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SHAPE SENSITIVITY ANALYSIS OF ELASTIC SHELLS WITH CRACKS

Edgardo Taroco
Raúl A. Feijóo

National Laboratory of Scientific Computing (LNCC/MCT), Petrópolis, Brasil
e-mail: etam@lncc.br; feij@lncc.br

This study concerns the application of shape sensitivity analysis as a systematic methodology to determine the energy release rate of cracked shells, within the framework of a linear elastic approach that takes into account the effect of transverse shear deformation. This methodology and the direct method of shape sensitivity analysis is applied to shells with an arbitrary middle surface and leads to an explicit general expression for the shape sensitivity of the total potential strain energy. In elastic shells with cracks, crack initiation is simulated by a change of shape characterized by a suitable tangential velocity distribution over the middle surface of the shell. In this case, a useful expression of energy release rate is expressed in terms of the strain-stress state and the adopted shape change velocity field. Finally, shape sensitivity analysis is applied to the circular cylindrical shell and thus the condition of null divergence of the corresponding Eshelby tensor is verified.

Key words: shape sensitivity analysis, elastic shells, linear elastic fracture mechanics

1. Introduction

It is well known that a curved sheet containing a through crack has a reduced resistance to fracture initiation. Moreover, it has been verified that cracks in shells can severely decrease their strength, their load-carrying capacity, and in limit situations can also cause sudden failure. For this reason, the interaction of flaws with shells curvature is a subject that has received careful attention. To ensure structural integrity failure criterions have been developed by the simultaneously application of shell theory and fracture mechanical concepts. To establish a proper failure criterion, the knowledge of the shell stress distribution due to the presence of cracks and the fracture initiation law are necessary, as it was pointed out by Folias (1974).

Concerning the shell stress distribution it is also well known in fracture mechanics that in the vicinity of the crack tip or on the crack surface, the transverse shear deformation can not be ignored. On the other hand, the approximate Kirchhoff boundary conditions used in the classical theory of plates and shells are inadequate to determine the stress field in the neighborhood of the crack tip or the displacement of the crack surface. Nevertheless, this difficulty can be overcome by selecting a shell theory that takes into account the effect of transverse deformation. Such a hypothesis leads to a more accurate solution for the stress analysis of cracked shells, as it was reported by Sih and Hagendorf (1974).

In reference to the fracture initiation laws, the earliest work of Griffith introduced the first statement of the energy balance criterion for crack initiation. Therefore, it has been accepted that the energy release rate provides the work required to create new fracture surface in elastic materials. Thus, the energy release rate has become a very useful parameter in linear elastic fracture mechanics.

Since the pioneer work of Griffith, the continued interest in developping the procedures to determine the energy release rate has been maintained up to the present time.

Regarding the conservation laws and path-independent integrals that have been widely used in the analysis of cracked bodies, we must go back to the classic papers of Eshelby (1956, 1975) in which the notion of a force on a lattice defect and the concept of the energy momentum tensor, were addressed. Particularly, the J-integral proposed by Rice (1968) exemplifies how successfully these invariant integrals can be applied in fracture mechanics and also indicates the usefulness of the J-integral in determining the energy release rate of cracked bodies.

In a subsequent paper, Bergez and Radenkovic (1973) extended the concept of path-independent integrals to the shell theory, even though they did not place any restrictions on the geometry of the middle surface. However, it is accepted today, that such integrals are not path-independent in general. Further, Lo (1980) has shown that J and related integrals, are path-independent for circular cylindrical shell in the context of the Koiter linear elastic theory. Later on, Kienzler and Golebiewska-Herrmann (1985) discussed the conservation laws in higher order shell theory. Recently, Li and Shyy (1997) derived several new invariant integrals for shallow shells within Marguerre's approach. More recently, Kienzler and Herrmann (2000) turned on to circular cylindrical shell theory that accounted transverse shear deformation and the associate Eshelby's tensor in the framework of material space.

Since the energy release rate is generally interpreted as the rate of the energy dissipated in the fracture process per unit crack propagation length, and due to the difficulty of obtaining expressions for the potential energy as explicit functions of the crack length, which enable us to obtain derivatives in a direct form, several procedures, both numerical and experimental, have been developed in fracture mechanics. Among them, the Shape Sensitivity Analysis (proposed originally by Cèa (1981) developed mainly by Zolésio (1981), Masmoudi (1987) and widely discussed by Haug *et al.* (1986), can be successfully applied. As shown in Feijóo *et al.* (2000), the crack growth is simulated as a *shape change* of a 3D cracked body. Then, using the well known results from the shape sensitivity analysis, the general expression for the energy release rate was obtained in that paper. Moreover, this general expression is a function of a velocity field describing the change of shape.

On the other hand, shape sensitivity analysis for curved elements was first applied by Chenais and Rousselet (1984) and later by Rousselet (1987) in the shape optimization of arches submitted to static loads. In addition, theoretical aspects of axisymmetric shells and numerical results were reported by Mota Soares *et al.* (1987).

In the case of arches, the analysis was performed along the mid line and in axisymmetric shells-along the meridian curve. In both cases the local system of coordinates is orthogonal and the vector base is given by the unit normal and the unit tangent to the curve (the mid line in the case of the arch and the meridian in the case of the axisymmetric shell).

However, in shells with arbitrary shape the coordinate curves over the middle surface generally are not orthogonal. Therefore the corresponding equations have been established by using curvilinear coordinates over the middle surface of the shell. Covariant and contravariant components of vectors and tensors have to be introduced and partial derivatives are to be replaced by covariant derivatives with the help of Christoffel's symbols.

In the case when the middle surface of a general shell is defined by a smooth mapping of a two-dimensional domain, shape derivatives may be performed by differentiation with respect to the mapping. In other words, the change of the shape can be seen as the change of the mapping.

This approach and the Lagrangian Method, which allows to carry out the derivative of any functional, were applied by Bernadou *et al.* (1991). Their aims were in shape optimization of a thin shell within the framework of Koiter's theory formulated in an arbitrary curvilinear coordinates and subjected to different kinds of loads over the middle surface of the shell. A comprehensive general analysis of shape sensitivity was included.

A rather simple and direct derivation in an orthogonal coordinate system, shell theories and shape sensitivity analysis become more involved in arbitrary curvilinear coordinates, making it difficult to follow the physical meaning of the model. To overcome this difficulty and to work independently of the coordinate system, we adopted in this paper the intrinsic base defined at each point of the middle surface by its unit normal vector and its tangent plane, cf. Valid (1981).

Among a number of different possible approaches to the theory of small-strain linear elastic shells, we have selected the one developed by Reissner (1941), that takes into account the effect of transverse shear deformation. Reissner's approach appears to us to be preferable to other since it uses relatively simple formulation and requires in the definition of strains only the first order gradient of displacements. Moreover, it leads to results of considerable generality and is also suitable for the applications that are in focus of attention.

Considering the application to fracture mechanics to be conducted later and to demonstrate the simplicity of the approach adopted in this work, we have limited ourselves to carry out exclusively the shape sensitivity analysis of the total potential energy of the shell submitted to static loads along its boundary.

2. Shell shape change

In the present section, we introduce the concept of shape change of the middle surface of the shell that will allow us to study the behavior of functions and functionals when the shape of a shell is modified. Proposed originally by Cèa (1981) and widely discussed by Haug *et al.* (1986), this approach simulates the change in shape by a *motion* from an *initial configuration* to a known *deformed configuration* characterized by the adopted velocity field defining the shape change.

On the other hand, the basic idea behind almost all theories of shells is to reduce the analysis over the middle surface by means of simplified assumptions. According to this, we may characterize the shape of the shell by the geometry of its middle surface.

Then, let us consider an elastic shell characterized by a smooth middle surface Ω_o, bounded by a curve that we also assume to be smooth and denoted by $\partial\Omega_o$. The shape change of the shell will be defined by a known smooth vector field $\boldsymbol{V}(\boldsymbol{X})$, $\boldsymbol{X} \in \Omega_o$. Using this approach, the shape change of the shell, and more precisely the shape change of its middle surface, can be described by the transformation χ_τ, given by

$$\begin{aligned} &\chi_\tau \;:\; \Omega_o \mapsto \Omega_\tau \\ &\boldsymbol{x}_\tau = \chi_\tau(\boldsymbol{X}) = \boldsymbol{X} + \tau \boldsymbol{V}(\boldsymbol{X}) \qquad \tau \in \mathbb{R}^+ \end{aligned} \tag{2.1}$$

for $\tau \in \mathbb{R}^+$ sufficiently small.

Thus, the shape change is a smooth one-parameter family of transformations where $\boldsymbol{V}(\boldsymbol{X})$ is the direction of the domain variation. This means that, for a given direction $\boldsymbol{V}(\boldsymbol{X})$, the shape change of Ω_o is uniquely determined by the parameter $\tau \in \mathbb{R}^+$.

The transformed domain Ω_τ might be considered as a *deformed configuration* of the initial domain Ω_o under the transformation from Ω_o to Ω_τ defined by (2.1). Furthermore, introducing the continuum mechanics terminology, Gurtin (1981), an analogy can be drawn between *change of shape* and *motion of a body*. From this point of view, $\boldsymbol{V}(\boldsymbol{X})$ can be seen as the *shape change velocity field*.

From now on and to simplify the notation, we will omit the subscript τ identifying Ω_τ $(\partial\Omega_\tau)$ with Ω $(\partial\Omega)$. Moreover, the surface Ω can be seen as the actual description of the middle surface at each value of τ. Therefore, the surface Ω might be considered as a perturbation of the initial surface Ω_o and the transformation from Ω_o to Ω, as a function of the point $\boldsymbol{X}$ and the parameter τ.

Since at each τ, the shape change is a one-to-one transformation from Ω_o to Ω, there is a unique inverse transformation χ_τ^{-1} of Ω to Ω_o.

Hence, any scalar, vector, or tensor field associated with the shape change can be expressed as a function over the initial surface Ω_o, or a function over the actual surface Ω. Within the continuum mechanics analogy, we call them *material* and *spatial descriptions*, respectively. For instance, in the particular case of the shape change velocity field, we may write for both descriptions

$$\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{X}) \qquad\qquad \boldsymbol{v} = \boldsymbol{v}(\tau; \boldsymbol{x}) \tag{2.2}$$

In this paper we shall carry out the analysis over the middle surface of the shell in the actual configuration. In other words we will adopt the spatial description, taking advantage of the well-known expressions of *the material or total (time) derivatives of spatial fields* developed in Continuum Mechanics, Gurtin (1981). From this analogy, the shape sensitivity of any regular functional characterized by its spatial description $\Psi(\tau;\boldsymbol{x})$, can formally be defined as

$$\frac{d\Psi}{d\tau} = \Big\{ \frac{\partial \Psi(\tau;\boldsymbol{x})}{\partial \tau}\Big|_{\boldsymbol{x}=\chi_\tau(\boldsymbol{X})} \Big\}\Big|_{\boldsymbol{X}=\chi_\tau^{-1}(\boldsymbol{x})} \tag{2.3}$$

Furthermore, in the shape sensitivity analysis of shells it is convenient to describe vector and tensor fields using the *intrinsic shell frame* defined at each point $\boldsymbol{x}$ of the middle surface Ω by its *unit normal* $\boldsymbol{n}$, and its *tangent plane* T_p.

In addition, we introduce the projection tensor operator over the tangent plane T_p and the projection tensor operator over the normal vector $\boldsymbol{n}$, respectively denoted by $\boldsymbol{\Pi}$ and $\boldsymbol{n}\otimes\boldsymbol{n}$. Hence, the unit tensor $\mathbf{I}$ may be described as

$$\mathbf{I} = \boldsymbol{\Pi} + (\boldsymbol{n}\otimes\boldsymbol{n}) \tag{2.4}$$

where $\otimes$ denotes the tensorial product of vectors.

Considering the application to cracked shells, we assume that the shape change velocity at each point of the middle surface lies over the corresponding tangent plane, thus the spatial description of this velocity is denoted by $\boldsymbol{v}_t$. Moreover, we assume that both the unit normal vector $\boldsymbol{n}$ and the tangent vector $\boldsymbol{v}_t$, are smooth fields on Ω.

We also define the *surface gradient* of spatial fields (Gurtin, 2000). This *surface gradient* can be seen as the restriction to T_p of the usual gradient

$$\operatorname{grad}_s(\cdot) = \operatorname{grad}(\cdot)|_{T_p} \tag{2.5}$$

Fore instance, the *surface gradient* of the spatial description of the velocity field, admits a unique decomposition into *tangential* and *normal* components

$$\operatorname{grad}_s\boldsymbol{v}_t = \boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{v}_t + \boldsymbol{n}\otimes(\operatorname{grad}_s\boldsymbol{v}_t)^\top\boldsymbol{n} \tag{2.6}$$

In particular, we denote by $\operatorname{grad}_s\boldsymbol{n}$, the surface gradient of the unit normal $\boldsymbol{n}$. This gradient, known in the literature as the curvature tensor, has a central place in the theory of surfaces and is concomitant with the formulation of theory of shells that reduces the analysis to the middle surface.

Since $\boldsymbol{n}$ is a unit vector, the surface gradient of the scalar product $\boldsymbol{n}\cdot\boldsymbol{n}=1$ leads to

$$(\operatorname{grad}_s\boldsymbol{n})^\top\boldsymbol{n} = \boldsymbol{0} \tag{2.7}$$

From (2.7), we conclude that $\operatorname{grad}_s\boldsymbol{n}$ lies on the tangent plane T_p at the point $\boldsymbol{x}$ under consideration. In addition, it can be easily verified that $\operatorname{grad}_s\boldsymbol{n}$ is a symmetric tensor (Gurtin, 2000), thus

$$\operatorname{grad}_s\boldsymbol{n} = (\operatorname{grad}_s\boldsymbol{n})^\top \tag{2.8}$$

On the other hand, as a consequence of the orthogonality between the vectors $\boldsymbol{v}_t$ and $\boldsymbol{n}$, the surface gradient of the scalar product $\boldsymbol{v}_t \cdot \boldsymbol{n} = 0$, leads to

$$(\operatorname{grad}_s \boldsymbol{v}_t)^\top \boldsymbol{n} = -(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \tag{2.9}$$

Subsequently, inserting (2.9) into (2.6), we may write

$$\operatorname{grad}_s \boldsymbol{v}_t = \boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{v}_t - \boldsymbol{n} \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \tag{2.10}$$

Here, the first term on the right-hand side represents the *tangential* component and the last term – the *normal* component of the surface gradient of the velocity.

It will be evident in the following sections of this paper that the surface gradient of the velocity plays an outstanding role in the shape sensitivity analysis of shells.

3. The potential energy of the shell

The purpose of this section is to introduce the mechanical model and the expression of the cost function, using the well-known terminology of optimization. As a first step we assume the selection of the mechanical model and the cost function. In spite of the fact that classical shell theories are quite appropriate for problems without stress singularities, when fracture mechanics must be included, more accurate theories are necessary to adequately model the behavior of the shell near the crack tip region. Among a number of different possible approaches to the analysis of elastic shells, we have selected one which appears to us to be preferable to others since it leads to results of considerable generality using only first order gradient in the strain-displacement relations. For simplicity we shall be concerned with a shell within the framework of a linear elastic small-strain approach that takes into account the effect of transverse shear deformation, known in the shell literature as Reissner's theory. Furthermore, considering the application to shells containing cracks to be accomplished later, we adopt as cost function the total potential energy of the shell under analysis, given by

$$\psi(\boldsymbol{u}_t, u_n, \boldsymbol{\vartheta}) = U - W = \int\limits_{\Omega} \phi(\boldsymbol{\varepsilon}^s, \boldsymbol{\gamma}, \boldsymbol{\kappa}^s)\, d\Omega - W \tag{3.1}$$

Here, ϕ denotes the specific strain energy of the shell, $\boldsymbol{u}_t$, u_n, $\boldsymbol{\vartheta}$ the kinematically admissible displacement fields, $\boldsymbol{\varepsilon}^s$, $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}^s$ the strains associated to the displacements by kinematical relations and the super index $(\bullet)^s$ the symmetric part of the tensor $(\bullet)$. The domain integral on the right-hand side of the above expression represents the total strain energy stored in the shell and W the external work.

Within the framework of Reissner's approach, the strain-displacement relations take the form

$$\begin{aligned} \boldsymbol{\varepsilon}^s &= (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t)^s + u_n\,\mathrm{grad}_s\boldsymbol{n} \\ \boldsymbol{\gamma} &= -\boldsymbol{\vartheta} + \mathrm{grad}_s u_n - (\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t \\ \boldsymbol{\kappa}^s &= (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{\vartheta})^s \end{aligned} \tag{3.2}$$

Here, the vector field $\boldsymbol{u}_t$ denotes the tangent displacement, the scalar field u_n is the normal displacement, the tangential vector field $\boldsymbol{\vartheta}$ is the rotational angle of the normal at any point of the middle surface.

Hence, from the above definition of the strains, stretching of the middle surface $\boldsymbol{\varepsilon}^s$ leads to a tangent second order tensor field, the transversal shearing $\boldsymbol{\gamma}$ to a tangent vector field, and the flexural strain $\boldsymbol{\kappa}^s$ to a tangent second order tensor field.

As it was noted, the strain-displacement relations of this approach, given by (3.2), involve all three displacements and require in its definition only the first order gradient.

Then, to throw additional light on Reissner's kinematical assumptions, the meaning of the surface gradient of scalar, tangential vector and unit normal fields are essential.

3.1. Total strain energy

According to Reissner's assumptions, the total elastic strain energy stored in the shell U may be expressed as the sum of the stretching U_ε, shearing U_γ and flexural strain energy U_κ

$$U = U_\varepsilon + U_\gamma + U_\kappa \tag{3.3}$$

given, respectively, by

$$U_\varepsilon = \int_\Omega \phi_\varepsilon\, d\Omega \qquad U_\gamma = \int_\Omega \phi_\gamma\, d\Omega \qquad U_\kappa = \int_\Omega \phi_\kappa\, d\Omega \tag{3.4}$$

where the scalars ϕ_ε, ϕ_γ and ϕ_κ denote, respectively, the specific elastic stretching, shearing and flexural shell energy.

3.2. External work

For simplicity we will not consider the body forces. In this case the external work is performed by a system of loads applied along the boundary, hence

$$W = \int\limits_{\partial\Omega_t} (\overline{\boldsymbol{t}} \cdot \boldsymbol{u}_t + \overline{q} u_n + \overline{\boldsymbol{m}} \cdot \boldsymbol{\vartheta})\, d\partial\Omega \tag{3.5}$$

Here, $\overline{\boldsymbol{t}}$, $\overline{q}$ and $\overline{\boldsymbol{m}}$ denote the loads prescribed on the boundary $\partial\Omega_t$, the vector field $\overline{\boldsymbol{t}}$ denotes the tangent force, the scalar field $\overline{q}$ – the shearing force and the tangential vector field $\overline{\boldsymbol{m}}$ the moment. These loads are compatible with the shell model under analysis.

4. Variational form of the equilibrium

Now, making use of the Principle of Virtual Power (which is equivalent to the Principle of Minimum Total Potential Energy due to the assumption adopted in this work), the equilibrium of the shell can be written in the following variational form:

- Find $\boldsymbol{u}_t$, u_n and $\boldsymbol{\vartheta} \in K_{in}$ such that

$$\begin{aligned} &\int\limits_{\Omega} \mathbf{N} \cdot (\boldsymbol{\Pi} \operatorname{grad}_s \widehat{\boldsymbol{u}}_t + \widehat{u}_n \operatorname{grad}_s \boldsymbol{n})\, d\Omega + \\ &+ \int\limits_{\Omega} \boldsymbol{Q} \cdot [-\widehat{\boldsymbol{\vartheta}} + \operatorname{grad}_s \widehat{u}_n - (\operatorname{grad}_s \boldsymbol{n}) \widehat{\boldsymbol{u}}_t]\, d\Omega + \int\limits_{\Omega} \mathbf{M} \cdot \boldsymbol{\Pi} \operatorname{grad}_s \widehat{\boldsymbol{\vartheta}}\, d\Omega - \\ &- \int\limits_{\partial\Omega_t} (\overline{\boldsymbol{t}} \cdot \widehat{\boldsymbol{u}}_t + \overline{q}\widehat{u}_n + \overline{\boldsymbol{m}} \cdot \widehat{\boldsymbol{\vartheta}})\, d\partial\Omega = 0 \end{aligned} \tag{4.1}$$

for all $\widehat{\boldsymbol{u}}_t$, $\widehat{u}_n$ and $\widehat{\boldsymbol{\vartheta}} \in K_{in}$, and where K_{in} is the space of admissible kinematical displacements.

We also assume that the fields $\boldsymbol{u}_t$, u_n and $\boldsymbol{\vartheta}$ are prescribed (null for simplicity) along the boundary $\partial\Omega_u$ $(\partial\Omega = \partial\Omega_u \cup \partial\Omega_t\,;\ \partial\Omega_u \cap \partial\Omega_t = \emptyset)$.

In the above variational form, the tangential symmetric second order tensor $\mathbf{N}$ denotes the membrane force of the shell, the tangential vector $\boldsymbol{Q}$ – the transverse shearing force and the tangential symmetric second order tensor $\mathbf{M}$

– the bending moment in the middle surface of the shell. They are given by the formulae

$$\mathbf{N} = \frac{\partial \phi_\varepsilon}{\partial \boldsymbol{\varepsilon}} \qquad \boldsymbol{Q} = \frac{\partial \phi_\gamma}{\partial \boldsymbol{\gamma}} \qquad \mathbf{M} = \frac{\partial \phi_\kappa}{\partial \boldsymbol{\kappa}} \tag{4.2}$$

Next, we insert the following surface tensor relations

$$\begin{aligned} \mathbf{N} \cdot \boldsymbol{\Pi} \operatorname{grad}_s \widehat{\boldsymbol{u}}_t &= \operatorname{div}_s(\mathbf{N}^\top \widehat{\boldsymbol{u}}_t) - \widehat{\boldsymbol{u}}_t \cdot \boldsymbol{\Pi} \operatorname{div}_s \mathbf{N} \\ \boldsymbol{Q} \cdot \operatorname{grad}_s \widehat{u}_n &= \operatorname{div}_s(\widehat{u}_n \boldsymbol{Q}) - \widehat{u}_n \operatorname{div}_s \boldsymbol{Q} \\ \mathbf{M} \cdot \boldsymbol{\Pi} \operatorname{grad}_s \widehat{\boldsymbol{\vartheta}} &= \operatorname{div}_s(\mathbf{M}^\top \widehat{\boldsymbol{\vartheta}}) - \widehat{\boldsymbol{\vartheta}} \cdot \boldsymbol{\Pi} \operatorname{div}_s \mathbf{M} \end{aligned} \tag{4.3}$$

Further, by the use of the surface divergence theorem

$$\begin{aligned} \int_\Omega \operatorname{div}_s(\mathbf{N}^\top \widehat{\boldsymbol{u}}_t)\, d\Omega &= \int_{\partial\Omega_t} \mathbf{N}\boldsymbol{m} \cdot \widehat{\boldsymbol{u}}_t \, d\partial\Omega \\ \int_\Omega \operatorname{div}_s(\widehat{u}_n \boldsymbol{Q})\, d\Omega &= \int_{\partial\Omega_t} (\boldsymbol{Q} \cdot \boldsymbol{m})\widehat{u}_n \, d\partial\Omega \\ \int_\Omega \operatorname{div}_s(\mathbf{M}^\top \widehat{\boldsymbol{\vartheta}})\, d\Omega &= \int_{\partial\Omega_t} \mathbf{M}\boldsymbol{m} \cdot \widehat{\boldsymbol{\vartheta}} \, d\partial\Omega \end{aligned} \tag{4.4}$$

the Principle of Virtual Power (4.1) can be rewritten as

$$\begin{aligned} &-\int_\Omega [\boldsymbol{\Pi} \operatorname{div}_s \mathbf{N} + (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{Q}] \cdot \widehat{\boldsymbol{u}}_t \, d\Omega - \int_\Omega [\operatorname{div}_s \boldsymbol{Q} - \mathbf{N} \cdot \operatorname{grad}_s \boldsymbol{n}]\widehat{u}_n \, d\Omega - \\ &-\int_\Omega [\boldsymbol{\Pi} \operatorname{div}_s \mathbf{M} + \boldsymbol{Q}] \cdot \widehat{\boldsymbol{\vartheta}} \, d\Omega + \\ &+\int_{\partial\Omega_t} [(\mathbf{N}\boldsymbol{m} - \overline{\boldsymbol{t}}) \cdot \widehat{\boldsymbol{u}}_t + (\boldsymbol{Q} \cdot \boldsymbol{m} - \overline{q})\widehat{u}_n + (\mathbf{M}\boldsymbol{m} - \overline{\boldsymbol{m}}) \cdot \widehat{\boldsymbol{\vartheta}}] \, d\partial\Omega = 0 \end{aligned} \tag{4.5}$$

where div_s denotes the spatial surface divergence of vector or tensor fields and $\boldsymbol{m}$ is the outward unit normal vector to the boundary curve $\partial\Omega$. This normal lies on the intersection of the tangent plane to the middle surface of the shell and the normal plane orthogonal to the unit tangent vector of the curve $\partial\Omega$ at the point under consideration. In the theory of surface curves, the unit tangent vector to the curve $\partial\Omega$ together with the normal vectors $\boldsymbol{n}$ and $\boldsymbol{m}$, mutually orthogonal, compose the intrinsic frame of $\partial\Omega$.

The above expression furnishes the Euler surface equations of the shell (strong form of the equilibrium) associated to the Principle of Virtual Power in Ω

$$\begin{aligned} &\boldsymbol{\Pi}\,\mathrm{div}_s\mathbf{N} + (\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{Q} = \mathbf{0} \\ &\mathrm{div}_s\boldsymbol{Q} - \mathbf{N}\cdot\,\mathrm{grad}_s\boldsymbol{n} = \mathbf{0} \\ &\boldsymbol{\Pi}\,\mathrm{div}_s\mathbf{M} + \boldsymbol{Q} = \mathbf{0} \end{aligned} \tag{4.6}$$

as well as the natural boundary conditions on $\partial\Omega_t$

$$\mathbf{N}\boldsymbol{m} = \overline{\boldsymbol{t}} \qquad \boldsymbol{Q}\cdot\boldsymbol{m} = \overline{q} \qquad \mathbf{M}\boldsymbol{m} = \overline{\boldsymbol{m}} \tag{4.7}$$

The coupled nature of the equilibrium equations is a direct consequence of the strain-displacement relations adopted.

5. Shape derivative of vectors

In this section we start applying the analogy between the material (total time) derivative and shape derivative to obtain the shape derivatives of the tangential vector $\boldsymbol{u}_t$ and the unit normal vector $\boldsymbol{n}$. When the direct method is used, these derivatives are useful to perform the shape derivative of the corresponding surface gradients.

In its general form, the material (total time) derivative of superficial fields defined by (2.3), may be rewritten as

$$\frac{d}{d\tau}\Psi(\tau;\boldsymbol{x}) = \Big\{\frac{\partial}{\partial\tau}\{\Psi(\tau;\boldsymbol{x})\}_m\Big\}_{sp} \tag{5.1}$$

Here, Ψ denotes a scalar, vector or tensor field, the subscript m – the material description and sp – the spatial description.

To this end, we focus our attention on shape changes characterized by the spatial description of a tangential velocity field given by $\boldsymbol{v}_t(\boldsymbol{x})$. Then, the shape change gradient $\mathbf{F}$ defined at each material point $\boldsymbol{X}$ $(d\boldsymbol{x} = \mathbf{F}d\boldsymbol{X})$ is such that its partial time (τ) derivative is given by

$$\frac{\partial}{\partial\tau}\mathbf{F} = \{\,\mathrm{grad}_s\boldsymbol{v}_t\}_m\mathbf{F} \tag{5.2}$$

Moreover, from $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$ we have

$$\frac{\partial}{\partial\tau}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\{\,\mathrm{grad}_s\boldsymbol{v}_t\}_m \tag{5.3}$$

Derivative of the tangent vector field $\boldsymbol{u}_t$

The spatial description of the tangent vector $\boldsymbol{u}_t$ and its material description denoted by $\widetilde{\boldsymbol{u}}_t$ are related as follows

$$\{\boldsymbol{u}_t\}_m = \mathbf{F}\widetilde{\boldsymbol{u}}_t \tag{5.4}$$

Next we apply the partial derivative with respect to τ on both sides of (5.4) and, inserting (5.2), we have

$$\frac{\partial}{\partial\tau}\{\boldsymbol{u}_t\}_m = \frac{\partial}{\partial\tau}(\mathbf{F}\widetilde{\boldsymbol{u}}_t) = \frac{\partial\mathbf{F}}{\partial\tau}\widetilde{\boldsymbol{u}}_t = \{\operatorname{grad}_s\boldsymbol{v}_t\}_m\mathbf{F}\widetilde{\boldsymbol{u}}_t \tag{5.5}$$

Further, inserting (5.5) into (5.2), we obtain

$$\frac{d\boldsymbol{u}_t}{d\tau} = \Big\{\{\operatorname{grad}_s\boldsymbol{v}_t\}_m\mathbf{F}\widetilde{\boldsymbol{u}}_t\Big\}_{sp} = (\operatorname{grad}_s\boldsymbol{v}_t)\boldsymbol{u}_t \tag{5.6}$$

thus, combining (2.10) and (5.6), the vector $d\boldsymbol{u}_t/d\tau$ can be written in the convenient form

$$\frac{d\boldsymbol{u}_t}{d\tau} = \boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau} - [(\operatorname{grad}_s\boldsymbol{n})\boldsymbol{v}_t\cdot\boldsymbol{u}_t]\boldsymbol{n} \tag{5.7}$$

Here, the first term on the right-hand side represents the tangential component and the second one – the normal component of the total derivative of $\boldsymbol{u}_t$.

Derivative of the normal vector field $\boldsymbol{n}$

Since $\boldsymbol{n}$ is a unit vector field, the first information about $d\boldsymbol{n}/d\tau$ may be found by differentiating the scalar product $\boldsymbol{n}\cdot\boldsymbol{n} = 1$ with respect to τ

$$\frac{d\boldsymbol{n}}{d\tau}\cdot\boldsymbol{n} = 0 \tag{5.8}$$

Thus, the total time derivative $d\boldsymbol{n}/d\tau$ results in a tangential vector.

Moreover, as the vectors $\boldsymbol{u}_t$ and $\boldsymbol{n}$ remain orthogonal, the differentiation of the scalar product $\boldsymbol{n}\cdot\boldsymbol{u}_t = 0$ with respect to τ, leads to

$$\frac{d\boldsymbol{n}}{d\tau}\cdot\boldsymbol{u}_t = -\boldsymbol{n}\cdot\frac{d\boldsymbol{u}_t}{d\tau} \tag{5.9}$$

From (5.7) and (5.9) it is finally obtained

$$\frac{d\boldsymbol{n}}{d\tau} = (\operatorname{grad}_s\boldsymbol{n})\boldsymbol{v}_t \tag{5.10}$$

As it will be seen later, the total derivative of $\boldsymbol{n}$ will be used in the approach of shape sensitivity analysis of shells presented in this work.

6. Shape derivative of surface gradients

While in the formulation of Reissner's shell model, the knowledge of first order surface gradients were necessary, in the shape sensitivity analysis the shape derivative of these gradients must be carried out.

For a superficial vector field, we may write

$$\{\operatorname{grad}_s \boldsymbol{u}\}_m = (\nabla_s \widetilde{\boldsymbol{u}})\mathsf{F}^{-1} \tag{6.1}$$

where $\boldsymbol{u}$ and $\widetilde{\boldsymbol{u}}$ denote respectively the spatial and material (referential) descriptions of the displacement, ∇_s is the material surface gradient and F – the shape change gradient.

From (6.1) and (5.4), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}[(\nabla_s\widetilde{\boldsymbol{u}})\mathsf{F}^{-1}] &= \Big(\nabla_s\frac{\partial\widetilde{\boldsymbol{u}}}{\partial\tau}\Big)\mathsf{F}^{-1} + (\nabla_s\widetilde{\boldsymbol{u}})\frac{\partial}{\partial\tau}\mathsf{F}^{-1} = \\ &= \Big\{\operatorname{grad}_s\frac{d\boldsymbol{u}}{d\tau}\Big\}_m - \{\operatorname{grad}_s\boldsymbol{u}\}_m\{\operatorname{grad}_s\boldsymbol{v}_t\}_m\end{aligned} \tag{6.2}$$

Thus, combining (5.1) and (6.2), we obtain

$$\frac{d}{d\tau}\operatorname{grad}_s\boldsymbol{u} = \operatorname{grad}_s\frac{d\boldsymbol{u}}{d\tau} - \operatorname{grad}_s\boldsymbol{u}\operatorname{grad}_s\boldsymbol{v}_t \tag{6.3}$$

On the other hand, in terms of intrinsic components of the displacement $(\boldsymbol{u}_t, u_n)$, we may write

$$\begin{aligned}&\boldsymbol{u} = \boldsymbol{u}_t + u_n\boldsymbol{n}\\ &\frac{d\boldsymbol{u}}{d\tau} = \frac{d\boldsymbol{u}_t}{d\tau} + \frac{du_n}{d\tau}\boldsymbol{n} + u_n\frac{d\boldsymbol{n}}{d\tau}\\ &\operatorname{grad}_s\boldsymbol{u} = \operatorname{grad}_s\boldsymbol{u}_t + u_n\operatorname{grad}_s\boldsymbol{n} + \boldsymbol{n}\otimes\operatorname{grad}_s u_n\\ &\operatorname{grad}_s\frac{d\boldsymbol{u}}{d\tau} = \operatorname{grad}_s\frac{d\boldsymbol{u}_t}{d\tau} + \frac{du_n}{d\tau}\operatorname{grad}_s\boldsymbol{n} + \boldsymbol{n}\otimes\operatorname{grad}_s\frac{du_n}{d\tau} +\\ &+u_n\operatorname{grad}_s\frac{d\boldsymbol{n}}{d\tau} + \frac{d\boldsymbol{n}}{d\tau}\otimes\operatorname{grad}_s u_n\end{aligned} \tag{6.4}$$

If we insert $(6.4)_{3,4}$ into (6.3), we obtain

$$\begin{aligned}\frac{d}{d\tau}\operatorname{grad}_s u_n &= \operatorname{grad}_s\frac{du_n}{d\tau} - (\operatorname{grad}_s\boldsymbol{v}_t)^\top\operatorname{grad}_s u_n\\ \frac{d}{d\tau}\operatorname{grad}_s\boldsymbol{n} &= \operatorname{grad}_s\frac{d\boldsymbol{n}}{d\tau} - \operatorname{grad}_s\boldsymbol{n}\operatorname{grad}_s\boldsymbol{v}_t\\ \frac{d}{d\tau}\operatorname{grad}_s\boldsymbol{u}_t &= \operatorname{grad}_s\frac{d\boldsymbol{u}_t}{d\tau} - \operatorname{grad}_s\boldsymbol{u}_t\operatorname{grad}_s\boldsymbol{v}_t\end{aligned} \tag{6.5}$$

Derivative of the gradient of scalar field u_n

In the case of the gradient of a scalar-valued field, we recall $(6.5)_1$. Again, since $\operatorname{grad}_s u_n$ is a tangent vector field, introduction (2.6) into $(6.5)_1$ yields

$$\frac{d}{d\tau}\operatorname{grad}_s u_n = \operatorname{grad}_s \frac{du_n}{d\tau} - (\mathbf{\Pi}\operatorname{grad}_s \boldsymbol{v}_t)^\top \operatorname{grad}_s u_n \tag{6.6}$$

Therefore, the total derivative of $\operatorname{grad}_s u_n$ leads also to a tangent vector.

Derivative of the gradient of the normal vector field $\boldsymbol{n}$

Now, we return to $(6.5)_2$. Upon inserting (5.10) into $(6.5)_2$ and making further use of the following tensorial relation

$$\operatorname{grad}_s[(\operatorname{grad}_s \boldsymbol{n})^\top \boldsymbol{v}_t] = (\operatorname{grad}_s \operatorname{grad}_s \boldsymbol{n})^\top \boldsymbol{v}_t + (\operatorname{grad}_s \boldsymbol{n})^\top \operatorname{grad}_s \boldsymbol{v}_t \tag{6.7}$$

the total derivative of $\operatorname{grad}_s \boldsymbol{n}$ becomes

$$\frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{n} = (\operatorname{grad}_s \operatorname{grad}_s \boldsymbol{n})^\top \boldsymbol{v}_t \tag{6.8}$$

Derivative of the gradient of the tangential vector field $\boldsymbol{u}_t$

For any superficial vector field, in particular for the tangential vector field $\boldsymbol{u}_t$, similarly to (2.10), the following relation holds

$$\mathbf{\Pi}\operatorname{grad}_s \boldsymbol{u}_t = \operatorname{grad}_s \boldsymbol{u}_t + \boldsymbol{n} \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t \tag{6.9}$$

Next, by differentiating with respect to τ, we obtain

$$\frac{d}{d\tau}(\mathbf{\Pi}\operatorname{grad}_s \boldsymbol{u}_t) = \frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{u}_t + \frac{d\boldsymbol{n}}{d\tau} \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t + \boldsymbol{n} \otimes \frac{d}{d\tau}[(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t] \tag{6.10}$$

If we apply the projector tensor $\mathbf{\Pi}$ on both sides, and introducing (5.10), the foregoing expression reduces to

$$\mathbf{\Pi}\frac{d}{d\tau}(\mathbf{\Pi}\operatorname{grad}_s \boldsymbol{u}_t) = \mathbf{\Pi}\frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{u}_t + (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t \tag{6.11}$$

From $(6.5)_3$ and (6.11), we may write

$$\begin{aligned}\mathbf{\Pi}\frac{d}{d\tau}(\mathbf{\Pi}\operatorname{grad}_s \boldsymbol{u}_t) &= \mathbf{\Pi}\operatorname{grad}_s \frac{d\boldsymbol{u}_t}{d\tau} - \mathbf{\Pi}\operatorname{grad}_s \boldsymbol{u}_t \operatorname{grad}_s \boldsymbol{v}_t + \\ &+ (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t\end{aligned} \tag{6.12}$$

Finally, substitution of (5.7) into (6.12) yields

$$\boldsymbol{\Pi}\frac{d}{d\tau}\Big(\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t\Big) = \boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau} - [(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{v}_t\cdot\boldsymbol{u}_t]\,\mathrm{grad}_s\boldsymbol{n} - \tag{6.13}$$
$$-\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t\,\mathrm{grad}_s\boldsymbol{v}_t + (\mathrm{grad}_s\boldsymbol{n})\boldsymbol{v}_t\otimes(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t$$

With the preceding results, we can now perform the shape derivative of the strains in Reissner's model.

7. Shape derivative of the strains

In order to obtain the total derivatives of the stretching strain $\boldsymbol{\varepsilon}$, transverse shearing strain $\boldsymbol{\gamma}$, and flexural strain $\boldsymbol{\kappa}$, with respect to the parameter τ, we return to the strain-displacement relations given by (3.2). An inspection of these equations shows us that we have to perform the total derivatives of the unit normal vector field $\boldsymbol{n}$, the tangent vector fields $\boldsymbol{u}_t$ and $\boldsymbol{\vartheta}$. Further we shall perform the total derivatives of the surface gradient of u_n, $\boldsymbol{n}$, $\boldsymbol{u}_t$ and $\boldsymbol{\vartheta}$. Within the continuum mechanics analogy and using the general expressions of the material (total time) derivative of superficial fields, we carry out in this section the shape derivatives following the definition given in (2.3).

Shape derivative of the stretching strain $\boldsymbol{\varepsilon}$

From the definition of stretching $(3.2)_1$, the total time derivative of $\boldsymbol{\varepsilon}$ yields

$$\frac{d\boldsymbol{\varepsilon}}{d\tau} = \frac{d}{d\tau}(\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t) + \frac{du_n}{d\tau}\,\mathrm{grad}_s\boldsymbol{n} + u_n\frac{d}{d\tau}\,\mathrm{grad}_s\boldsymbol{n} \tag{7.1}$$

As it will be seen later, in the present approach the tangential component of $d\boldsymbol{\varepsilon}/d\tau$ will be relevant, thus

$$\boldsymbol{\Pi}\frac{d\boldsymbol{\varepsilon}}{d\tau} = \boldsymbol{\Pi}\frac{d}{d\tau}(\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t) + \frac{du_n}{d\tau}\,\mathrm{grad}_s\boldsymbol{n} + u_n\boldsymbol{\Pi}\frac{d}{d\tau}\,\mathrm{grad}_s\boldsymbol{n} \tag{7.2}$$

Upon introducing (6.13) into (7.2), the tangential component of the total derivative of the stretching strain can be written as

$$\boldsymbol{\Pi}\frac{d\boldsymbol{\varepsilon}}{d\tau} = -\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t\,\mathrm{grad}_s\boldsymbol{v}_t + u_n\boldsymbol{\Pi}\frac{d}{d\tau}\,\mathrm{grad}_s\boldsymbol{n} - [(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{v}_t\cdot\boldsymbol{u}_t]\,\mathrm{grad}_s\boldsymbol{n} + \tag{7.3}$$
$$+(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{v}_t\otimes(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + \boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau} + \frac{du_n}{d\tau}\,\mathrm{grad}_s\boldsymbol{n}$$

Shape derivative of the shearing strain $\boldsymbol{\gamma}$

Likewise, from $(3.2)_2$, the total derivative of $\boldsymbol{\gamma}$, takes the form

$$\frac{d\boldsymbol{\gamma}}{d\tau} = -\frac{d\boldsymbol{\vartheta}}{d\tau} + \frac{d}{d\tau}\operatorname{grad}_s u_n - \Big(\frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{n}\Big)\boldsymbol{u}_t - (\operatorname{grad}_s \boldsymbol{n})\frac{d\boldsymbol{u}_t}{d\tau} \tag{7.4}$$

By inserting the derivative of $\operatorname{grad}_s u_n$ from (6.6) into the above expression, the total derivative of the shearing strain leads to

$$\begin{aligned}\frac{d\boldsymbol{\gamma}}{d\tau} &= -(\boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{v}_t)^{\top}\operatorname{grad}_s u_n - \Big(\frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{n}\Big)\boldsymbol{u}_t - \frac{d\boldsymbol{\vartheta}}{d\tau} + \\ &+ \operatorname{grad}_s \frac{du_n}{d\tau} - (\operatorname{grad}_s \boldsymbol{n})\frac{d\boldsymbol{u}_t}{d\tau}\end{aligned} \tag{7.5}$$

Shape derivative of the flexural strain $\boldsymbol{\kappa}$

From $(3.2)_3$, the tangential component of the total derivative of $\boldsymbol{\kappa}$ may be written as

$$\boldsymbol{\Pi}\frac{d\boldsymbol{\kappa}}{d\tau} = \boldsymbol{\Pi}\frac{d}{d\tau}\boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{\vartheta} \tag{7.6}$$

Since the rotation $\boldsymbol{\vartheta}$ is a tangent vector, the evaluation of its total time derivative will be entirely similar to the evaluation of $\boldsymbol{\Pi} d(\boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{u}_t)/d\tau$ shown in (6.13), thus (7.6) may be rewritten as

$$\begin{aligned}\boldsymbol{\Pi}\frac{d\boldsymbol{\kappa}}{d\tau} &= -\boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{\vartheta}\operatorname{grad}_s \boldsymbol{v}_t - [(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \cdot \boldsymbol{\vartheta}]\operatorname{grad}_s \boldsymbol{n} + \\ &+ (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \otimes (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{\vartheta} + \boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau}\end{aligned} \tag{7.7}$$

8. Shell shape sensitivity

Let us begin the present section with differentiating the cost function with respect to the parameter τ. Due to the approach adopted in this work, in which the potential energy is chosen as the cost function, combining (3.1) and (3.3), we may rewrite $d\psi/d\tau$ as

$$\frac{d\psi}{d\tau} = \frac{d}{d\tau}U_\varepsilon + \frac{d}{d\tau}U_\gamma + \frac{d}{d\tau}U_\kappa - \frac{d}{d\tau}W \tag{8.1}$$

Thus, to perform the total derivative of the potential energy of the shell, it will be required to calculate the total derivative of each one of its terms.

Shape derivative of the stretching strain energy U_ε

Let us first consider the elastic stretching strain energy U_ε, given by $(3.4)_1$. Upon the application of Reynolds' transport theorem, the total derivative of U_ε with respect to τ yields

$$\frac{d}{d\tau}U_\varepsilon = \int\limits_\Omega \Big(\frac{d\phi_\varepsilon}{d\tau} + \phi_\varepsilon \operatorname{div}_s \boldsymbol{v}_t\Big)\, d\Omega = \int\limits_\Omega \Big(\frac{\partial \phi_\varepsilon}{\partial \boldsymbol{\varepsilon}^s} \cdot \frac{d\boldsymbol{\varepsilon}^s}{d\tau} + \phi_\varepsilon \operatorname{div}_s \boldsymbol{v}_t\Big)\, d\Omega \tag{8.2}$$

where $\operatorname{div}_s \boldsymbol{v}_t$ represents the surface divergence of the shape change velocity, defined by

$$\operatorname{div}_s \boldsymbol{v}_t = \mathbf{I}_p \cdot \boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{v}_t \tag{8.3}$$

where $\mathbf{I}_p$ denotes the identity tensor over the tangent plane.

Expression (8.2) can be rewritten as

$$\frac{d}{d\tau}U_\varepsilon = \int\limits_\Omega \Big(\mathbf{N} \cdot \boldsymbol{\Pi} \frac{d\boldsymbol{\varepsilon}}{d\tau} + \phi_\varepsilon \operatorname{div}_s \boldsymbol{v}_t\Big)\, d\Omega \tag{8.4}$$

Furthermore, by substituting (7.3) and (8.3) into (8.4), the total derivative of the stretching energy may be rewritten as

$$\begin{aligned}\frac{d}{d\tau}U_\varepsilon = &\int\limits_\Omega [\phi_\varepsilon \mathbf{I}_p - (\boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{u}_t)^\top \mathbf{N}] \cdot \operatorname{grad}_s \boldsymbol{v}_t d\Omega + \int\limits_\Omega u_n \mathbf{N} \cdot \boldsymbol{\Pi}\Big(\frac{d}{d\tau} \operatorname{grad}_s \boldsymbol{n}\Big) d\Omega - \\ &- \int\limits_\Omega [(\mathbf{N} \cdot \operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t - \mathbf{N}(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t] \cdot (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \, d\Omega + \\ &+ \int\limits_\Omega \mathbf{N} \cdot \Big[\boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{\Pi} \frac{d\boldsymbol{u}_t}{d\tau} + \frac{du_n}{d\tau} \operatorname{grad}_s \boldsymbol{n}\Big]\, d\Omega\end{aligned} \tag{8.5}$$

Shape derivative of the shearing strain energy U_γ

In the same manner as before, the total derivative of U_γ given in $(3.4)_2$ can be written as

$$\frac{d}{d\tau}U_\gamma = \int\limits_\Omega \Big(\frac{d\phi_\gamma}{d\tau} + \phi_\gamma \operatorname{div}_s \boldsymbol{v}_t\Big)\, d\Omega = \int\limits_\Omega \Big(\boldsymbol{Q} \cdot \frac{d\boldsymbol{\gamma}}{d\tau} + \phi_\gamma \operatorname{div}_s \boldsymbol{v}_t\Big)\, d\Omega \tag{8.6}$$

Thus, from (7.5), (8.3) and (8.6), the total derivative of the shearing energy becomes

$$\begin{aligned}\frac{d}{d\tau}U_\gamma = &\int_\Omega [\phi_\gamma \mathbf{I}_p - (\operatorname{grad}_s u_n \otimes \boldsymbol{Q})] \cdot \operatorname{grad}_s \boldsymbol{v}_t \, d\Omega - \\ &- \int_\Omega (\boldsymbol{Q} \otimes \boldsymbol{u}_t) \cdot \Big(\frac{d}{d\tau}\operatorname{grad}_s \boldsymbol{n}\Big) \, d\Omega + \\ &+ \int_\Omega \boldsymbol{Q} \cdot \Big[-\frac{d\boldsymbol{\vartheta}}{d\tau} + \operatorname{grad}_s \frac{du_n}{d\tau} - (\operatorname{grad}_s \boldsymbol{n})\frac{d\boldsymbol{u}_t}{d\tau}\Big] \, d\Omega\end{aligned} \tag{8.7}$$

Shape derivative of the flexural strain energy U_κ

Similarly, from $(3.4)_3$, we obtain

$$\frac{d}{d\tau}U_\kappa = \int_\Omega \Big(\frac{d\phi_\kappa}{d\tau} + \phi_\kappa \operatorname{div}_s \boldsymbol{v}_t\Big) \, d\Omega = \int_\Omega \Big(\mathbf{M} \cdot \boldsymbol{\Pi}\frac{d\boldsymbol{\kappa}}{d\tau} + \phi_\kappa \operatorname{div}_s \boldsymbol{v}_t\Big) \, d\Omega \tag{8.8}$$

Finally, from (7.7) and (8.3), the total derivative of the flexural energy equals

$$\begin{aligned}\frac{d}{d\tau}U_\kappa = &\int_\Omega [\phi_\kappa \mathbf{I}_p - (\boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{\vartheta})^\top \mathbf{M}] \cdot \operatorname{grad}_s \boldsymbol{v}_t \, d\Omega - \\ &- \int_\Omega [(\mathbf{M} \cdot \operatorname{grad}_s \boldsymbol{n})\boldsymbol{\vartheta} - \mathbf{M}(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{\vartheta}] \cdot (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{v}_t \, d\Omega + \\ &+ \int_\Omega \mathbf{M} \cdot \boldsymbol{\Pi}\operatorname{grad}_s \boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau} \, d\Omega\end{aligned} \tag{8.9}$$

Shape derivative of the external work W

Proceeding as before and assuming that the prescribed load at the boundary remains unchanged, the material derivative of W may be expressed as

$$\frac{d}{d\tau}W = \int_{\partial\Omega_t} \Big[\overline{\boldsymbol{t}} \cdot \frac{d}{d\tau}\boldsymbol{u}_t + \overline{q}\frac{d}{d\tau}u_n + \overline{\boldsymbol{m}} \cdot \frac{d}{d\tau}\boldsymbol{\vartheta} + (\overline{\boldsymbol{t}} \cdot \boldsymbol{u}_t + \overline{q}u_n + \overline{\boldsymbol{m}} \cdot \boldsymbol{\vartheta}) \operatorname{div}_s \boldsymbol{v}_t\Big] \, d\partial\Omega \tag{8.10}$$

In what follows we also assume that the shape change of the shell, characterized by the velocity field $\boldsymbol{v}_t$, is such that $\operatorname{div}_s \boldsymbol{v}_t = 0$ on $\partial\Omega$ and $\boldsymbol{v}_t = \mathbf{0}$ on

$\partial\Omega_u$. Hence, the preceding expression takes the more simple form

$$\begin{aligned}\frac{d}{d\tau}W &= \int\limits_{\partial\Omega_t}\Big[\overline{\boldsymbol{t}}\cdot\frac{d}{d\tau}\boldsymbol{u}_t+\overline{q}\frac{d}{d\tau}u_n+\overline{\boldsymbol{m}}\cdot\frac{d}{d\tau}\boldsymbol{\vartheta}\Big]\,d\partial\Omega = \\ &= \int\limits_{\partial\Omega_t}\Big[\overline{\boldsymbol{t}}\cdot\boldsymbol{\Pi}\frac{d}{d\tau}\boldsymbol{u}_t+\overline{q}\frac{d}{d\tau}u_n+\overline{\boldsymbol{m}}\cdot\boldsymbol{\Pi}\frac{d}{d\tau}\boldsymbol{\vartheta}\Big]\,d\partial\Omega\end{aligned} \tag{8.11}$$

Furthermore, by combination of (8.1), (8.5), (8.7), (8.9) and (8.11), the shape derivative of the potential energy of the shell can be written as

$$\begin{aligned}\frac{d\psi}{d\tau} &= \int\limits_{\Omega}\Big[(\phi_\varepsilon+\phi_\gamma+\phi_\kappa)\mathbf{I}_p-(\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{u}_t)^\top\mathbf{N}-\operatorname{grad}_s u_n\otimes\boldsymbol{Q}- \\ &-(\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{\vartheta})^\top\mathbf{M}\Big]\cdot\operatorname{grad}_s\boldsymbol{v}_t\,d\Omega+\int\limits_{\Omega}(u_n\mathbf{N}-\boldsymbol{Q}\otimes\boldsymbol{u}_t)\cdot\frac{d}{d\tau}\operatorname{grad}_s\boldsymbol{n}d\Omega- \\ &-\int\limits_{\Omega}\Big[(\mathbf{N}\cdot\operatorname{grad}_s\boldsymbol{n})\boldsymbol{u}_t-\mathbf{N}(\operatorname{grad}_s\boldsymbol{n})\boldsymbol{u}_t+(\mathbf{M}\cdot\operatorname{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}- \\ &-\mathbf{M}(\operatorname{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big]\cdot(\operatorname{grad}_s\boldsymbol{n})\boldsymbol{v}_t\,d\Omega+\int\limits_{\Omega}\mathbf{N}\cdot\Big(\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau}+\frac{du_n}{d\tau}\operatorname{grad}_s\boldsymbol{n}\Big)\,d\Omega+ \\ &+\int\limits_{\Omega}\boldsymbol{Q}\cdot\Big[-\frac{d\boldsymbol{\vartheta}}{d\tau}+\operatorname{grad}_s\frac{du_n}{d\tau}-(\operatorname{grad}_s\boldsymbol{n})\frac{d\boldsymbol{u}_t}{d\tau}\Big]\,d\Omega+\int\limits_{\Omega}\mathbf{M}\cdot\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau}\,d\Omega- \\ &-\int\limits_{\partial\Omega_t}\Big(\overline{\boldsymbol{t}}\cdot\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau}+\overline{q}\frac{du_n}{d\tau}+\overline{\boldsymbol{m}}\cdot\boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau}\Big)\,d\partial\Omega\end{aligned} \tag{8.12}$$

In addition, since the shell is in equilibrium with the prescribed loads along the boundary and taking into account that $\boldsymbol{\Pi}d\boldsymbol{u}_t/d\tau$, $du_n/d\tau$ and $\boldsymbol{\Pi}d\boldsymbol{\vartheta}/d\tau$ belong to K_{in}, the Principle of Virtual Power (4.1), leads to

$$\begin{aligned}&\int\limits_{\Omega}\mathbf{N}\cdot\Big(\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau}+\frac{du_n}{d\tau}\operatorname{grad}_s\boldsymbol{n}\Big)\,d\Omega+ \\ &+\int\limits_{\Omega}\boldsymbol{Q}\cdot\Big[-\frac{d\boldsymbol{\vartheta}}{d\tau}+\operatorname{grad}_s\frac{du_n}{d\tau}-(\operatorname{grad}_s\boldsymbol{n})\frac{d\boldsymbol{u}_t}{d\tau}\Big]\,d\Omega+ \\ &+\int\limits_{\Omega}\mathbf{M}\cdot\boldsymbol{\Pi}\operatorname{grad}_s\boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau}\,d\Omega-\int\limits_{\partial\Omega_t}\Big(\overline{\boldsymbol{t}}\cdot\boldsymbol{\Pi}\frac{d\boldsymbol{u}_t}{d\tau}+\overline{q}\frac{du_n}{d\tau}+\overline{\boldsymbol{m}}\cdot\boldsymbol{\Pi}\frac{d\boldsymbol{\vartheta}}{d\tau}\Big)\,d\partial\Omega=0\end{aligned} \tag{8.13}$$

Further, by combination of (8.12) and (8.13), the shape derivative of the potential energy of the shell reduces to

$$\begin{aligned}\frac{d\psi}{d\tau} &= \int_{\Omega} \Big[(\phi_\varepsilon + \phi_\gamma + \phi_\kappa)\mathbf{I}_p - (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t)^\top \mathbf{N} - \mathrm{grad}_s u_n \otimes \boldsymbol{Q} - \\ &-(\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{\vartheta})^\top \mathbf{M}\Big] \cdot \mathrm{grad}_s\boldsymbol{v}_t \, d\Omega + \int_{\Omega} (u_n\mathbf{N} - \boldsymbol{Q}\otimes\boldsymbol{u}_t) \cdot \frac{d}{d\tau}\mathrm{grad}_s\boldsymbol{n} d\Omega - \\ &- \int_{\Omega} \Big[(\mathbf{N}\cdot \mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot \mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} - \\ &-\mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big] \cdot (\mathrm{grad}_s\boldsymbol{n})\boldsymbol{v}_t \, d\Omega\end{aligned} \tag{8.14}$$

The expression in brackets in the first integral on the right-hand side of the above equation we denote as

$$\mathbf{\Sigma} = (\phi_\varepsilon + \phi_\gamma + \phi_\kappa)\mathbf{I}_p - (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t)^\top \mathbf{N} - \mathrm{grad}_s u_n \otimes \boldsymbol{Q} - (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{\vartheta})^\top \mathbf{M} \tag{8.15}$$

An inspection of this expression enables us to recognize the similarity between $\mathbf{\Sigma}$ and the energy momentum tensor, introduced by Eshelby (1975) in the analysis of defects in three-dimensional elasticity in the context of infinitesimal deformation. Thus $\mathbf{\Sigma}$ could be viewed as an extension of Eshelby's tensor used for the analysis of elastic shells within Reissner's approach. The energy momentum tensor $\mathbf{\Sigma}$ yields a tangential tensor and to point out the effects of the stretching, transversal shearing and flexural strain, it can be expressed as the sum of three terms

$$\mathbf{\Sigma} = \mathbf{\Sigma}_\varepsilon + \mathbf{\Sigma}_\gamma + \mathbf{\Sigma}_\kappa \tag{8.16}$$

where

$$\begin{aligned}&\mathbf{\Sigma}_\varepsilon = \phi_\varepsilon\mathbf{I}_p - (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{u}_t)^\top \mathbf{N} \qquad \mathbf{\Sigma}_\gamma = \phi_\gamma\mathbf{I}_p - \mathrm{grad}_s u_n \otimes \boldsymbol{Q} \\ &\mathbf{\Sigma}_\kappa = \phi_\kappa\mathbf{I}_p - (\mathbf{\Pi}\,\mathrm{grad}_s\boldsymbol{\vartheta})^\top \mathbf{M}\end{aligned} \tag{8.17}$$

Next, we insert (6.8) and (8.15) into (8.14), to obtain

$$\begin{aligned}\frac{d\psi}{d\tau} &= \int_{\Omega} \mathbf{\Sigma} \cdot \mathrm{grad}_s\boldsymbol{v}_t \, d\Omega + \int_{\Omega} (u_n\mathbf{N} - \boldsymbol{Q}\otimes\boldsymbol{u}_t) \cdot [(\mathrm{grad}_s\,\mathrm{grad}_s\boldsymbol{n})^\top \boldsymbol{v}_t] \, d\Omega - \\ &- \int_{\Omega} \Big[(\mathbf{N}\cdot \mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot \mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} - \\ &-\mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big] \cdot (\mathrm{grad}_s\boldsymbol{n})^\top\boldsymbol{v}_t \, d\Omega\end{aligned} \tag{8.18}$$

Finally, taking into account the definition of the transpose of the second and third order tensors, this expression may be written in a more suitable form

$$\begin{aligned}\frac{d\psi}{d\tau} &= \int_{\Omega} \boldsymbol{\Sigma} \cdot \operatorname{grad}_s \boldsymbol{v}_t \, d\Omega + \int_{\Omega} (\operatorname{grad}_s \operatorname{grad}_s \boldsymbol{n})(u_n \mathbf{N} - \boldsymbol{Q} \otimes \boldsymbol{u}_t) \cdot \boldsymbol{v}_t \, d\Omega - \\ &- \int_{\Omega} (\operatorname{grad}_s \boldsymbol{n}) \Big[(\mathbf{N} \cdot \operatorname{grad}_s \boldsymbol{n}) \boldsymbol{u}_t - \mathbf{N} (\operatorname{grad}_s \boldsymbol{n}) \boldsymbol{u}_t + (\mathbf{M} \cdot \operatorname{grad}_s \boldsymbol{n}) \boldsymbol{\vartheta} - \\ &- \mathbf{M} (\operatorname{grad}_s \boldsymbol{n}) \boldsymbol{\vartheta} \Big] \cdot \boldsymbol{v}_t \, d\Omega \end{aligned} \tag{8.19}$$

It should be emphasized that the shape derivative of the potential strain energy of the shell, given by the foregoing expression, is exclusively a function of the strain-stress state and the adopted *shape change velocity field* $\boldsymbol{v}_t$.

9. Boundary integral

Let us review in this section the shape derivative of the total potential energy of the shell in the light of the expression of the Reynolds' theorem that allows us to rewrite the mentioned derivative as a boundary integral.

To do so, we assume that the shape change of the shell is given by the tangential velocity field $\boldsymbol{v}_t$ defined along its boundary $\partial\Omega_t$ ($\boldsymbol{v}_t = 0$ along $\partial\Omega - \partial\Omega_t$). Then, we recall the definition of the potential energy of the shell that, in the present analysis, takes the following form

$$\psi(\boldsymbol{u}_t, u_n, \boldsymbol{\vartheta}) = U - W = \int_{\Omega} \phi(\boldsymbol{\varepsilon}^s, \boldsymbol{\gamma}, \boldsymbol{\kappa}^s) \, d\Omega - \int_{\partial\Omega_t} (\overline{\boldsymbol{t}} \cdot \boldsymbol{u}_t + \overline{q} u_n + \overline{\boldsymbol{m}} \cdot \boldsymbol{\vartheta}) \, d\partial\Omega \tag{9.1}$$

Next, consider the first term on the right-hand side of the above expression and by using Reynolds' theorem, the shape derivative of the strain energy of the shell yields

$$\frac{d}{d\tau} U = \int_{\Omega} \frac{\partial \phi}{\partial \tau} \, d\Omega + \int_{\partial\Omega} \phi \boldsymbol{v}_t \cdot \boldsymbol{m} \, d\partial\Omega = \int_{\Omega} \frac{\partial \phi}{\partial \tau} \, d\Omega + \int_{\partial\Omega_t} \phi \boldsymbol{v}_t \cdot \boldsymbol{m} \, d\partial\Omega \tag{9.2}$$

Further consider the second term on the right-hand side of (9.1). In the same manner as before and with the assumption that the prescribed loads at the boundary remain unchanged ($\overline{\boldsymbol{t}} d\partial\Omega$, $\overline{q} d\partial\Omega$ and $\overline{\boldsymbol{m}} d\partial\Omega$ are independent of the parameter τ), the shape derivative of the external work may be expressed as

$$\frac{d}{d\tau}W = \int\limits_{\partial\Omega_t} \overline{\boldsymbol{t}} \cdot \boldsymbol{\Pi}\Big[\frac{\partial}{\partial\tau}\boldsymbol{u}_t + (\operatorname{grad}_s \boldsymbol{u}_t)\boldsymbol{v}_t\Big]\, d\partial\Omega + \\ + \int\limits_{\partial\Omega_t} \overline{q}\Big[\frac{\partial}{\partial\tau}u_n + (\operatorname{grad}_s u_n)\cdot \boldsymbol{v}_t\Big]\, d\partial\Omega + \int\limits_{\partial\Omega_t} \overline{\boldsymbol{m}} \cdot \boldsymbol{\Pi}\Big[\frac{\partial}{\partial\tau}\boldsymbol{\vartheta} + (\operatorname{grad}_s \boldsymbol{\vartheta})\boldsymbol{v}_t\Big]\, d\partial\Omega \tag{9.3}$$

Upon combining (9.1), (9.2), (9.3), the total derivative of the potential energy becomes

$$\frac{d\psi}{d\tau} = \int\limits_{\Omega} \frac{\partial\phi}{\partial\tau}\, d\Omega - \int\limits_{\partial\Omega_t} \Big[\overline{\boldsymbol{t}} \cdot \boldsymbol{\Pi}\frac{\partial}{\partial\tau}\boldsymbol{u}_t + \overline{q}\frac{\partial}{\partial\tau}u_n + \overline{\boldsymbol{m}} \cdot \boldsymbol{\Pi}\frac{\partial}{\partial\tau}\boldsymbol{\vartheta}\Big]\, d\partial\Omega + \\ + \int\limits_{\partial\Omega_t} \Big[\phi\boldsymbol{v}_t \cdot \boldsymbol{m} - \overline{\boldsymbol{t}} \cdot (\boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{u}_t)\boldsymbol{v}_t - \overline{q}(\operatorname{grad}_s u_n)\cdot \boldsymbol{v}_t - \overline{\boldsymbol{m}} \cdot (\boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{\vartheta})\boldsymbol{v}_t\Big]\, d\partial\Omega \tag{9.4}$$

Moreover, as the shell is in equilibrium with the applied loads along its boundary, from the Principle of Minimum Total Potential Energy, we have

$$\int\limits_{\Omega} \frac{\partial\phi}{\partial\tau}\, d\Omega - \int\limits_{\partial\Omega_t} \Big[\overline{\boldsymbol{t}} \cdot \boldsymbol{\Pi}\frac{\partial}{\partial\tau}\boldsymbol{u}_t + \overline{q}\frac{\partial}{\partial\tau}u_n + \overline{\boldsymbol{m}} \cdot \boldsymbol{\Pi}\frac{\partial}{\partial\tau}\boldsymbol{\vartheta}\Big]\, d\partial\Omega = 0 \tag{9.5}$$

Therefore, from (9.4), (9.5) and the natural boundary conditions (4.7), the shape derivative of the total potential energy of the shell, becomes

$$\frac{d\psi}{d\tau} = \int\limits_{\partial\Omega_t} \boldsymbol{\Sigma}\boldsymbol{m} \cdot \boldsymbol{v}_t\, d\partial\Omega = \int\limits_{\partial\Omega} \boldsymbol{\Sigma}\boldsymbol{m} \cdot \boldsymbol{v}_t\, d\partial\Omega \tag{9.6}$$

This expression points out that when the change of the shape of the shell is performed by a tangential velocity defined along the boundary, according to our assumption, the shape derivative of the potential energy leads to a path integral that represents the flux of the Eshelby energy momentum tensor along the boundary of the shell.

The foregoing result allows us to know something more about the Eshelby's tensor for elastic shells.

10. Eshelby's momentum energy tensor

This section is devoted to perform the surface divergence of Eshelby's tensor within the framework of Reissner's shell theory. It should be noted that the surface divergence of a tangential tensor results in a vector with tangential and normal components. Nevertheless we will now focus our attention on the tangential component; the application to cracked shells will be conducted in the next section.

To do so, let us compare the shape derivative of the total potential energy expressed as a domain integral (8.19) with the same shape derivative carried out in (9.6). Thus, we may write

$$
\begin{aligned}
\frac{d\psi}{d\tau} &= \int\limits_{\partial\Omega} \boldsymbol{\Sigma}\boldsymbol{m}\cdot\boldsymbol{v}_t \, d\partial\Omega = \int\limits_{\Omega} \boldsymbol{\Sigma}\cdot\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{v}_t\, d\Omega + \\
&+ \int\limits_{\Omega} (\mathrm{grad}_s\,\mathrm{grad}_s\boldsymbol{n})(u_n\mathbf{N} - \boldsymbol{Q}\otimes\boldsymbol{u}_t)\cdot\boldsymbol{v}_t\, d\Omega - \int\limits_{\Omega} (\mathrm{grad}_s\boldsymbol{n})\Big[(\mathbf{N}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \\
&-\mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} - \mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big]\cdot\boldsymbol{v}_t\, d\Omega
\end{aligned}
\tag{10.1}
$$

Subsequently, we introduce the following tensor relation

$$
\mathrm{div}_s(\boldsymbol{\Sigma}^\top\boldsymbol{v}_t) = \boldsymbol{\Pi}\,\mathrm{div}_s\boldsymbol{\Sigma}\cdot\boldsymbol{v}_t + \boldsymbol{\Sigma}\cdot\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{v}_t \tag{10.2}
$$

Next, we integrate the preceding expression over the domain Ω and further we make use of the divergence theorem to obtain

$$
\int\limits_{\partial\Omega} \boldsymbol{\Sigma}\boldsymbol{m}\cdot\boldsymbol{v}_t\, d\partial\Omega = \int\limits_{\Omega} \boldsymbol{\Sigma}\cdot\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{v}_t\, d\Omega + \int\limits_{\Omega} \boldsymbol{\Pi}\,\mathrm{div}_s\boldsymbol{\Sigma}\cdot\boldsymbol{v}_t\, d\Omega \tag{10.3}
$$

Then, from (10.1) and (10.3), we have

$$
\begin{aligned}
&\int\limits_{\Omega} \Big\{\boldsymbol{\Pi}\,\mathrm{div}_s\boldsymbol{\Sigma} - (\mathrm{grad}_s\,\mathrm{grad}_s\boldsymbol{n})(u_n\mathbf{N} - \boldsymbol{Q}\otimes\boldsymbol{u}_t) + (\mathrm{grad}_s\boldsymbol{n})\Big[(\mathbf{N}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \\
&-\mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} - \mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big]\Big\}\cdot\boldsymbol{v}_t\, d\Omega = 0
\end{aligned}
\tag{10.4}
$$

Since the above equation should hold for arbitrary tangential velocity $\boldsymbol{v}_t$, the quantity in the parentheses must vanish, then

$$
\begin{aligned}
\boldsymbol{\Pi}\,\mathrm{div}_s\boldsymbol{\Sigma} &= \boldsymbol{\Pi}(\mathrm{grad}_s\,\mathrm{grad}_s\boldsymbol{n})(u_n\mathbf{N} - \boldsymbol{Q}\otimes\boldsymbol{u}_t) - (\mathrm{grad}_s\boldsymbol{n})\Big[(\mathbf{N}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \\
&-\mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} - \mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big]
\end{aligned}
\tag{10.5}
$$

Consequently, we have obtained the expression of the projection over the tangent plane of the surface divergence of Eshelby's tensor of elastic shells within the framework of Reissner's approach. On the other hand, using (10.5) together with the shape derivative of the potential energy expressed as a domain integral (8.19), we arrive to the equivalent expression of shape derivative as a path integral (9.6).

10.1. Circular cylindrical shell

Now, the shape derivative of the potential energy will be applied to a circular cylindrical shell. At the begining we carry out the first and second order surface gradient of $\boldsymbol{n}$, that characterize the geometry of the middle surface of the shell.

For circular cylindrical shell, one of the principal radii of curvature is infinite and the other is constant. Consequently, the second order surface gradient of $\boldsymbol{n}$ vanishes

$$\operatorname{grad}_s \operatorname{grad}_s \boldsymbol{n} = 0 \tag{10.6}$$

Moreover, if we assume over the tangent plane orthogonal base vectors $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_x$, respectively following the circumferential and the longitudinal directions, the only non-vanishing component of $\operatorname{grad}_s \boldsymbol{n}$ is given by

$$(\operatorname{grad}_s \boldsymbol{n})_{\phi\phi} = \boldsymbol{e}_\phi \cdot (\operatorname{grad}_s \boldsymbol{n})\boldsymbol{e}_\phi = r^{-1} \tag{10.7}$$

where r denotes the radius of curvature.

If we assume that (10.7) and (10.6) hold, it can be easily verified that the right-hand side of (10.5) vanishes, thus

$$\begin{aligned}&\boldsymbol{\Pi}(\operatorname{grad}_s \operatorname{grad}_s \boldsymbol{n})(u_n \mathbf{N} - \boldsymbol{Q} \otimes \boldsymbol{u}_t) - (\operatorname{grad}_s \boldsymbol{n})\Big[(\mathbf{N} \cdot \operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t - \\ &-\mathbf{N}(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M} \cdot \operatorname{grad}_s \boldsymbol{n})\boldsymbol{\vartheta} - \mathbf{M}(\operatorname{grad}_s \boldsymbol{n})\boldsymbol{\vartheta}\Big] = \mathbf{0}\end{aligned} \tag{10.8}$$

Then, combining (10.5) and (10.8) we obtain

$$\boldsymbol{\Pi} \operatorname{div}_s \boldsymbol{\Sigma} = \mathbf{0} \tag{10.9}$$

Therefore, for displacement fields $\boldsymbol{u}_t$, u_n and $\boldsymbol{\vartheta}$ in equilibrium with the applied loads on the boundary of a circular cylindrical shell, the surface divergence of Eshelby's tensor projected over the tangent plane vanishes.

Finally, combining (10.1) and (10.8), the following relations holds

$$\frac{d\psi}{d\tau} = \int_{\partial\Omega} \boldsymbol{\Sigma}\boldsymbol{m} \cdot \boldsymbol{v}_t \, d\partial\Omega = \int_{\Omega} \boldsymbol{\Sigma} \cdot \boldsymbol{\Pi} \operatorname{grad}_s \boldsymbol{v}_t \, d\Omega \tag{10.10}$$

11. Cracked elastic shells

In the present section we apply the analysis to the case of an elastic shell containing cracks.

Since Griffith provided the primary criterion for crack extension in linearly elastic bodies, the energy release rate has played an essential role in fracture mechanics. Therefore, our aim in this section is to evaluate the energy release rate of an elastic shell with arbitrary smooth middle surface containing a crack.

To do this, let us consider a plane $\mathcal{P}$ cutting the middle surface of the shell along the (smooth) curve $\mathcal{C}$, see Fig. 1. The crack is a part of this curve and its faces are denoted by $\mathcal{C}_c^+$ and $\mathcal{C}_c^-$, respectively. We assume that the shell is in equilibrium with a given traction at the boundary. For simplicity, body forces will not be considered and null traction along the faces of the crack will be assumed.

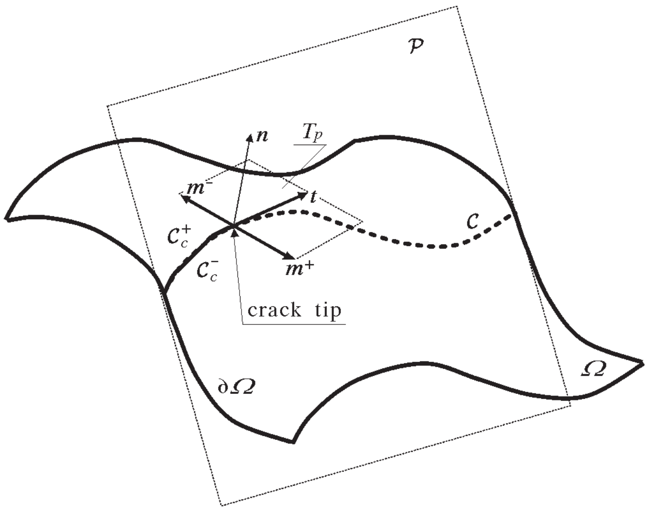

Fig. 1. Cracked shell

Let us also assume that the crack advances in a such form that the crack tip remains over the curve $\mathcal{C}$. Then, the crack initiation can be simulated as a shape change of the shell by choosing a suitable tangential velocity field, $\boldsymbol{v}_t$, over the cracked domain Ω (see Fig. 1). This tangential velocity function must be smooth, takes unitary value at the crack tip and remains tangent to the faces of the crack, $\boldsymbol{v}_t \cdot \boldsymbol{m}^{\pm} = 0$, and also vanishes along the boundary of the uncracked shell domain $(\partial\Omega)$.

With the aid of the previously developed shape sensitivity analysis, we can easily obtain the expression for the derivative of the potential strain energy with respect to crack advance. This derivative with negative sign, traditionally denoted by G, is known in fracture mechanics as the energy release rate. From (10.1) follows

$$
\begin{aligned}
G = &-\int_{\Omega} \boldsymbol{\Sigma}\cdot\boldsymbol{\Pi}\,\mathrm{grad}_s\boldsymbol{v}_t\;d\Omega - \int_{\Omega}(\mathrm{grad}_s\,\mathrm{grad}_s\boldsymbol{n})(u_n\mathbf{N}-\boldsymbol{Q}\otimes\boldsymbol{u}_t)\cdot\boldsymbol{v}_t\;d\Omega +\\
&+\int_{\Omega}(\mathrm{grad}_s\boldsymbol{n})\Big[(\mathbf{N}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t - \mathbf{N}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{u}_t + (\mathbf{M}\cdot\,\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta} -\\
&-\mathbf{M}(\mathrm{grad}_s\boldsymbol{n})\boldsymbol{\vartheta}\Big]\cdot\boldsymbol{v}_t\;d\Omega
\end{aligned}
\tag{11.1}
$$

11.1. Circular cylindrical shell with a crack

In this section, the above expression for G will be written for the case of a circular cylindrical shell containing a crack through its thickness, Yahsi and Erdogan (1983). The cutting plane $\mathcal{P}$ is inclined by an arbitrary angle θ to the circumferential direction (see Fig. 2).

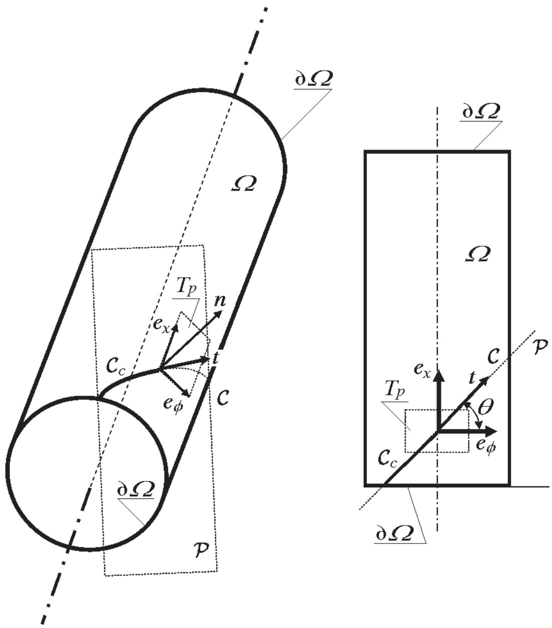

Fig. 2. Circular cylindrical shell with a crack

For this shell, the tangent plane at any point $\boldsymbol{x}$ of the middle surface Ω ($\boldsymbol{x} \in \Omega$), can be described by the intrinsic base system $\{\boldsymbol{e}_x, \boldsymbol{e}_\phi\}$. Then, the tangential unit vector $\boldsymbol{t}$, always tangent to the curve $\mathcal{C}$, can be defined as

$$\boldsymbol{t} = \cos\theta \boldsymbol{e}_\phi + \sin\theta \boldsymbol{e}_x \tag{11.2}$$

From the above definition, the shape change velocity field $\boldsymbol{v}_t$ takes the form

$$\boldsymbol{v}_t = \alpha \boldsymbol{t} \tag{11.3}$$

where $\alpha(\boldsymbol{x}) \in [0,1]$ is a smooth scalar (realvalued) field. At the tip of the crack $\alpha(\boldsymbol{x}) = 1$, on both crack surfaces, $0 < \alpha < 1$, and along the boundary of the domain $(\partial\Omega)$ $\alpha(\boldsymbol{x}) = 0$.

From the definition of $\boldsymbol{v}_t$ follows

$$\operatorname{grad}_s \boldsymbol{v}_t = \boldsymbol{t} \otimes \operatorname{grad}_s \alpha + \alpha \operatorname{grad}_s \boldsymbol{t} \tag{11.4}$$

Then, projection of this expression on the tangent plane is

$$\mathbf{\Pi} \operatorname{grad}_s \boldsymbol{v}_t = \boldsymbol{t} \otimes \operatorname{grad}_s \alpha + \alpha \mathbf{\Pi} \operatorname{grad}_s \boldsymbol{t} \tag{11.5}$$

From (10.10) the energy release rate as a domain integral can be expressed by

$$G = -\frac{d\psi}{d\tau} = -\int_\Omega \mathbf{\Sigma} \cdot \operatorname{grad}_s \boldsymbol{v}_t \, d\Omega = -\int_\Omega \mathbf{\Sigma} \cdot (\boldsymbol{t} \otimes \operatorname{grad}_s \alpha + \alpha \mathbf{\Pi} \operatorname{grad}_s \boldsymbol{t}) \, d\Omega =$$
$$= -\int_\Omega (\boldsymbol{t} \cdot \mathbf{\Sigma} \operatorname{grad}_s \alpha + \alpha \mathbf{\Sigma} \cdot \mathbf{\Pi} \operatorname{grad}_s \boldsymbol{t}) \, d\Omega \tag{11.6}$$

As a path integral, the energy release rate becomes

$$G = -\frac{d\psi}{d\tau} = J = \int_\Gamma \alpha \mathbf{\Sigma} \boldsymbol{m} \cdot \boldsymbol{t} \, d\Gamma \tag{11.7}$$

Here Γ is any contour around the tip of the crack over the middle surface of the shell and whose two end points lie on the crack faces $\mathcal{C}_c^+$ and $\mathcal{C}_c^-$.

12. Final remarks

The present paper shows a straightforward use of the (continuous) variational formulation linked to the direct method of sensitivity analysis to obtain the shape derivative of the total potential energy stored in a shell within the framework of Reissner's theory.

To perform the shape derivative, the analogy with material (total time) derivative of Continuum Mechanics is widely explored. In fact, the spatial description of this derivative and the use of some well-known expressions of mechanics vastly simplify this task.

The intrinsic surface frame composed of both the unit normal vector and the tangent plane in each point of the middle surface of the shell is employed. Moreover, the procedure and the results are presented in a compact notation (independent of the coordinate system) to point out the advantage of this formulation. By doing so the physical meaning of the model and the shape derivatives are preserved and the resulting expressions are not obscured by an excess of notations.

In dealing with general elastic shells containing through cracks, if crack advance is simulated by a suitable change of shape, the shape sensitivity analysis can be used as a systematic methodology to obtain the energy release rate.

Moreover, the energy release rate expression obtained in the present work requires the evaluation of the displacement $(\boldsymbol{u}_t, u_n, \boldsymbol{\vartheta})$ solution of the state equation (equilibrium equation in our case) and the definition of the shape change velocity field $\boldsymbol{v}_t$. In practical evaluation of the energy release rate, as we are free to select the velocity, we can take advantage of choosing the more convenient distribution over the middle surface of the shell. Thus, the energy release rate expression for cracked shells conducted in the present study is meaningful in both the theoretical and practical aspects.

In shells with arbitrary middle surface, this procedure led to a surface integral in which the Eshelby's tensor naturally appears. It was also verified that, in spite of the considered null body force in the analysis of the shell, the divergence of Eshelby's tensor did not vanish. However, in the particular case of circular cylindrical shell, the divergence of Eshelby's tensor vanishes. In this case it is simple to show the equivalence between the surface integral and the integral along a contour around the crack tip lying over the middle surface of the shell. This integral, well known in fracture mechanics as the Rice-Eshelby-Cherepanov J-integral, remains path independent and also represents a useful alternative to evaluate the energy release rate of circular cylindrical shells containing through cracks.

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Analiza wrażliwości kształtu podatnych powłok z pęknięciami

Streszczenie

Badania opisane w pracy dotyczą zastosowania analizy wrażliwości kształtu jako systematycznej metodologii wyznaczania tempa uwalnianej energii powłok z pęknięciami w ramach liniowego podejścia uwzględniającego efekt deformacji od ścinania poprzecznego. Ta metodologia i bezpośrednia analiza wrażliwości kształtu została zastosowana do powłok o dowolnej powierzchni środkowej, pozwalając na znalezienie

jawnego i ogólnego wyrażenia na pochodną całkowitej energii odkształcenia. W podatnych powłokach z pęknięciami symulację inicjacji pęknięcia dokonano na podstawie zmiany kształtu określonej odpowiednim rozkładem prędkości powierzchni środkowej powłoki. W takim przypadku użyteczną formułę określającą tempo uwalnianej energii wyznaczono w funkcji stanu naprężenia i odkształcenia oraz zmian rozkładu pola prędkości powierzchni środkowej. Na koniec, analizę wrażliwości kształtu zastosowano do szczególnego przypadku powłoki cylindrycznej, gdzie warunek zerowej dywergencji odpowiadającego tensora Eshelby'ego został potwierdzony.

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