline{\nabla}h^A$ remain finite. In this limit case, the unknowns $\boldsymbol{v}^A$ can be eliminated from (2.8) and we arrive at a single equation for the averaged displacement $\boldsymbol{w}$. At the same time, periodic cell problem (2.5) reduces to the well known periodic cell problem of the homogenization theory, cf. Bensoussan *et al.* (1978), Jikov

et al. (1994). It follows that from the physical point of view the model obtained by the tolerance averaging technique can be treated as a certain generalization of the homogenized model of a linear-elastic periodic solid.

- The example discussed in Section 4 shows that the obtained model equations can be successfully applied to the analysis of wave propagation problems including the dispersion phenomena caused by the heterogeneous biperiodic structure of a solid.

References

1. AUGUSTOWSKA L., WIERZBICKI E., 2002, Two approaches to the formation of tolerance averaged equations for elastodynamics of periodic solids, *Prace Naukowe Inst. Mat. Inf. Politechniki Czestochowskiej*, 5-14

2. BARON E., WOŹNIAK C., 1995, On the microdynamics of composite plates, *Arch. Appl. Mech.*, **66**, 126-133

3. BENSOUSSAN A., LIONS J.L., PAPANICOLAOU G., 1978, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam

4. DELL'ISOLA F., ROSA L., WOŹNIAK C., 1998, A micro-structural continuum modelling compacting fluid-saturated grounds, *Acta Mech.*, **127**, 165-182

5. FICHERA G., 1992, Is the Fourier theory of heat propagation paradoxical?, *Rend. Circolo Mat. Palermo*, Ser. II, **XLI**, 5-28

6. IGNACZAK J., 1998, Saint-Venant type decay estimates for transient heat conduction in a composite rigid semispace, *J. Therm. Stresses*, **21**, 185-204

7. JĘDRYSIAK J., 1999, Dynamics of thin periodic plates resting on a periodically inhomogeneous Winkler foundation, *Arch. Appl. Mech.*, **69**, 345-356

8. JĘDRYSIAK J., 2000, On the stability of thin periodic plates, *Eur. J. Mech. A/Solids*, **19**, 487-502

9. JIKOV V.V., KOZLOV C.M., OLEINIK O.A., 1994, *Homogenization of Differential Operators and Integral Functionals*, Springer Verlag, Berlin-Heidelberg

10. MAZUR-ŚNIADY K., 2001, A kinematic internal variable approach to dynamics of beams with a periodic-like structure, *J. Theor. Appl. Mech.*, **39**, 175-194

11. MATYSIAK S.J., NAGÓRKO W., 1995, On the wave propagation in periodically laminated composites, *Bull. Pol. Ac. Sci.; Sci. Tech.*, **43**, 1-12

12. MICHALAK B., 1998, Stability of elastic slightly wrinkled plates, *Acta Mech.*, **130**, 111-119

13. MICHALAK B., 2000, *Vibrations of plates with initial geometrical imperfections interacting with a periodic elastic foundation*, *Arch. Appl. Mech.*, **70**, 508-518

14. WIERZBICKI E., WOŹNIAK C., WOŹNIAK M., 2001, *On the modelling of transient micro-motions and near-boundary phenomena in a stratified elastic layer*, *Int. J. Engng Sci.*, **39**, 1429-1441

15. WOŹNIAK C., 1999, *On dynamics of substructured shells*, *J. Theor. Appl. Mech.*, **37**, 255-265

16. WOŹNIAK M., 1996, *2D-dynamics of a stratified elastic subsoil layer*, *Arch. Appl. Mech.*, **66**, 284-290

17. WOŹNIAK C., WIERZBICKI E., 2000, *Averaging Techniques in Thermomechanics of Composite Solids. Tolerance Averaging Versus Homogenization*, Wydawnictwo Politechniki Częstochowskiej, Częstochowa

18. WOŹNIAK C., WIERZBICKI E., 2002, *On the macroscopic modelling of elastic/viscoplastic composites*, *Arch. Mech.*, **54**, 551-564

19. WOŹNIAK M., WIERZBICKI E., WOŹNIAK C., 2002, *A macroscopic model for the diffusion and heat transfer process in a periodically micro-stratified solid layer*, *Acta Mech.*, **157**, 175-185

20. ŻMIJEWSKI K.H., 1987, *Numeryczna realizacja metody parametrów mikrolokalnych*, *VIII Konferencja Metody Komputerowe w Mechanice Konstrukcji*, Jadwisin, 471-480

O elastodynamice dwuperiodycznych kompozytów

Streszczenie

W artykule przedstawiono dwa problemy. Po pierwsze, sformułowano matematyczny model umożliwiający badanie propagacji fal w liniowo-sprężystych kompozytach, dla których każda płaszczyzna prostopadła do pewnej prostej ma taką samą dwuwymiarową strukturę periodyczną. Po drugie, otrzymany model zastosowano do analizy propagacji fal w laminacie o periodycznie pofałdowanych warstwach. Dla przypadku tego wyznaczono niższą i wyższą prędkość fazową propagacji, które w przypadku fal długich dają się wyrazić za pomocą prostych formuł asymptotycznych.