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AFFINE TENSORS IN SHELL THEORY

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Resultant force and moment are structured as a single object called the torsor. Excluding all metric notions, we define the torsors as skew-symmetric bilinear mappings operating on the linear space of the affine vector-valued functions. Torsors are a particular family of affine tensors. On this ground, we define an intrinsic differential operator called the affine covariant divergence. Next, we claim that the torsor field characterizing the behavior of a continuous medium is affine covariant divergence free. Applying this general principle to the dynamics of three-dimensional media, Euler's equations are recovered. Finally, we investigated more thoroughly the dynamics of shells. Using adapted coordinates, this general principle provides a consistent way to obtain new equations with non-expected terms involving Coriolis's effects and the time evolution of the surface.

Key words: tensorial analysis, continuum mechanics, dynamics of shells

1. Introduction

Our starting point is closely related to a new setting developed in mechanics by Souriau (1992, 1997a) on the ground of two key ideas: a new definition of torsors and the crucial part played by the affine group of $\boldsymbol{R}^n$. This group forwards on a manifold an intentionally poor geometrical structure. Indeed, this choice is guided by the fact that it contains both Galileo and Poincaré groups (Souriau, 1997b), that allows to involve the classical and relativistic mechanics at one go. This viewpoint implies that we do not use the trick of the Riemannian structure. In particular, the linear tangent space cannot be identified to its dual one and tensorial indices may be neither lowered nor raised.

To each group corresponds a class of tensors. The components of these tensors are transformed according to the action of the considered group. The standard tensors discussed in the literature are those of the linear group of $\boldsymbol{R}^n$. We will call them linear tensors. A fruitful standpoint consists in considering the class of the affine tensors, corresponding to the affine group. To each group a family of connections allowing one to define covariant derivatives for the corresponding classes of tensors is associated. The connections of the linear group are known through Christoffel's coefficients. They represent, as usual, infinitesimal motions of the local basis. From a physical viewpoint, these coefficients are force fields such as gravity or Coriolis's force. To construct the connection of the affine group, we need Christoffel's coefficients stemming from the linear group and additional ones describing infinitesimal motions of the origin of the affine space associated with the linear tangent space.

2. Affine tensors

Notations

Let $\mathcal{T}$ be a linear space (or vector space) of the dimension n, and $(\vec{e}_\alpha)$ be a basis of $\mathcal{T}$. The associated co-basis $(\overleftarrow{e}^{\,\alpha})$ is such that $\overleftarrow{e}^{\,\alpha}(\vec{e}_{\,\beta}) = \delta^\alpha_\beta$. A new basis $\vec{e}_{\alpha'} = P^\beta_{\alpha'}\,\vec{e}_\beta$ can be defined through the transformation matrix $P = (P^\beta_{\alpha'})$. We denote the inverse matrix P^{-1}.

Tensor class

In a previous paper (de Saxcé, 2002), we proposed a generalization of the usual concept of the tensor, relevant for the mechanics: *the tensors are objects the components of which are changed by a given group of transformations* (more precisely, they are changed by a linear representation of the considered group). Considering the linear group $\boldsymbol{GL}(n)$, we recover the class of linear tensors. Nevertheless, other choices of transformation groups are possible. In many applications, people customarily handle the orthogonal group $\boldsymbol{O}(n)$, a subgroup of $\boldsymbol{GL}(n)$, that leads to the class of the Euclidean tensors. On the other hand, considering the affine group $\boldsymbol{A}(n)$, an extension of $\boldsymbol{GL}(n)$ obtained by adding the translations, we define the class of affine tensors.

Affine space

To define an origin Q of the affine space $A_{\mathcal{T}}$ associated to $\mathcal{T}$, we can use the column vector V_0 collecting the components V_0^α, in the basis $(\vec{e}_\alpha)$, of the vector $\overrightarrow{Q0}$ joining Q to the zero of $\mathcal{T}$ considered as a point of $A_{\mathcal{T}}$. By the choice of this affine frame $r = (V_0, (\vec{e}_\alpha))$, any point V of $A_{\mathcal{T}}$ can be identified to the vector $\overrightarrow{QV} = V^\alpha\, \vec{e}_\alpha$. Now, let $r' = (V_0', (\vec{e}_{\alpha'}))$ be a new affine frame of the origin Q' and basis $(\vec{e}_{\alpha'})$. Let C' be the column vector collecting the components $C^{\alpha'}$ of the translation $\overrightarrow{Q'Q}$ in the new basis. The set of all affine transformations $a = (C', P)$ is the affine group $\boldsymbol{A}(n)$. The transformation law for the affine components V^β of the point V is

$$V^{\alpha'} = C^{\alpha'} + (P^{-1})^{\alpha'}_{\beta}\, V^\beta \tag{2.1}$$

Affine functions

Any affine mapping ψ from $A_{\mathcal{T}}$ into $\boldsymbol{R}$ is called an affine function of $A_{\mathcal{T}}$. The affine mapping ψ is represented in an affine frame r by

$$\psi(V) = \chi + \varPhi_\alpha V^\alpha$$

where $\chi = \psi(Q)$ and $\varPhi_\alpha$ are the components, in the co-basis $(\overleftarrow{e}^\alpha)$, of the unique covector $\overleftarrow{\varPhi}$ associated to ψ. We will call $(\varPhi_\alpha, \chi)$ the affine components of ψ. After a change of the affine frame, they are given by

$$\varPhi_{\alpha'} = \varPhi_\beta P^\beta_{\alpha'} \qquad\qquad \chi' = \chi - \varPhi_\beta P^\beta_{\alpha'} C^{\alpha'}$$

as it can be easily verified. The set $A^*_{\mathcal{T}}$ of such functions is a linear space of the dimension $(n+1)$.

Vector-valued torsors

Let then $\mathcal{R}$ be a linear space of the dimension $p \leqslant n$. We call the torsor any bilinear skew-symmetric mapping $(\psi, \widehat{\psi}) \mapsto \vec{\mu}(\psi, \widehat{\psi}) \in \mathcal{R}$ from $A^*_{\mathcal{T}} \times A^*_{\mathcal{T}}$ into $\mathcal{R}$. Following Souriau (1992) all the tensorial indices related to $\mathcal{R}$ will be located at the left hand of the tensor. With respect to an affine frame $r = (V_0, (\vec{e}_\alpha))$ of $A_{\mathcal{T}}$ and a basis $({}_\rho\vec{\eta})$ of $\mathcal{R}$, the torsor is represented by

$$\vec{V} = \vec{\mu}(\psi, \widehat{\psi}) = {}^\gamma\mu(\psi, \widehat{\psi})\; {}_\gamma\vec{\eta} = \Big({}^\gamma J^{\alpha\beta}\, \varPhi_\alpha\, \widehat{\varPhi}_\beta + {}^\gamma T^\alpha (\chi \widehat{\varPhi}_\alpha - \widehat{\chi}\varPhi_\alpha)\Big)\, {}_\gamma\vec{\eta}$$

with ${}^{\gamma}J^{\alpha\beta} = -{}^{\gamma}J^{\beta\alpha}$. Let $r' = (V_0', (\vec{e}_{\alpha'}))$ be a new frame of $A_{\mathcal{T}}$ and $\overrightarrow{{}_{\gamma'}\eta} = {}^{\rho}_{\gamma'}Q \, \overrightarrow{{}_{\rho}\eta}$ be a new basis of $\mathcal{R}$. The corresponding transformation law is found to be

$$
\begin{aligned}
{}^{\gamma'}T^{\alpha'} &= {}^{\gamma'}_{\rho}(Q^{-1}) \, (P^{-1})^{\alpha'}_{\mu} \, {}^{\rho}T^{\mu} \\
{}^{\gamma'}J^{\alpha'\beta'} &= \Big((P^{-1})^{\alpha'}_{\mu} \, (P^{-1})^{\beta'}_{\nu} \, {}^{\rho}J^{\mu\nu} + C^{\alpha'} ((P^{-1})^{\beta'}_{\mu} \, {}^{\rho}T^{\mu}) - \\
&\quad - ((P^{-1})^{\alpha'}_{\mu} \, {}^{\rho}T^{\mu}) \, C^{\beta'} \Big) \, {}^{\gamma'}_{\rho}(Q^{-1})
\end{aligned}
\tag{2.2}
$$

Proper frames and intrinsic torsors

An affine frame will be called a *proper frame* if the zero vector of $\mathcal{T}$ is taken as the origin of the tangent affine space $A_{\mathcal{T}} : \ V_0 = 0$. Any change of proper frames is a linear transformation (no translation $C' = 0$). Restricting the analysis to linear transformations, we define the class of *intrinsic torsors*. For any intrinsic torsor $\overrightarrow{\mu}_0$, transformation law (2.2) degenerates into

$$
\begin{aligned}
{}^{\gamma'}T^{\alpha'} &= {}^{\gamma'}_{\rho}(Q^{-1}) \, (P^{-1})^{\alpha'}_{\mu} \, {}^{\rho}T^{\mu} \\
{}^{\gamma'}J^{\alpha'\beta'} &= {}^{\gamma'}_{\rho}(Q^{-1}) \, (P^{-1})^{\alpha'}_{\mu} \, (P^{-1})^{\beta'}_{\nu} \, {}^{\rho}J^{\mu\nu}
\end{aligned}
$$

The components ${}^{\gamma}T^{\alpha}$ are clearly components of a linear vector-valued tensor given by a linear mapping from $\mathcal{T}^*$ into $\mathcal{R}$ that we call the *linear momentum*. The components ${}^{\gamma}J^{\alpha\beta}$ of the intrinsic torsor can be interpreted as the components of a linear vector-valued tensor given by a bilinear mapping from $\mathcal{T}^* \times \mathcal{T}^*$ into $\mathcal{R}$ that we call the *intrinsic angular momentum* or *spin* (Misner *et al.*, 1973). The intrinsic tensor being given, the affine components of $\overrightarrow{\mu}$ in any other affine frame $r' = (V_0', (\vec{e}_{\alpha'}))$ are deduced from general transformation law (2.2). Thus, ${}^{\gamma'}J^{\alpha'\beta'}$ is obtained as the sum of the component of the spin and of the additional term

$$
{}^{\gamma'}J_C^{\alpha'\beta'} = \Big(C^{\alpha'} ((P^{-1})^{\beta'}_{\mu} \, {}^{\rho}T^{\mu}) - ((P^{-1})^{\alpha'}_{\mu} \, {}^{\rho}T^{\mu}) \, C^{\beta'} \Big)^{\gamma'}_{\rho} (Q^{-1})
$$

called the *orbital angular momentum* (Misner *et al.*, 1973). In conclusion, there is a one-to-one correspondence $\overrightarrow{\mu}_0 \mapsto \overrightarrow{\mu}$ between the intrinsic torsors and torsors.

3. Affine connection

Affine tangent space

While the difference of the components of two points of an affine space is defined without ambiguousness, the difference of the coordinates of two points in a manifold *has no meaning* in general. To get round this difficulty, the key idea is to consider that $A_{\mathcal{T}}$ is the affine space associated to the tangent linear space $T_X\mathcal{M}$, denoted $AT_X\mathcal{M}$ and called the *affine tangent space* at X. As observed by Cartan (1923): *"The affine space at point* $\boldsymbol{m}$ *could be seen as the manifold itself that would be perceived in an affine manner by an observer located at* $\boldsymbol{m}$*"*.

Linear connection as sliding

Let X be a point of a manifold $\mathcal{M}$ and $X' = X + dX$ be another point in the vicinity of X. Let us denote the zero vector at X as 0 and at X' as $0'$. For constructing a linear connection, we need to compare the tangent linear spaces at X and X' by a suitable identification. A linear space is the corresponding affine space with the zero vector as the particular origin Q. Hence, a linear connection is obtained by a smooth *sliding* on the manifold of the origin $Q = 0$ of $AT_X\mathcal{M}$ onto the origin $Q' = 0'$ of $AT_{X'}\mathcal{M}$, as depicted in Figure 1. Infinitesimal motions of the basis are specified through the connection $\nabla\, \vec{e}_\alpha = \omega^\beta_\alpha\, \vec{e}_\beta$.

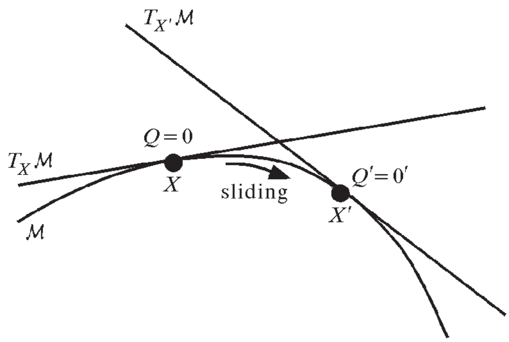

Fig. 1. Linear connecting (sliding)

Affine connection as rolling

On the other hand, let us consider a *rolling* of the affine tangent space $AT_X\mathcal{M}$ and let us work on proper frames $(Q = 0,\, Q' = 0')$. The identification

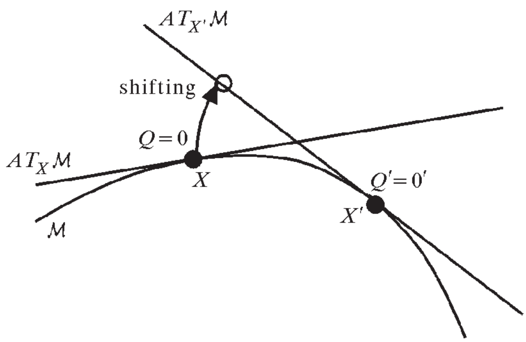

Fig. 2. Affine connection (rolling with initial origin at 0)

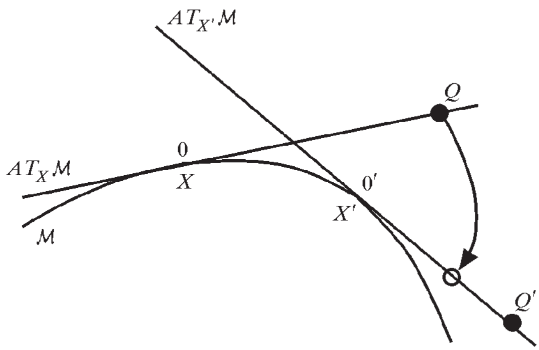

Fig. 3. Affine connection (rolling with arbitrary origin)

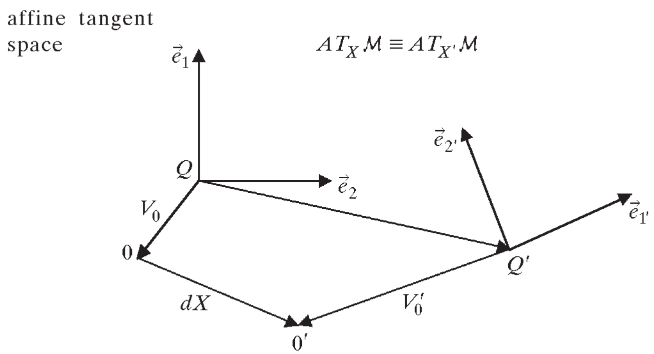

Fig. 4. Calculation of the affine connection by identifying two neighboring affine tangent spaces

with the neighboring affine tangent space $AT_{X'}\mathcal{M}$ shifts the origin $Q = 0$ onto a distinct point from $Q' = 0'$, as shown in Figure 2. Working more generally in arbitrary affine frames, an affine connection is constructed by rolling of $AT_X\mathcal{M}$ with the origin Q shifted onto a distinct point from the origin Q' of $AT_{X'}\mathcal{M}$, according to Figure 3. We define the affine connection ω_C^α as the components of the infinitesimal displacement $\overrightarrow{dQ} = \overrightarrow{QQ'}$ of the origin when identifying both neighboring affine tangent spaces (Figure 4). Because of rolling, it holds

$$\overrightarrow{00'} = dX^\alpha \, \vec{e}_\alpha$$

Hence, the displacement is decomposed as follows

$$\begin{aligned}\omega_C^\alpha \, \vec{e}_\alpha &= \overrightarrow{QQ'} = \overrightarrow{Q0} + \overrightarrow{00'} + \overrightarrow{0'Q'} = (V_0^\alpha + dX^\alpha) \, \vec{e}_\alpha - V_0^{\alpha'} \, \vec{e}_{\alpha'} = \\ &= dX^\alpha \, \vec{e}_\alpha - \nabla(V_0^\alpha \, \vec{e}_\alpha)\end{aligned}$$

that gives

$$\omega_C^\alpha = dX^\alpha - \nabla V_0^\alpha \tag{3.1}$$

In short, the affine connection provides a smooth variation of the moving affine frames

$$X \mapsto r(X) = \big(V_0(X), (\vec{e}_\alpha(X))\big)$$

The usual connection matrix ω_β^α gives a smooth variation of the basis, while the ω_C^α specify the motion of the affine space origin. The affine connections are due to Cartan (1923). In the present section, the key ideas are explained for readers interested in the mechanical science but who are not necessarily aware of advanced concepts of the differential geometry. A presentation using the geometry of principal bundles can be found in (de Saxcé, 2002) but, in any case, the final result is the same, whether it is obtained by the principal bundle theory or as before.

Affine covariant derivative

On this ground, we are able to calculate the intrinsic covariant derivative of the affine tensors of any type, that we call the *affine covariant derivative* and denote $\widetilde{\nabla}$ (in opposition to the usual covariant derivative ∇ which should be called the linear covariant derivative). First of all, let us consider a field of the tangent vector. As a member of the linear space $T_X\mathcal{M}$, it has a linear covariant derivative, but as a member of the associated affine space $AT_X\mathcal{M}$, it has also an affine covariant derivative denoted $\widetilde{\nabla}V$. By the choice of an affine frame $r = (V_0, (\vec{e}_\alpha))$, any point V of $AT_X\mathcal{M}$ can be identified to the vector $\overrightarrow{QV} = V^\alpha\,\vec{e}_\alpha$. Thus, its linear covariant derivative is

$$\nabla\overrightarrow{V} = \overrightarrow{Q'V'} - \overrightarrow{QV} = \overrightarrow{Q'Q} + \overrightarrow{QV'} - \overrightarrow{QV}$$

The difference between the two last terms represents the infinitesimal variation of the field V as point of the affine space, i.e. its affine covariant derivative. Hence, we obtain the relation

$$\nabla\overrightarrow{V} = (-\omega_C^\alpha + \tilde{\nabla}V^\alpha)\,\vec{e}_\alpha$$

from which one we deduce

$$\tilde{\nabla}V^\alpha = \nabla V^\alpha + \omega_C^\alpha \tag{3.2}$$

Next, we calculate the affine covariant derivative of the affine functions. According to the rule of differentiating a product, one has

$$\tilde{\nabla}(\psi(V)) = \tilde{\nabla}(\chi + \Phi_\alpha V^\alpha) = \tilde{\nabla}\chi + (\tilde{\nabla}\Phi_\alpha)V^\alpha + \Phi_\alpha(\tilde{\nabla}V^\alpha)$$

As the components Φ_α represent a linear object, the covector $\overleftarrow{\Phi}$ associated to ψ, its affine derivative is just its linear one. Owing to equation (3.2), it holds

$$\tilde{\nabla}(\psi(V)) = \tilde{\nabla}\chi + (\nabla\Phi_\alpha)V^\alpha + \Phi_\alpha(\nabla V^\alpha + \omega_C^\alpha)$$

Hence, one has

$$\tilde{\nabla}(\psi(V)) = \tilde{\nabla}\chi + \nabla(\Phi_\alpha V^\alpha) + \Phi_\alpha\omega_C^\alpha = \tilde{\nabla}\chi + \nabla(\psi(V)) - \nabla\chi + \Phi_\alpha\omega_C^\alpha$$

On the other hand, for any field V, we have to satisfy

$$\tilde{\nabla}(\psi(V)) = d(\psi(V)) = \nabla(\psi(V))$$

Finally, the affine covariant derivatives of the components of ψ are given by

$$\tilde{\nabla}\Phi_\alpha = \nabla\Phi_\alpha \qquad\qquad \tilde{\nabla}\chi = \nabla\chi - \Phi_\alpha\omega_C^\alpha \tag{3.3}$$

4. Affine covariant divergence of vector-valued torsors

Thin body

Let $\mathcal{M}$ be a manifold of the dimension n representing the physical space in statics $(n = 3)$ and the space-time in dynamics $(n = 4)$. The mapping

$\mathcal{N} \to \mathcal{M} : \ \xi \mapsto X = f(\xi)$ defines a sub-manifold of the dimension p enable one to represent three-dimensional bodies $(p = n)$ or thin ones $(p < n)$. In the sequel, $\mathcal{R}$ will be the tangent space $T_\xi\mathcal{N}$ at ξ, while $A_\mathcal{T}$ will be the *affine tangent space* $AT_X\mathcal{M}$, that is the tangent space $T_X\mathcal{M}$ at $X = f(\xi)$ endowed with the structure of the affine space. By a choice of the coordinate systems (X^α) on $\mathcal{M}$ and (ξ^β) on $\mathcal{N}$, the tangent mapping to f is given by

$$ {}_\beta U^\alpha = \frac{\partial X^\alpha}{\partial \xi^\beta} \tag{4.1}$$

Linear covariant divergence

It is assumed that $\mathcal{N}$ is equipped with a symmetric connection ${}^{\alpha}_{\beta}\omega = {}^{\alpha}_{\rho\beta}\gamma d\xi^\rho$, where ${}^{\alpha}_{\rho\beta}\gamma = {}^{\alpha}_{\beta\rho}\gamma$ are Christoffel's connection coefficients. The covariant derivative of the tangent vector field $\overrightarrow{V} = {}^\gamma V \, {}_\gamma\overrightarrow{\eta}$ on $\mathcal{N}$ is given by

$$\nabla\, {}^\gamma V = d\xi^\beta \ {}_\beta\nabla\, {}^\gamma V \qquad\qquad {}_\beta\nabla\, {}^\gamma V = \frac{\partial^\gamma V}{\partial \xi^\beta} + {}^{\gamma}_{\beta\rho}\gamma\ {}^\rho V \tag{4.2}$$

The manifold $\mathcal{M}$ is equipped with a symmetric connection

$$\omega^\alpha_\beta = \Gamma^\alpha_{\rho\beta}\, dX^\beta \tag{4.3}$$

using Christoffel's connection coefficients $\Gamma^\alpha_{\rho\beta} = \Gamma^\alpha_{\beta\rho}$. The covariant derivative of any covector field $\overleftarrow{\Phi} = \Phi_\alpha \overleftarrow{e}^{\,\alpha}$ on $\mathcal{M}$ is

$$\nabla\Phi_\alpha = dX^\beta\ \nabla_\beta\Phi_\alpha \qquad\qquad \nabla_\beta\Phi_\alpha = \frac{\partial\Phi_\alpha}{\partial X^\beta} - \Gamma^\rho_{\beta\alpha}\,\Phi_\rho$$

Considering the restriction to the sub-manifold $\mathcal{N}$, it holds

$$ {}_\gamma\nabla\Phi_\alpha = \frac{\partial X^\beta}{\partial\xi^\gamma}\nabla_\beta\Phi_\alpha = \frac{\partial\Phi_\alpha}{\partial\xi^\gamma} - {}_\gamma U^\beta\ \Gamma^\rho_{\beta\alpha}\Phi_\rho \tag{4.4}$$

Now, we consider a tensor field $\xi \mapsto T(\xi)$ of the components ${}^\gamma T^\alpha$ as defined before. We hope to calculate its linear covariant derivative, namely ${}_\gamma\nabla\, {}^\gamma T^\alpha$. According to the rule of differentiating a product, one has for any covector field on $\mathcal{M}$ of the components Φ_α

$$ {}_\gamma\nabla({}^\gamma T^\alpha\Phi_\alpha) = ({}_\gamma\nabla\, {}^\gamma T^\alpha)\Phi_\alpha + {}^\gamma T^\alpha({}_\gamma\nabla\Phi_\alpha)$$

The left hand member, representing the divergence of a vector field on $\mathcal{N}$, can be developed using (4.2), while, in the right hand member, the last term is transformed owing to (4.4). After simplification, it remains

$$\frac{\partial^\gamma T^\alpha}{\partial\xi^\gamma}\Phi_\alpha + {}^{\gamma}_{\gamma\rho}\gamma\ {}^\rho T^\alpha\ \Phi_\alpha = ({}_\gamma\nabla\, {}^\gamma T^\alpha)\Phi_\alpha - {}^\gamma T^\alpha\ {}_\gamma U^\beta\ \Gamma^\rho_{\beta\alpha}\,\Phi_\rho$$

In the last term, replacing α, β, ρ in turn by β, ρ, α, we obtain

$$({}_{\gamma}\nabla\ {}^{\gamma}T^{\alpha})\Phi_{\alpha} = \Big(\frac{\partial^{\gamma}T^{\alpha}}{\partial\xi^{\gamma}} + {}^{\gamma}_{\gamma\rho}\gamma\ {}^{\rho}T^{\alpha} + {}^{\gamma}T^{\beta}\ {}_{\gamma}U^{\rho}\ \Gamma^{\alpha}_{\rho\beta}\Big)\Phi_{\alpha}$$

With the covector field Φ_{α} being arbitrary, the previous relation is satisfied if and only if

$$_{\gamma}\nabla\ {}^{\gamma}T^{\alpha} = \frac{\partial^{\gamma}T^{\alpha}}{\partial\xi^{\gamma}} + {}^{\gamma}_{\gamma\rho}\gamma\ {}^{\rho}T^{\alpha} + {}^{\gamma}T^{\beta}\ {}_{\gamma}U^{\rho}\ \Gamma^{\alpha}_{\rho\beta} \tag{4.5}$$

This formula allows one to calculate the linear divergence of this class of tensors.

Affine covariant divergence

Now, we are able to calculate the affine covariant derivative of a vector-valued torsor. For any covector field $\overleftarrow{F} = {}_{\gamma}F\ {}^{\gamma}\overleftarrow{\eta}$ on $\mathcal{N}$, we have

$$\widetilde{\nabla}\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \widetilde{\nabla}\Big({}_{\gamma}F\ {}^{\gamma}\mu(\psi,\widehat{\psi})\Big) = \widetilde{\nabla}\Big(({}^{\gamma}J^{\alpha\beta}\ \Phi_{\alpha}\ \widehat{\Phi}_{\beta} + {}^{\gamma}T^{\alpha}(\chi\widehat{\Phi}_{\alpha} - \widehat{\chi}\Phi_{\alpha}))_{\gamma}F\Big)$$

As the affine derivatives of the components ${}_{\gamma}F$, Φ_{α}, $\widehat{\Phi}_{\beta}$ and ${}^{\gamma}T^{\alpha}$ representing linear objects are equal to their linear derivatives, it holds, according to the rule of differentiating products

$$\begin{aligned}\widetilde{\nabla}\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \Big[&(\widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta})\Phi_{\alpha}\widehat{\Phi}_{\beta} + {}^{\gamma}J^{\alpha\beta}\ \nabla(\Phi_{\alpha}\widehat{\Phi}_{\beta}) + (\nabla\ {}^{\gamma}T^{\alpha})(\chi\widehat{\Phi}_{\alpha} - \widehat{\chi}\Phi_{\alpha}) + \\ &+ {}^{\gamma}T^{\alpha}\Big((\widetilde{\nabla}\chi)\widehat{\Phi}_{\alpha} - (\widetilde{\nabla}\widehat{\chi})\Phi_{\alpha}\Big) + {}^{\gamma}T^{\alpha}\Big(\chi(\nabla\widehat{\Phi}_{\alpha}) - \widehat{\chi}(\nabla\Phi_{\alpha})\Big)\Big]\ {}_{\gamma}F + {}^{\gamma}\mu(\psi,\widehat{\psi})\nabla\ {}_{\gamma}F\end{aligned}$$

Taking into account expression (3.3) of the derivative of the affine functions, we obtain after some rearrangements

$$\begin{aligned}\widetilde{\nabla}\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \Big[&(\widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta})\Phi_{\alpha}\widehat{\Phi}_{\beta} + {}^{\gamma}J^{\alpha\beta}\ \nabla(\Phi_{\alpha}\widehat{\Phi}_{\beta}) + \\ &+ \nabla\Big({}^{\gamma}T^{\alpha}(\chi\widehat{\Phi}_{\alpha} - \widehat{\chi}\Phi_{\alpha})\Big) + ({}^{\gamma}T^{\alpha}\ \omega^{\beta}_{C} - \omega^{\alpha}_{C}\ {}^{\gamma}T^{\beta})\Big]\ {}_{\gamma}F + {}^{\gamma}\mu(\psi,\widehat{\psi})\nabla\ {}_{\gamma}F\end{aligned}$$

which can be simplified as follows

$$\begin{aligned}\widetilde{\nabla}\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \Big[&(\widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta})\Phi_{\alpha}\widehat{\Phi}_{\beta} + \nabla({}^{\gamma}\mu(\psi,\widehat{\psi})) - (\nabla\ {}^{\gamma}J^{\alpha\beta})\Phi_{\alpha}\widehat{\Phi}_{\beta} + \\ &+ ({}^{\gamma}T^{\alpha}\ \omega^{\beta}_{C} - \omega^{\alpha}_{C}\ {}^{\gamma}T^{\beta})\Phi_{\alpha}\widehat{\Phi}_{\beta}\Big]\ {}_{\gamma}F + {}^{\gamma}\mu(\psi,\widehat{\psi})\nabla\ {}_{\gamma}F\end{aligned}$$

On the other hand, because the value of $\overleftarrow{F}$ for $\overrightarrow{\mu}$ is a scalar field, we have

$$\widetilde{\nabla}\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = d\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \nabla\Big(\overleftarrow{F}(\overrightarrow{\mu}(\psi,\widehat{\psi}))\Big) = \\ = \nabla({}^{\gamma}\mu(\psi,\widehat{\psi}))\ {}_{\gamma}F + {}^{\gamma}\mu(\psi,\widehat{\psi})\nabla\ {}_{\gamma}F$$

Hence, for any ψ, $\widehat{\psi}$ and $\overleftarrow{F}$, it holds

$$\Big(\widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta} - \nabla\ {}^{\gamma}J^{\alpha\beta} - \omega^{\alpha}_{C}\ {}^{\gamma}T^{\beta} + {}^{\gamma}T^{\alpha}\ \omega^{\beta}_{C}\Big)\ {}_{\gamma}F\Phi_{\alpha}\ \widehat{\Phi}_{\beta} = 0$$

With the affine functions and covectors being arbitrary, we obtain an expression of the affine covariant derivative of the vector-valued torsors

$$\widetilde{\nabla}\ {}^{\gamma}T^{\alpha} = \nabla\ {}^{\gamma}T^{\alpha} \qquad\qquad \widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta} = \nabla\ {}^{\gamma}J^{\alpha\beta} + \omega^{\alpha}_{C}\ {}^{\gamma}T^{\beta} - {}^{\gamma}T^{\alpha}\ \omega^{\beta}_{C} \tag{4.6}$$

By analogy with (4.3), we introduce the *affine connection coefficients* $\Gamma^{\alpha}_{\rho C}$ such that

$$\omega^{\alpha}_{C} = \Gamma^{\alpha}_{\rho C}\ dX^{\rho} = \Gamma^{\alpha}_{\rho C}\ {}_{\gamma}U^{\rho}\ d\xi^{\gamma}$$

Owing to (3.1), one has

$$\Gamma^{\alpha}_{\rho C} = \delta^{\alpha}_{\rho} - \frac{\partial V^{\alpha}_{0}}{\partial X^{\rho}} - \Gamma^{\alpha}_{\rho\beta}V^{\beta}_{0} \tag{4.7}$$

As a particular case of (4.6), we obtain the affine covariant divergence of a vector-valued torsor

$$\begin{aligned} {}_{\gamma}\widetilde{\nabla}\ {}^{\gamma}T^{\alpha} &= {}_{\gamma}\nabla\ {}^{\gamma}T^{\alpha} \\ {}_{\gamma}\widetilde{\nabla}\ {}^{\gamma}J^{\alpha\beta} &= {}_{\gamma}\nabla\ {}^{\gamma}J^{\alpha\beta} + {}_{\gamma}U^{\rho}\ \Gamma^{\alpha}_{\rho C}\ {}^{\gamma}T^{\beta} - {}^{\gamma}T^{\alpha}\ {}_{\gamma}U^{\rho}\ \Gamma^{\beta}_{\rho C} \end{aligned} \tag{4.8}$$

where the divergence of ${}^{\gamma}T^{\alpha}$ is given by (4.5) and

$${}_{\gamma}\nabla\ {}^{\gamma}J^{\alpha\beta} = \frac{\partial^{\gamma}J^{\alpha\beta}}{\partial\xi^{\gamma}} + {}^{\gamma}J^{\rho\beta}\ {}_{\gamma}U^{\mu}\ \Gamma^{\alpha}_{\mu\rho} + {}^{\gamma}J^{\alpha\rho}\ {}_{\gamma}U^{\mu}\ \Gamma^{\beta}_{\mu\rho} + {}^{\gamma}_{\gamma\rho}\gamma\ {}^{\rho}J^{\alpha\beta} \tag{4.9}$$

which can be easily obtained by reasoning as for the proof of (4.5).

5. Dynamics of three-dimensional bodies

Three-dimensional bodies

Let a continuous medium (a solid or a fluid) occupying an open domain $\Omega \subset \boldsymbol{R}^3$ that we call also a body. In order to model its evolution between the instants t_0 and t_1, we consider the sub-manifold of the space-time

$f :]t_0, t_1[\times \Omega \to \mathcal{M}$. For convenience, we choose *the same coordinate system on* $\mathcal{N}$ *and* $\mathcal{M}$. Hence, the local expression of f is the identity mapping $X^\alpha = \xi^\alpha$. The distinction between the left and right hand indices becomes irrelevant and, in the present section, we put all the indices at the right hand as usual. Thus, we have ${}_\beta U^\alpha = \delta^\alpha_\beta$ and ${}_\gamma\widetilde{\nabla} = \widetilde{\nabla}_\gamma$. Moreover, we write ${}^\gamma T^\alpha = T^{\alpha\gamma}$ and ${}^\gamma J^{\alpha\beta} = J^{\alpha\beta\gamma}$. The behavior of the continuous medium is described by a torsor field $X \mapsto \overrightarrow{\mu}(X)$. We claim that the balance (or conservation law) of momentum of the continuous medium says that *the torsor field is affine covariant divergence free*

$$\widetilde{\nabla}_\gamma T^{\alpha\gamma} = 0 \qquad\qquad \widetilde{\nabla}_\gamma J^{\alpha\beta\gamma} = 0 \tag{5.1}$$

Following Souriau (1992, 1997a), the first equation traduces the balance of linear momentum. The second one is a full covariant version of the balance of angular momentum as presented by Misner *et al.* (1973, p. 156).

Balance of the angular momentum

Three-dimensional continuous media are mainly considered in the literature as *non polarized* media, according to Cauchy's famous theory.

Let $(T^{\alpha\gamma}, J^{\alpha\beta\gamma})$ be the affine components of the unique intrinsic torsor field $X \mapsto \overrightarrow{\mu}_0(X)$, associated to $X \mapsto \overrightarrow{\mu}(X)$, in a moving proper frame $X \mapsto r(X) = (0, (\overrightarrow{e}_\alpha(X))$. We claim that the intrinsic torsor is *spin free* $J^{\alpha\beta\gamma} = 0$. Thus, accounting for (4.7), $(4.8)_2$, the balance of angular momentum $(5.1)_2$ leads to

$$\widetilde{\nabla}_\gamma J^{\alpha\beta\gamma} = T^{\beta\alpha} - T^{\alpha\beta} = 0$$

In this symmetry condition of the linear momentum, the reader can clearly recognize the classical hypothesis of Cauchy's media.

Galilean tensors

All what has been said so far may be applied as much for the general relativity theory as for the classical mechanics. Henceforth, we shall restrict the analysis to the latter theory. In the sequel, Greek indices are 0 to 3 while Latin ones run from 1 to 3 (associated to the space coordinates only). Any point X of the space-time $\mathcal{M}$ represents an event occurring at position r and time t. With an appropriate coordinate system, it is represented by $X^i = r^i$, $X^0 = t$. Let us consider Galileo's group, a subgroup of the affine group $\boldsymbol{A}(4)$, collecting the Galilean transformations, that is the affine transformations $a = (C', P)$ such that (Souriau, 1997b)

$$C' = \begin{bmatrix} \tau \\ k \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 \\ u & R \end{bmatrix}$$

where $u \in \boldsymbol{R}^3$ is a Galilean boost, $R \in \boldsymbol{SO}(3)$ is a rotation, $k \in \boldsymbol{R}^3$ is a spatial translation and $\tau \in \boldsymbol{R}$ is a clock change. Any coordinate change representing a rigid body motion and a clock change

$$r' = (R(t))^\top (r - r_0(t)) \qquad t' = t + \tau_0$$

where $t \mapsto R(t) \in \boldsymbol{SO}(3)$ and $t \mapsto r_0(t) \in \boldsymbol{R}^3$ are smooth mappings and $\tau_0 \in \boldsymbol{R}$ is a constant, is called a *Galilean coordinate change*. Indeed, the corresponding Jacobean matrix is a linear Galilean transformation

$$P = \frac{\partial X'}{\partial X} = \begin{bmatrix} 1 & 0 \\ u & R \end{bmatrix}$$

where $u = \varpi(t) \times (r - r_0(t)) + \dot{r}_0(t)$, is the well-known *velocity of transport*. It involves Poisson's vector ϖ such that $\dot{R} = j(\varpi)R$, where $j(\varpi)$ (sometimes also denoted $ad(\varpi)$) is the skew-symmetric matrix representing the cross-product by $\varpi :\ \forall v \in \boldsymbol{R}^3,\ j(\varpi)v = \varpi \times v$.

Galilean connections

At each group of the transformation G a family of connections and the corresponding geometry (called the G-structure by Dieudonné (1971)) is associated. We call Galilean connections the symmetric connections associated to Galileo's group. In a Galilean coordinate system, they are given by

$$\omega = \begin{bmatrix} 0 & 0 \\ j(\Omega)dr - gdt & j(\Omega)dt \end{bmatrix} \tag{5.2}$$

where g is a column-vector collecting the $g^j = -\Gamma^j_{00}$ and identified to the gravity (Cartan, 1923), while Ω is a column-vector associated by the mapping j^{-1} to the skew-symmetric matrix the elements of which are $\Omega^i_j = \Gamma^i_{j0}$ and interpreted as Coriolis's effects (Souriau, 1997a).

Balance of the linear momentum

In the present sub-section, we shall follow the reasoning proposed by Souriau (1992, 1997a). Let be an Eulerian representation of the continuous medium in which any event is represented in a Galilean coordinate system by *Euler's coordinates* $X^i = r^i$ and $X^0 = t$. On the other hand, let be a Lagrangean representation in which the same event is represented by *Lagrange's coordinates* of the material particle $X^{i'} = s^{i'}$ and $X'^0 = t'$, which are not in general Galilean coordinates. They are related to the previous representation by a smooth coordinate change like

$$r^i = \varphi^i(s^{j'}, t') \qquad\qquad t = t'$$

We introduce the deformation gradient $F^i_{j'} = \partial r^i / \partial s^{j'}$, and the velocity $u^i = \partial r^i / \partial t$.

The particle of the coordinates $(s^{j'})$ being at rest in the Lagrangean representation has the trajectory such that $ds^{j'} = 0$. Differentiating, we obtain

$$dX = \begin{bmatrix} dt \\ dr \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u & F \end{bmatrix} \begin{bmatrix} dt' \\ ds \end{bmatrix} = P\, dX' \tag{5.3}$$

To adjust to usual convention in the continuum mechanics (compressive stresses are negative), we put $S^{i'j'} = -T^{i'j'}$. The components $S^{i'j'}$ are generally recognized as representing the internal forces or stresses in the Lagrangean representation, and are customarily called *symmetric Piola-Kirchhoff stresses*. The component $\rho = T'^{00}$ can be interpreted as the mass density. The particles being at rest in this particular representation, the components $T^{0i'} = T^{i'0}$, interpreted as the linear momentum, are supposed to vanish. In Euler's coordinates, it holds, owing to $(2.2)_1$

$$T = PT'P^\top = \begin{bmatrix} 1 & 0 \\ u & F \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} 1 & u^\top \\ 0 & F^\top \end{bmatrix} = \begin{bmatrix} \rho & \rho u^\top \\ \rho u & -\sigma + \rho u u^\top \end{bmatrix} \tag{5.4}$$

where $\sigma_k = \rho u u^\top$ collects kinetic stresses and, according to Simo (1988) $\sigma = FSF^\top$ collects *Cauchy's stresses*. In Euler's representation, the events being given by a Galilean coordinate system, the connection matrix is given by (5.2). Thus, accounting for $(4.8)_1$, equation $(5.1)_1$ can be interpreted as *Euler's equations* of the continuous medium

$$\frac{\partial}{\partial r^j}(\rho u^j) + \frac{\partial \rho}{\partial t} = 0$$

$$\rho\Big(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial r^j}\Big) = \frac{\partial \sigma^{ij}}{\partial r^j} + \rho(g^i - 2\Omega^i_j u^j)$$

6. Adapted coordinates

Moving surface

We want to model problems of the dynamics of shells and plates within the frame of the classical mechanics. We have to represent the time evolution of a smooth material surface.

Hence, we are in the case $n = 4$ and $p = 3 < n$. In the sequel, the indices a, b, c, e associated to the surface parameters takes only values 1 or 2, while the other Latin indices such that i, j, k are running from 1 to 3 as before. In the Eulerian representation, let us consider a *Galilean coordinate system* (X^α) on the space-time $\mathcal{M}$. Interpreting the last coordinate ξ^0 on the submanifold $\mathcal{N}$ as the time, let us suppose that the mapping $\mathcal{N} \to \mathcal{M} : \ \xi \mapsto X = f(\xi)$ is represented in local coordinates by given equations

$$X^i = r^i = p^i(\xi^\gamma) = p^i(\xi^a, \xi^0) = p^i(\xi^a, t) \qquad\qquad X^0 = t = \xi^0$$

Adapted coordinates

A classical tool of the theory of surfaces is the tangent plane to the current point. In order to separate in-plane and off-plane components of the torsor and the balance of momentum, we introduce another coordinates $(X^{\beta'})$ of the space-time. The new spatial coordinates are X'^a, denoted θ^a, and X'^3, denoted θ^3. The time coordinate X'^0, denoted t', is unchanged. The key-idea is to choose the new coordinates in such a way that the equation of the material surface at a given time is merely

$$\theta^3 = 0$$

In these adapted coordinates, the local representation $\xi \mapsto X'$ of the mapping f defining the sub-manifold $\mathcal{N}$ is

$$\theta^a = \xi^a \qquad\qquad \theta^3 = 0 \qquad\qquad t' = \xi^0 \tag{6.1}$$

The adapted coordinates are related to the previous ones through equations like

$$r^i = p^i(\theta^a, t) + \theta^3 n^i(\theta^a, t) \qquad\qquad t = t' \tag{6.2}$$

For convenience, following for instance (Naghdi, 1972), we choice n^i such that

$$\sum_{i=1}^{3} \pi_a^i n^i = 0 \qquad\qquad \sum_{i=1}^{3} n^i n^i = 1 \tag{6.3}$$

with

$$\pi_a^i = \frac{\partial p^i}{\partial \theta^a} \qquad v^i = \frac{\partial p^i}{\partial t} \qquad \beta_a^i = \frac{\partial n^i}{\partial \theta^a} \qquad w^i = \frac{\partial n^i}{\partial t}$$

Let n be a column vector of the components n^i and π (resp. β) be a matrix the element of which at the ath row and ith column is π_a^i (resp. β_a^i). As usual, n is interpreted as the unit vector normal to the material surface and π as the projector onto the tangent plane (to the material surface at the current time) (Valid, 1995). The uniqueness of n is ensured by conditions (6.3). The column vector v, collecting the components v^i, represents the velocity and the column vector w, collecting the components w^i, represents the time rate of the unit normal vector. Hence equations (6.3) read

$$\pi n = 0 \qquad n^\top n = 1 \tag{6.4}$$

Differentiating $(6.4)_2$ leads to $n^\top dn = 0$. Thus

$$\beta n = 0 \qquad n^\top w = 0 \tag{6.5}$$

Calculation of the connection matrix

Notice that the new coordinates $(X^{\beta'})$ are not generally Galilean. Hence the connection matrix ω' in this coordinate system does not have the standard form of (5.2). Differentiating (6.2) gives successively

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 & 0 \\ v + \theta^3 w & \pi^\top + \theta^3 \beta^\top & n \end{bmatrix} \\ dP &= \begin{bmatrix} 1 & 0 & 0 \\ dv + w d\theta^3 + \theta^3 dw & \pi^\top + \beta^\top d\theta^3 + \theta^3 d\beta^\top & dn \end{bmatrix} \end{aligned} \tag{6.6}$$

Putting $\theta^3 = 0$ on the material surface, leads to

$$P = \begin{bmatrix} 1 & 0 & 0 \\ v & \pi^\top & n \end{bmatrix} \qquad dP = \begin{bmatrix} 1 & 0 & 0 \\ dv + w d\theta^3 & \pi^\top + \beta^\top d\theta^3 & dn \end{bmatrix} \tag{6.7}$$

Owing to (6.4), the inverse transformation matrix is

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -v_t & a^{-1}\pi \\ -v^3 & n^\top \end{bmatrix} \tag{6.8}$$

where, the symmetric matrix $a = \pi\pi^\top$, represents the *first fundamental form* of the material surface, $v_t = a^{-1}\pi v$ is the in-plane velocity and $v^3 = n^\top v$ is the off-plane component of the velocity. Because of the classical transformation law

$$\omega' = P^{-1}\omega P + P^{-1}dP$$

and taking into account (5.2) and (6.7-8), the connection matrix in the adapted coordinate system is

$$\omega' = \begin{bmatrix} 0 & 0 & 0 \\ a^{-1}\pi A_1 & a^{-1}\pi(A_2 + \beta^\top d\theta^3) & a^{-1}\pi(dn + j(\Omega)ndt) \\ n^\top A_1 & n^\top A_2 & 0 \end{bmatrix} \tag{6.9}$$

where

$$\begin{aligned} A_1 &= j(\Omega)(dr + vdt) - gdt + dv + wd\theta^3 \\ A_2 &= d\pi^\top + j(\Omega)\pi^\top dt \end{aligned}$$

Its elements have to be expressed with respect to the differential of the adapted coordinates $d\theta^i$ and dt'. For convenience, we introduce the column vector $d\theta_t$ collecting $d\theta^a$, and we put $dt' = dt$. By introducing the linear operators

$$d_{\theta_t} = d\theta^a \frac{\partial}{\partial\theta^a} \qquad\qquad \frac{\mathfrak{d}}{\mathfrak{d}t} = I_3\frac{\partial}{\partial t} + j(\Omega)$$

where I_3 is the 3×3 identity matrix, we have

$$\begin{aligned} d\pi^\top + j(\Omega)\pi^\top dt &= d_{\theta_t}\pi^\top + \frac{\mathfrak{d}\pi^\top}{\mathfrak{d}t}dt \\ dn + j(\Omega)ndt &= d_{\theta_t}n + \frac{\mathfrak{d}n}{\mathfrak{d}t}dt \end{aligned}$$

Let g_p be a column vector representing the acceleration of transport

$$g_p = \frac{\partial v}{\partial t} = \frac{\partial^2 p}{\partial t^2} \qquad \text{and} \qquad \frac{\partial v}{\partial\theta_t} = \frac{\partial^2 p}{\partial\theta_t\partial t} = \frac{\partial\pi^\top}{\partial t}$$

Accounting for (6.5), it holds

$$\begin{aligned} &j(\Omega)(dr + vdt) - gdt + dv + wd\theta^3 = j(\Omega)(\pi^\top d\theta_t + nd\theta^3 + 2vdt) - gdt + \\ &+g_p dt + \frac{\partial\pi^\top}{\partial t}d\theta_t + \frac{\partial n}{\partial t}d\theta^3 = \frac{\mathfrak{d}\pi^\top}{\mathfrak{d}t}d\theta_t + \frac{\mathfrak{d}n}{\mathfrak{d}t}d\theta^3 - g^* dt \end{aligned}$$

where $g^* = g - 2\Omega \times v - g_p$, is the gravity in the adapted coordinate system, obtained by subtracting Coriolis's acceleration $g_c = 2\Omega \times v$ and the acceleration of transport g_p from the gravity in the Galilean coordinate system. Because of $(6.5)_2$, one has

$$n^\top \frac{\partial n}{\partial t} = n^\top w + n^\top j(\Omega) n = 0$$

The connection matrix (6.9) becomes

$$\omega' = \begin{bmatrix} 0 & 0 & 0 \\ a^{-1}\pi\left(A_3 + \frac{\partial n}{\partial t} d\theta^3\right) & a^{-1}\pi\left(A_4 + \beta^\top d\theta^3\right) & a^{-1}\pi\left(d_{\theta_t} n + \frac{\partial n}{\partial t} dt\right) \\ n^\top A_3 & n^\top A_4 & 0 \end{bmatrix} \tag{6.10}$$

where

$$A_3 = \frac{\partial \pi^\top}{\partial t} d\theta_t - g^* dt \qquad\qquad A_4 = d_{\theta_t} \pi^\top + \frac{\partial \pi^\top}{\partial t} dt$$

As we shall be working latter on in the adapted coordinate system, for sake of easiness, we cancel the prime symbol. It clearly results from (6.1) that

$$\begin{aligned} &{}_aU^b = \delta_a^b \quad {}_aU^3 = 0 \quad {}_0U^i = 0 \\ &{}_aU^0 = 0 \quad {}_0U^0 = 1 \end{aligned} \tag{6.11}$$

7. Shell variables

Balance of momentum

Let a continuous medium of an arbitrary dimension $p \leqslant n$, the behavior of which is described by a torsor field $\xi \mapsto \overrightarrow{\mu}(\xi)$ on $\mathcal{N}$. Generalizing the approach of Section 5, we claim that the balance of momentum says that this torsor field is *affine covariant divergence free*

$${}_\gamma\widetilde{\nabla}\,{}^\gamma T^\alpha = 0 \qquad\qquad {}_\gamma\widetilde{\nabla}\,{}^\gamma J^{\alpha\beta} = 0 \tag{7.1}$$

Now, we want to particularize this general principle to the shells.

Discussion

Any torsor has pn components ${}^{\gamma}T^{\alpha}$ and, owing to the skew-symmetry with respect to the right hand side indices, $pn(n-1)/2$ independent components ${}^{\gamma}J^{\alpha\beta}$. Then, it has $pn(n+1)/2$ affine components. On the other hand, a torsor field is subjected to $n(n+1)/2$ independent scalar equations (7.1). In the statics of shells $(p=2, n=3)$, only 6 equations are available to determine 12 variables. In the dynamics $(p=3, n=4)$, the difficulty is higher with 10 equations for 30 variables.

Fortunately, it is possible in the Galilean setting to reduce the redundancy of the shell by introducing additional hypothesis related to the modeling. A resisting material surface can be seen as an approximation of a three dimensional medium, as a consequence of the fact that it is thin in the normal direction to the middle surface. There is a broad variety of situations such as a smooth curved sheet or a composite laminate but also a smooth surface approximating a corrugated sheet, a lattice or a fluid moving between two close sheets and so on. Although the general modeling proposed as before is relevant to represent this wide range of situations, it would be a heavy task to examine every one of them. Hence, we only wish to illustrate our method by focussing the attention on the most simple case of an homogeneous curved thin sheet of the current thickness h.

The behavior of a three dimensional body is characterized by a spin free torsor field with components $T^{\alpha\gamma}$, as discussed in Section 5. We hope to build a shell torsor field with the affine components ${}^{\gamma}T^{\alpha}$ and ${}^{\gamma}J^{\alpha\beta}$ by a suitable integration over the thickness. The *shell variables* ${}^{\gamma}T^{\alpha}$, ${}^{\gamma}J^{\alpha\beta}$ are related to an infinitesimal surface element modeling a piece of the three dimensional body occupying the volume over and above the surface element in the thickness direction, and called the *shell element*.

Three dimensional continuum torsor

In a first draft, we adopt two usual hypotheses. On the shell element scale, the surface curvature is neglected and the strain is small (but not necessarily the displacements and rotations). A more sophisticated method using the concept of the affine transport is proposed in (de Saxcé, 2002a), but we shall not follow this way in the present work. According to the approach of Section 5, let $X^{i'} = s^{i'}$ be Lagrange's coordinates of the material particles of the three dimensional continuum and $X'^{0} = t'$. Neglecting the strains, the motion of the shell element can be locally approximated by a rigid motion

$$r = \varphi(s,t) = p(t) + R(t)s \qquad\qquad t = t'$$

where $p(t) \in \boldsymbol{R}^3$ represents the position of the point on the middle surface (in short, the *middle point*) and the time dependent rotation matrix $R(t) \in \boldsymbol{SO}(3)$ describes the rigid motion of the shell element around this point. Relation (5.3) degenerates into

$$dX = \begin{bmatrix} dt \\ dr \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u & R \end{bmatrix} \begin{bmatrix} dt' \\ ds \end{bmatrix} = P dX'$$

involving the velocity of transport of the middle point $v = \dot{p}$ and of any point of the shell element

$$u = v + \varpi \times (r - p)$$

where ϖ is Poisson's vector $\dot{R} = j(\varpi) R$. Taking into account (6.2), it holds

$$u = v + \theta^3 \varpi \times n \tag{7.2}$$

On the other hand, (6.6) says that

$$u = v + \theta^3 w \tag{7.3}$$

Identifying (7.2) to (7.3), leads to

$$w = \frac{\partial n}{\partial t} = \varpi \times n \tag{7.4}$$

In the Eulerian representation, the torsor components $T^{\alpha\gamma}$ are given by (5.4). On the considered scale, the surface curvature effects can be neglected. The inverse transformation matrix is approximated by its value (6.8) on the middle surface. Thus, in the adapted coordinates, the new components $T^{\alpha'\gamma'}$ are given by

$$T' = P^{-1} T P^{-\top} = \begin{bmatrix} 1 & 0 \\ -v_t & a^{-1}\pi \\ -v^3 & n^\top \end{bmatrix} \begin{bmatrix} \rho & \rho u^\top \\ \rho u & -\sigma + \rho u u^\top \end{bmatrix} \begin{bmatrix} 1 & v_t^\top & -v^3 \\ 0 & (a^{-1}\pi)^\top & n \end{bmatrix}$$

Accounting for (6.5) and (6.2)-(6.3), it holds

$$T' = \begin{bmatrix} \rho & \rho\theta^3 (a^{-1}\pi w)^\top & 0 \\ \rho\theta^3 a^{-1}\pi w & a^{-1}\pi\big(-\sigma + \rho(\theta^3)^2 w w^\top\big)(a^{-1}\pi)^\top & -a^{-1}\pi\sigma n \\ 0 & -n^\top \sigma (a^{-1}\pi)^\top & -n^\top \sigma n \end{bmatrix}$$

Therefore, in the adapted coordinates, the stress components are

$$\sigma'^{ab} = \sum_{i,j=1}^{3} a^{ae}\, \pi_e^i\, \sigma^{ij}\, \pi_c^j\, a^{cb} \qquad\qquad \sigma'^{33} = \sum_{i,j=1}^{3} n^i\, \sigma^{ij}\, n^j$$

$$\sigma'^{b3} = \sigma'^{3b} = \sum_{i,j=1}^{3} a^{ae}\, \pi_e^i\, \sigma^{ij}\, n^j$$

while the new velocity components are

$$w'^a = \sum_{i=1}^{3} a^{ae}\, \pi_e^i\, w^i$$

Canceling the prime symbol for sake of easiness, we obtain the torsor components of the three dimensional medium in the adapted coordinates

$$\begin{aligned} &T^{ab} = -\sigma^{ab} + \rho(\theta^3)^2 w^a w^b \quad && T^{a0} = T^{0a} = \rho\theta^3 w^a && \\ &T^{a3} = T^{3a} = -\sigma^{a3} && T^{30} = T^{03} = 0 && T^{00} = \rho \end{aligned} \tag{7.5}$$

This result has been obtained according to the two classical previous hypotheses. They could be eliminated by a more pervasive analysis using for instance tools developed in (Hamdouni *et al.*, 1999) by considering a shell as a stacking-up of curve sheets.

Integration over the thickness

If the torsor components are calculated with respect to the current point r of the three dimensional medium in a proper frame, the spin components vanish.

It is recalled that every point X of the manifold $\mathcal{M}$ corresponds to an event occurring at a given position and time. If we neglect once again the curvature effects on the shell element scale, the manifold $\mathcal{M}$ can be approximated by the affine tangent space $AT_X\mathcal{M}$ at the current point X. Naturally, we are working with the basis $(\vec{e}_\alpha)$ associated to the considered adapted coordinate system. The current point X of coordinates $(X^0 = t, X^a = \theta^a, X^3 = \theta^3)$ is identified to the origin of the proper frame, that is zero of the tangent linear space $T_X\mathcal{M}$. Concerning the position of the middle point at the same time, the point X' of the coordinates $(X^0 = t, X^a = \theta^a, X^3 = 0)$ is represented by the point of the affine tangent space $AT_X\mathcal{M}$ with the affine coordinates

$$V = \begin{bmatrix} t \\ \theta^1 \\ \theta^2 \\ 0 \end{bmatrix} - \begin{bmatrix} t \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\theta^3 \end{bmatrix}$$

Let us take this point as a new origin. In the non-proper frame $r' = (V, (\vec{e}_\alpha))$, the considered point has vanishing coordinates $V' = 0$. According to transformation law (2.1), we consider the affine transformation $a = (C', P)$ with P being the identity of $\boldsymbol{R}^4$, and

$$C' = V' - V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta^3 \end{bmatrix} \tag{7.6}$$

Owing to transformation law (2.2), the components of the linear momentum are unchanged while those of the angular momentum do not vanish due to the existence of the orbital angular momentum any longer

$$J^{\alpha'\beta'\gamma'} = C^{\alpha'} T^{\beta'\gamma'} - T^{\alpha'\gamma'} C^{\beta'}$$

For easy notations, we cancel the prime symbol. Owing to (7.6), the non-vanishing components of the angular momentum in the new frame are

$$J^{3b\gamma} = -J^{b3\gamma} = \theta^3 T^{b\gamma} \qquad\qquad J^{30\gamma} = -J^{03\gamma} = \theta^3 T^{0\gamma} \tag{7.7}$$

Under the previous approximations, the torsor at every point of the shell and at the considered time is given by its components with respect to the affine frame associated to the middle point. By integrating them over the thickness, we obtain the shell variables

$$^{\gamma}T^{\alpha} = \int_{-h/2}^{h/2} T^{\alpha\gamma}\, d\theta^3 \qquad\qquad ^{\gamma}J^{\alpha\beta} = \int_{-h/2}^{h/2} J^{\alpha\beta\gamma}\, d\theta^3 \tag{7.8}$$

Combining (7.5), (7.7) and (7.8) leads to

$$\begin{gathered} ^{a}T^{b} = -\int_{-h/2}^{h/2} \sigma^{ab}\, d\theta^3 + \frac{\rho h^3}{12} w^a w^b \qquad ^{a}T^{3} = -\int_{-h/2}^{h/2} \sigma^{a3}\, d\theta^3 \qquad ^{0}T^{0} = \rho h \\ ^{a}J^{b3} = -{}^{a}J^{3b} = \int_{-h/2}^{h/2} \theta^3 \sigma^{ab}\, d\theta^3 \qquad ^{0}J^{3b} = -{}^{0}J^{b3} = \frac{\rho h^3}{12} w^b \end{gathered} \tag{7.9}$$

the other variables being zero. The previous result is neither general nor exact but it illustrates a method to construct the shell variables and provides their physical interpretation. Accounting for the symmetry of Cauchy's stress tensor, it remains 11 independent non-zero shell variables in the Galilean setting, instead of 30 in the general affine geometry.

8. Usual theory of plates and shells in static equilibrium

Shell variables

Let us assume that all the points of the three dimensional body are *at rest* in the Galilean coordinate system (X^α) at any time. The function n^i does not explicitly depend on $t = \xi^0$ and, consequently, the velocity w vanishes. The components $N^{ab} = -{}^aT^b$ are line densities of membrane forces, and the components $Q^a = -{}^aT^3$ are line densities of transverse shear forces. The component $\rho_s = {}^0T^0$ is interpreted as a surface density of mass. The components $M^{ab} = {}^aJ^{b3}$ can be interpreted as line densities of bending and twisting couples. Shell variables (7.9) are reduced to

$$N^{ab} = -{}^aT^b = \int_{-h/2}^{h/2} \sigma^{ab}\, d\theta^3 \qquad Q^a = -{}^aT^3 = \int_{-h/2}^{h/2} \sigma^{a3}\, d\theta^3$$

$$\rho_s = {}^0T^0 = \rho h \qquad M^{ab} = {}^aJ^{b3} = -{}^aJ^{3b} = \int_{-h/2}^{h/2} \theta^3 \sigma^{ab}\, d\theta^3 \tag{8.1}$$

the other variables are zero. Taking into account the symmetry of the stress tensor, it remains 9 independent variables in statics, instead of 11 in dynamics.

Static equilibrium

Let us assume that all the points of the material surface are *at rest* in the Galilean coordinate system (X^α) at any time. The functions r^i does not explicitly depend on $t = \xi^0$ and, consequently, the velocity v and the acceleration of transport g_p vanish. Besides, we suppose that Coriolis's effects are absent $\Omega = 0$. Therefore, it holds

$$\frac{\partial \pi^\top}{\partial t} = 0 \qquad \frac{\partial n}{\partial t} = 0 \qquad g^* = g$$

Connection matrix (6.10) becomes

$$\omega' = \begin{bmatrix} 0 & 0 & 0 \\ -g_t dt & a^{-1}\pi d_{\theta_t}\pi^\top + \beta^\top d\theta^3 & a^{-1}\pi d_{\theta_t} n \\ -g^n dt & n^\top d_{\theta_t}\pi^\top & 0 \end{bmatrix}$$

with: $g_t = a^{-1}\pi g$, $g^3 = n^\top g$. Moreover, we assume that the connection on the submanifold $\mathcal{N}$ is the connection induced by the one of $\mathcal{M}$. The element of the matrix a^{-1} at the bth row and eth column will be denoted a^{be}. For convenience, we put

$$c_i^a = a^{ae} \pi_e^i$$

The following Christoffel's coefficients are generated

$$\mathit{\Gamma}_{bc}^a = {}_{bc}^{a}\gamma = c_i^a \frac{\partial \pi_b^i}{\partial \theta^c} \qquad \mathit{\Gamma}_{3b}^a = c_i^a \frac{\partial n^i}{\partial \theta^b} = -b_b^a \qquad \mathit{\Gamma}_{00}^a = -c_i^a g^i = -g_t^a$$

$$\mathit{\Gamma}_{ab}^3 = \sum_{i=1}^{3} n^i \frac{\partial \pi_b^i}{\partial \theta^a} = b_{ab} \qquad \mathit{\Gamma}_{00}^3 = -g^3 \tag{8.2}$$

with the other ones being zero. As usual, the coefficients b_{ab} define the *2nd fundamental form* of the surface.

Let us examine the particular form of the balance of linear momentum $(7.1)_1$ in the adapted coordinates. Accounting for (6.11), (8.1) and (8.2), many terms disappear. For the in-plane translation equilibrium, it remains

$$-{}_\gamma\nabla\ {}^\gamma T^a = N^{ba}\Big|_b - b_b^a Q^b + \rho_s g_t^a = 0 \tag{8.3}$$

where

$$N^{ba}\Big|_b = \frac{\partial N^{ba}}{\partial \theta^b} + \mathit{\Gamma}_{bc}^a N^{bc} + \mathit{\Gamma}_{cb}^c N^{ba}$$

is the linear covariant divergence with respect the connection on the material surface. Similarly, for the off-plane translation equilibrium equation, it holds

$$-{}_\gamma\nabla\ {}^\gamma T^3 = b_{ab} N^{ab} + Q^b\Big|_b + \rho_s g^3 = 0 \tag{8.4}$$

where

$$Q^b\Big|_b = \frac{\partial Q^b}{\partial \theta^b} + \mathit{\Gamma}_{bc}^c Q^b$$

The last equation provides the balance of mass in the Lagrangean representation

$${}_\gamma\nabla\ {}^\gamma T^0 = \frac{\partial \rho_s}{\partial t} = 0 \tag{8.5}$$

For the balance of angular momentum $(7.1)_2$, we need to evaluate first the linear covariant divergence by (4.9), next the affine one by (4.8). For the in-plane rotation equilibrium, we have

$${}_\gamma\nabla\ {}^\gamma J^{b3} = \frac{\partial M^{ab}}{\partial \theta^a} + \mathit{\Gamma}_{ac}^b M^{ac} + \mathit{\Gamma}_{ac}^c M^{ab} = M^{ab}\Big|_a$$

Thus, it holds

$$ {}_\gamma\widetilde{\nabla}\ {}^\gamma J^{b3} = M^{ab}\Big|_a - Q^b = 0 \tag{8.6} $$

For the off-plane rotation equilibrium, one has successively

$$ \begin{aligned} &{}_\gamma\nabla\ {}^\gamma J^{21} = -b^1_a M^{a2} + b^2_a M^{a1} \\ &{}_\gamma\widetilde{\nabla}\ {}^\gamma J^{21} = {}_\gamma\nabla\ {}^\gamma J^{21} - N^{21} + N^{12} = (N^{12} - b^1_a M^{a2}) - (N^{21} - b^2_a M^{a1}) = 0 \end{aligned} \tag{8.7} $$

Let us define the symbol ε_{cb} such that

$$ \varepsilon_{12} = -\varepsilon_{21} = 1 \qquad\qquad \varepsilon_{11} = \varepsilon_{22} = 0 $$

Equation (8.7) reads

$$ {}_\gamma\widetilde{\nabla}\ {}^\gamma J^{21} = \varepsilon_{cb}(N^{cb} - b^c_a M^{ab}) = 0 \tag{8.8} $$

which can be recognized as the classical *symmetry relation* of the usual shell theory (Valid, 1995). We recover the standard system of equilibrium equations of shells (8.3)-(8.6), (8.8) (Green and Zerna, 1968) but as the expression in an adapted coordinate system of the free affine covariante torsor principle (7.1). Nevertheless, these 7 scalar equations are not sufficient to determine the 9 independent shell variables (8.1), and we need additional conditions, the constitutive laws, but this topic will not be treated here.

9. Consistent formulation of the dynamics of plates and shells

Dynamic equilibrium

We start again with general expression (6.10) of the connection matrix in the adapted coordinates. In addition to (8.2), new Christoffel's coefficients arise, due to both Coriolis's effects and the time evolution of the surface

$$ \begin{aligned} &\varGamma^a_{b0} = c^a_i\Big(\frac{\partial \pi^i_b}{\partial t} + \varOmega^i_j \pi^j_b\Big) = \varPhi^a_b \qquad\qquad \varGamma^a_{30} = c^a_i\Big(\frac{\partial n^i}{\partial t} + \varOmega^i_j n^j\Big) = \varPhi^a \\ &\varGamma^3_{b0} = \sum_{i=1}^{3} n^i\Big(\frac{\partial \pi^i_b}{\partial t} + \varOmega^i_j \pi^j_b\Big) = \varPhi_b \end{aligned} \tag{9.1} $$

According to definitions (8.1), non-zero shell variables (7.9) are

$$
\begin{aligned}
&{}^aT^b = -N^{ab} + \frac{\rho h^3}{12} w^a w^b \quad {}^aT^3 = -Q^a \qquad {}^0T^0 = \rho h \\
&{}^aJ^{b3} = -{}^aJ^{3b} = M^{ab} \qquad {}^0J^{3b} = -{}^0J^{b3} = \frac{\rho h^3}{12} w^b
\end{aligned}
\tag{9.2}
$$

In the adapted coordinates, the balance of linear momentum $(7.1)_1$ gives the following equations. For the in-plane translation equation, it holds

$$
-{}_\gamma\nabla\ {}^\gamma T^a = \Big(N^{ba} - \frac{\rho h^3}{12} w^a w^b\Big)\Big|_b - b^a_b Q^b + \rho_s c^a_i\Big(g^i - 2\Omega^i_j v^j\Big) - \rho_s \frac{\partial v^i}{\partial t} = 0 \tag{9.3}
$$

The off-plane translation equation is

$$
-{}_\gamma\nabla\ {}^\gamma T^3 = b_{ab}\Big(N^{ab} - \frac{\rho h^3}{12} w^a w^b\Big) + Q^b\Big|_b + \sum_{i=1}^{3} n^i\Big(\rho_s(g^i - 2\Omega^i_j v^j) - \rho_s \frac{\partial v^i}{\partial t}\Big) = 0 \tag{9.4}
$$

By comparison with corresponding static equilibrium equations (8.3) and (8.4), we obtain additional terms reflecting expected Coriolis's and inertia effects. Less classical are kinetic terms similar to those arising in the equations of three-dimensional continuous medium given in Section 5. In mechanics of solids, they are generally neglected, but we have to notice that their magnitude could become significant at the high velocity, for instance in the case of impact. Finally, the last equation provides the balance of mass

$$
{}_\gamma\nabla\ {}^\gamma T^0 = \frac{\partial \rho_s}{\partial t} + \Phi^a_a \rho_s = 0 \tag{9.5}
$$

Next, we calculate the linear covariant divergence of the angular momentum by (4.9). The non-zero components are

$$
\begin{aligned}
{}_\gamma\nabla\ {}^\gamma J^{21} &= -b^1_a M^{a2} + b^2_a M^{a1} + \Phi^{10} J^{23} - \Phi^{20} J^{13} \\
{}_\gamma\nabla\ {}^\gamma J^{b3} &= \frac{\partial\, {}^\gamma J^{b3}}{\partial \xi^\gamma} + \Gamma^b_{ac}\, {}^c J^{a3} + \Gamma^c_{\rho c}\, {}^\rho J^{b3} + \Phi^{b0}_a J^{a3}
\end{aligned}
\tag{9.6}
$$

The affine covariant divergence of the angular momentum is given by (4.8). Let us determine the particular form of the balance of angular momentum $(7.1)_2$ in the adapted coordinates. The off-plane rotation equation is

$$
{}_\gamma\tilde{\nabla}\ {}^\gamma J^{b3} = M^{ab}\Big|_a - Q^b - \frac{\rho h^3}{12}\Big(\frac{\partial w^b}{\partial t} + \Phi^b_a w^a + \Phi^c_c w^b\Big) = 0 \tag{9.7}
$$

The first two terms are the same as in static equilibrium equation (8.6). The third one represents inertia effects and is expected. On the other hand, the last two terms are non-standard in the literature as subtly resulting from the time evolution of the surface geometry through new Christoffel's coefficients (9.1). Besides, owing to (9.6), the in-plane rotation equation

$$_{\gamma}\widetilde{\nabla}\ {}^{\gamma}J^{21} = -\varepsilon_{cb}({}^{c}T^{b} + b^{c}_{a}{}^{a}J^{b3} - \Phi^{c0}J^{b3}) = 0$$

generalizes the *symmetry relation* of the usual shell theory to the dynamics. Owing to (9.2), one has equivalently

$$_{\gamma}\widetilde{\nabla}\ {}^{\gamma}J^{21} = \varepsilon_{cb}\Big(N^{cb} - b^{c}_{a}M^{ab} - \frac{\rho h^3}{12}(\Phi^c + w^c)w^b\Big) = 0 \tag{9.8}$$

Once again, new terms appear, taking into account kinetic terms and the time evolution of the surface geometry. Finally, it can be seen that the remaining equations are automatically satisfied

$$_{\gamma}\widetilde{\nabla}\ {}^{\gamma}J^{03} = 0 \qquad\qquad _{\gamma}\widetilde{\nabla}\ {}^{\gamma}J^{b0} = 0$$

In short, the behavior of the torsor field is governed by a system of 7 scalar equations, namely (9.3)-(9-5) and (9.7)-(9.8). On the other hand, we have 11 independent unknowns, Q^b, N^{ab}, M^{ab}, h, w^b linked to shell variables (9.2) and 3 additional variables v^i. Initially equal to $30 - 10 = 20$ in the most general case, the redundancy degree is now reduced to $14 - 7 = 7$. To get rid of this indeterminacy, we must introduce additional assumptions concerning the constitutive laws.

What about the $30-11 = 19$ other shell variables? Under various assumptions introduced in Sections 6 to 9, they are in a way asleep. Nevertheless, their existence is predicted by the general theory within the frame of the affine group geometry. They could be waken up by considering other idealization than the one of a smooth curved thin sheet undergoing small deformation, and above all in a relativistic context.

10. Conclusions

Firstly, although the affine geometry could appear as a poverty-stricken mathematical frame, we think it is sufficient to describe the fundamental tools of the continuum mechanics. It leads to a definition of the torsors which is completely relieved of all metric features. Next, we developed the corresponding Affine Tensor Analysis enabling one to propose a general principle of an affine divergence free torsor. We showed that this principle allows one to recover the balance equations of the statics of three dimensional bodies. For the dynamics of shells, we revealed the existence of new terms, depending on velocities and involving time variation of the surface geometry. Of course, some open problems deserve more pervasive investigations. We have to develop at first applications of the new shell theory in order to assess the magnitude of the predicted terms. Other subjects of interest would be the dynamics of beams. Finally, we would like to point out related topics. In the previous work, the first author proved that for the dynamics of particles and rigid bodies, the well-known theorem of angular momentum is also a consequence of our principle of balance (de Saxcé, 2002). Besides, there is a subtle link with the symplectic mechanics which leads to a nice extension of Kirillov-Kostant-Souriau theorem.

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Tensory afiniczne w teorii powłok

Streszczenie

Strukturę formalna wypadkowej siły i momentu można ująć w postaci pojedynczego obiektu zwanego torsorem. Wyłączając wszystkie pojęcia metryczne, torsory definiujemy jako skośno-symetryczne dwuliniowe odwzorowania w przestrzeni liniowej w dziedzinie funkcji wektorowych. Torsory stanowią szczególną rodzinę afinicznych tensorów. Na tej podstawie zdefiniowano wewnętrzny operator różniczkowania zwany afiniczną kowariantną dywergencją. Następnie wysunięto postulat, że zachowanie się ośrodka ciągłego opisane polem torsorowym posiada zerową taką dywergencję. Zastosowawszy tę ogólną zasadę, użyto równań Eulera w opisie dynamiki ciał trójwymiarowych. W dalszej części pracy skoncentrowano się na dynamice powłok. Poprzez użycie odpowiednio zaadaptowanych współrzędnych wykazano, że zastosowanie tej ogólnej zasady stanowi spójną metodę otrzymywania równań z nieoczekiwanie pojawiającymi się członami odpowiedzialnymi za efekty przyspieszenia Coriolisa oraz zmian powierzchni powłoki w czasie.

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