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SIMILARITY SOLUTIONS TO BOUNDARY LAYER EQUATIONS FOR THIRD-GRADE NON-NEWTONIAN FLUID IN SPECIAL COORDINATE SYSTEM

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Two dimensional equations of steady motion for third order fluids are expressed in a special coordinate system generated by the potential flow corresponding to an inviscid fluid. For the inviscid flow around an arbitrary object, the streamlines are the ϕ-coordinates and the velocity potential lines are ψ-coordinates which form an orthogonal curvilinear set of coordinates. The outcome, boundary layer equations, is then shown to be independent of the body shape immersed into the flow. As the first approximation, it is assumed that the second grade terms are negligible compared to the viscous and third grade terms. The second grade terms spoil scaling transformation which is the only transformation leading to similarity solutions for a third grade fluid. By using Lie's group methods, infinitesimal generators of boundary layer equations are calculated. The equations are transformed into an ordinary differential system. Numerical solutions to the outcoming nonlinear differential equations are found by using a combination of the Runge-Kutta algorithm and a shooting technique.

Key words: boundary layer equations, Lie's groups, third grade fluids

1. Introduction

As a non-Newtonian fluid model, Rivlin-Ericksen fluids gained much acceptance from both theorists and experimenters. Special cases of the model, which is the fluid of the third grade, are extensively used, and a lot of works have been done on the subject. Several boundary layer equations are derived for different non-Newtonian models. For the sake of brevity, we mentioned only a few examples. Acrivos *et al.* (1960) and Pakdemirli (1996) derived boundary layer equations for power-law fluids. For the rate type of fluids, the works due to Beard and Walters (1964) and Astin *et al.* (1973) are of significant importance. The multiple deck boundary layer concept has been applied to the second and third grade fluids by Pakdemirli (1994). Yürüsoy and Pakdemirli (1999) considered boundary layer equations for third grade fluids over a stretching sheet in Cartesian coordinates.

We choose a convenient coordinate system first purposed by Kaplun (1954), which makes the equations independent of the body shape immersed into the flow. The coordinate system is an orthogonal curvilinear system in which ϕ-coordinates are the streamlines and ψ-coordinates are the velocity potentials of the inviscid flow past a two-dimensional arbitrary profile. The boundary layer equations of Newtonian fluids in this coordinate system are given by Kevorkian and Cole (1981). The boundary layer equations of the second-grade fluids in this coordinate system are derived by Pakdemirli and Suhubi (1992a), and the general symmetry groups for the equations are calculated using exterior calculus by the same authors (1992b). They showed that second-grade boundary layer equations accept only scaling transformation, and they presented a similarity solution corresponding to this transformation. For the fluids of grade three Pakdemirli (1992) showed that the additional term, due to the third grade, prevent the applicability of the scaling transformation, hence no similarity solutions exist.

First of all, in this article, in deriving boundary layer equations we use a fluid of grade three as a non-Newtonian fluid model. It is shown by Rivlin and Ericksen (1955) that the stress tensor is given by following relation

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1 \tag{1.1}$$

where p is the pressure, μ is the viscosity, α_1 and α_2 are the second grade fluid terms, β is the third grade fluid term, and $\mathbf{A}_1$, $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors given by the relations

$$\begin{aligned} &\mathbf{L} = \operatorname{grad}\boldsymbol{v} \qquad\qquad \mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top \\ &\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^\top\mathbf{A}_1 \end{aligned} \tag{1.2}$$

where $\boldsymbol{v}$ is the velocity vector. Rivlin and Ericksen (1955) showed that making equation (1.1) compatible with the thermodynamics and minimizing the free energy when the fluid is at rest, the material constants should satisfy the relations

$$
\begin{aligned}
&\mu \geqslant 0 \qquad && \alpha_1 \geqslant 0 \\
&\beta \geqslant 0 \qquad && |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta}
\end{aligned}
\tag{1.3}
$$

The dimensionless form of the equations of motion for a third grade fluid are (Pakdermirli, 1992)

$$
\begin{aligned}
&\frac{1}{2}\operatorname{grad}|\boldsymbol{q}|^2 + \boldsymbol{\omega}\times\boldsymbol{q} = -\operatorname{grad}p + \varepsilon\nabla^2\boldsymbol{q} + \varepsilon_1(\nabla^2\boldsymbol{\omega}\times\boldsymbol{q}) + \varepsilon_1\operatorname{grad}(\boldsymbol{q}\cdot\nabla^2\boldsymbol{q}) + \\
&+\frac{1}{4}(2\varepsilon_1+\varepsilon_2)\operatorname{grad}|A_1|^2 + (\varepsilon_1+\varepsilon_2)\Big[A_1\cdot\nabla^2\boldsymbol{q} + 2\operatorname{div}\Big(\operatorname{grad}\boldsymbol{q}(\operatorname{grad}\boldsymbol{q})^\top\Big)\Big] + \\
&+\varepsilon_3\mathbf{A}_1\cdot\operatorname{grad}|\mathbf{A}_1|^2 + \varepsilon_3|\mathbf{A}_1|^2\nabla^2\boldsymbol{q} \\
&\operatorname{div}\boldsymbol{q} = 0
\end{aligned}
\tag{1.4}
$$

where $\boldsymbol{q}$ is the dimensionless velocity vector, ∇ denotes Laplacian, $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{q}$ and the dimensionless coefficients are defined as follows

$$
\begin{aligned}
&\varepsilon = \frac{\mu}{\rho UL} = \frac{1}{\mathrm{Re}} \qquad && \varepsilon_1 = \frac{\alpha_1}{\rho L^2} \\
&\varepsilon_2 = \frac{\alpha_2}{\rho L^2} \qquad && \varepsilon_3 = \frac{\beta U}{\rho L^3}
\end{aligned}
\tag{1.5}
$$

where L and U are some reference length and velocity, respectively, ρ is the density, Re is the Reynolds number.

2. Coordinate system

A special coordinate system making the equations independent of the body shape is chosen (Fig. 1). The ϕ coordinate is related to streamlines and the ψ coordinate to velocity potential lines of the inviscid flow past an arbitrary object.

If we defined a complex function

$$
F(z) = \phi + \mathrm{i}\psi
\tag{2.1}
$$

we can easily write the well-known formulas

$$
\begin{aligned}
&\boldsymbol{q}_0 = u_0\boldsymbol{i} + v_0\boldsymbol{j} \qquad && u_0 - \mathrm{i}v_0 = F'(z) \\
&u_0 = \phi_x = \psi_y \qquad && v_0 = \phi_y = -\psi_x
\end{aligned}
\tag{2.2}
$$

where $\boldsymbol{q}_0$ is the potential velocity field. The pressure follows from Bernoulli's equation as

$$p = -\frac{1}{2}q_0^2 + C \tag{2.3}$$

where C is a constant. The metric for the system can be then defined as

$$dz = \frac{dF}{F'} \qquad (dx)^2 + (dy)^2 = \frac{(d\phi)^2 + (d\psi)^2}{|F'(z)|^2} = \frac{(d\phi)^2 + (d\psi)^2}{q_0^2} \tag{2.4}$$

To simplify the equations of motion, we introduce new velocity components as follows

$$W_\phi = \frac{q_\phi}{q_0} \qquad W_\psi = \frac{q_\psi}{q_0} \tag{2.5}$$

The velocity and gradient operator in this coordinate system are given by

$$\boldsymbol{q} = q_0(W_\phi \boldsymbol{i}_\phi + W_\psi \boldsymbol{i}_\psi) \qquad \nabla = \Big(q_0\frac{\partial}{\partial\phi}, q_0\frac{\partial}{\partial\psi}\Big) \tag{2.6}$$

In our case the Christoffel symbols are

$$\begin{aligned} &\Gamma^\phi_{\phi\phi} = \Gamma^\psi_{\psi\phi} = -\frac{\partial Q}{\partial\phi} \qquad &&\Gamma^\phi_{\psi\phi} = \Gamma^\psi_{\psi\psi} = -\frac{\partial Q}{\partial\psi} \\ &\Gamma^\phi_{\psi\psi} = \frac{\partial Q}{\partial\phi} &&\Gamma^\psi_{\phi\phi} = \frac{\partial Q}{\partial\psi} \end{aligned} \tag{2.7}$$

where $Q = \log q_0$.

3. Boundary layer equations

We have now necessary tools to obtain the boundary layer equations for a special third grade fluid. As the first approximation, the assume that the second grade terms are negligible compared to the viscous and third grade terms. The second grade terms spoil the scaling transformations which is the only transformation leading to similarity solutions for third grade fluids (see Pakdemirli, 1992). Equation $(1.4)_1$ is reduced to that of a third grade fluid if we take $\varepsilon_1 = \varepsilon_2 = 0$.

We assume that ε_3 is proportional to ε^2

$$\varepsilon_3 = k\varepsilon^2 \tag{3.1}$$

The method of matched asymptotic expansions will be used in the derivation. We have to construct an inner expansion inside the boundary layer and outer expansion outside out it. Letting the perturbation parameter $\varepsilon \to 0$, we have to obtain the limit flow which is inviscid and irrotational when

$$W_\phi = 1 \qquad W_\psi = 0 \qquad p = -\frac{1}{2}q_0^2 + C \tag{3.2}$$

The outer expansion will then consist of the first terms in (3.2) and of corrections due to the boundary layer as follows

$$\begin{aligned} W_\phi(\phi,\psi;\varepsilon) &= 1 + \beta(\varepsilon)W_\phi^1(\phi,\psi) + ... \\ W_\psi(\phi,\psi;\varepsilon) &= \beta(\varepsilon)W_\psi^1(\phi,\psi) + ... \\ p(\phi,\psi;\varepsilon) &= -\frac{1}{2}q_0^2 + C + \beta(\varepsilon)P^1(\phi,\psi) + ... \end{aligned} \tag{3.3}$$

where $\beta(\varepsilon)$ is as yet an unknown coefficient to be determined from matching with the restriction that $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The inner expansion variable is defined by stretching the ψ coordinate

$$\psi^* = \frac{1}{\delta(\varepsilon)}\psi \tag{3.4}$$

with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then the inner expansion will be

$$\begin{aligned} W_\phi(\phi,\psi;\varepsilon) &= W_\phi(\phi,\psi^*) + \delta(\varepsilon)W_\phi^1(\phi,\psi^*) + ... \\ W_\psi(\phi,\psi;\varepsilon) &= \delta(\varepsilon)W_\psi(\phi,\psi^*) + \delta^2(\varepsilon)W_\psi^1(\phi,\psi^*) + ... \\ p(\phi,\psi;\varepsilon) &= P(\phi,\psi^*) + \delta(\varepsilon)P^1(\phi,\psi) + ... \end{aligned} \tag{3.5}$$

The form above leads to a nontrivial continuity equation, and the inviscid velocity inside the boundary layer will approach the velocity on the surface as follows

$$q_0(\phi,\psi) = q_B(\phi) + O(\delta(\varepsilon)) \tag{3.6}$$

If we substitute equations (3.4)-(3.6) into equations of motion (1.4) and retain relatively larger terms in each group, we obtain the equations

$$\begin{aligned} & W_\phi\frac{\partial W_\phi}{\partial\phi} + W_\psi\frac{\partial W_\phi}{\partial\psi} + W_\phi^2\frac{dQ_B}{d\phi} = -\frac{1}{q_B^2}\frac{\partial P}{\partial\phi} + \frac{\varepsilon}{\delta^2}\frac{\partial^2 W_\phi}{\partial\psi^2} + \\ & +6k\frac{\varepsilon}{\delta^4}q_B^4(\phi)\frac{\partial^2 W_\phi}{\partial\psi^2}\Big(\frac{\partial W_\phi}{\partial\psi}\Big)^2 \\ & 0 = -\frac{1}{\delta}\frac{1}{q_B^2}\frac{\partial P}{\partial\psi} + \varepsilon\Big(O\Big(\frac{1}{\delta^3}\Big)\Big) \\ & \frac{\partial W_\phi}{\partial\phi} + \frac{\partial W_\psi}{\partial\psi} = 0 \end{aligned} \tag{3.7}$$

where δ, the boundary layer thickness, is a small parameter and ε is the perturbation parameter. We eliminate the pressure in equation $(3.7)_1$ by using the equation $P=-\frac{1}{2}q_B^2+C$. Now we can assume that ε is of order δ^2 and k of order δ^2. On these assumptions, we finally write the boundary layer equations for a special third grade fluid and the boundary conditions as follows

$$
\begin{gathered}
\frac{\partial W_\phi}{\partial \phi}+\frac{\partial W_\psi}{\partial \psi}=0 \\
W_\phi \frac{\partial W_\phi}{\partial \phi}+W_\psi \frac{\partial W_\phi}{\partial \psi}+(W_\phi^2-1)\frac{dQ_B}{d\phi}=\frac{\partial^2 W_\phi}{\partial \psi^2}+6kq_B^4(\phi)\frac{\partial^2 W_\phi}{\partial \psi^2}\Big(\frac{\partial W_\phi}{\partial \psi}\Big)^2 \\
W_\phi(\phi,0)=W_\psi(\phi,0)=0 \qquad W_\phi(\phi,\infty)=1
\end{gathered}
\tag{3.8}
$$

where $Q_B'=q_B'/q_B$, k are the third grade fluid coefficients. $k=0$ corresponds to the Newtonian flow. Note that the final equations are valid for arbitrary profiles because the inviscid surface velocity distribution q_B appears as an arbitrary function ϕ in the equations. It is therefore much more straightforward to draw a general conclusion from equations. Lie's group transformations may be then useful in investigating particular forms of q_B so that the partial differential equations could be reduced to ordinary differential equations via the similarity transformation.

Lie's group theory is applied to the equations. The equations admit a scaling symmetry. The scaling symmetry is used to transform the system of partial differential equations into a system of ordinary differential ones. Numerical solutions to the resulting nonlinear ordinary differential equations are found by using a combination of the Runge-Kutta algorithm and a shooting technique.

4. Equations determining infinitesimals generators

To find all possible exact solutions to equations $(3.8)_{1,2}$, we prefer using the general method of Lie's group analysis rather than using special group transformations. Details on the application of Lie's groups to solutions to differential equations can be found by Fosdick amd Rajagopal (1980), Bluman and Kumei (1989).

A one parameter Lie's group of transformations and the corresponding generator X is defined as follows

$$
\begin{aligned}
\phi^* &= \phi + \varepsilon\xi_1(\phi, \psi, W_\phi, W_\psi) \\
\psi^* &= \psi + \varepsilon\xi_2(\phi, \psi, W_\phi, W_\psi) \\
W_\phi^* &= W_\phi + \varepsilon\eta_1(\phi, \psi, W_\phi, W_\psi) \\
W_\psi^* &= W_\psi + \varepsilon\eta_2(\phi, \psi, W_\phi, W_\psi)
\end{aligned}
\tag{4.1}
$$

$$
X = \xi_1\frac{\partial}{\partial\phi} + \xi_2\frac{\partial}{\partial\psi} + \eta_1\frac{\partial}{\partial W_\phi} + \eta_2\frac{\partial}{\partial W_\psi}
\tag{4.2}
$$

By carrying out a straightforward and tedious algebra, we obtained the following infinitesimals and equations

$$
\begin{aligned}
&\xi_1 = \xi_1(\phi) && \xi_2 = c_2(\phi)\psi + \alpha(\phi) \\
&\eta_1 = c_1(\phi)W_\phi && \eta_2 = W_\phi(\psi c_2' + \alpha') + W_\psi(c_1 + c_2 - \xi_1')
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&2c_1\frac{q_B'}{q_B} + c_1' = 0 && c_1' + c_2' = 0 \\
&c_1 + 2c_2 - \xi_1' = 0 && 4c_2 - c_1 - \xi_1' - 4\xi_1\frac{q_B'}{q_B} = 0
\end{aligned}
\tag{4.4}
$$

From the above equations, we conclude that either $q_B' = 0$ or $c_1 = 0$. The two cases will be considered separately.

i) $q_B =$ const

Solving equations (4.3) and (4.4), we finally obtain the form of the so-called infinitesimals

$$
\begin{aligned}
&\xi_1 = 3a\phi + b && \xi_2 = a\psi + \alpha(\phi) \\
&\eta_1 = aW_\phi && \eta_2 = -aW_\psi + W_\phi\alpha'
\end{aligned}
\tag{4.5}
$$

These results agree with the ones by Yürüsoy and Pakdemirli (1999).

ii) $c_1 = 0$

Solving equations (4.3) and (4.4), we finally obtain the infinitesimals

$$
\begin{aligned}
&\xi_1 = 2a\phi + b && \xi_2 = a\psi + \alpha(\phi) \\
&\eta_1 = 0 && \eta_2 = -aW_\psi + W_\phi\alpha'
\end{aligned}
\tag{4.6}
$$

Solving equation $(4.4)_4$, we find that

$$q_B = c\sqrt[4]{2a\phi + b} \tag{4.7}$$

If we consider case **i**, the problem can be transformed to the conventional boundary layer problem, which was discussed by Yürüsoy and Pakdemirli (1999). Therefore, it supplies no new information. Only case **ii**, which is a scaling transformation, supplies useful information leading to the similarity solutions.

Imposing the restrictions from the boundaries and from equation $(3.8)_3$ on the boundary conditions on the infinitesimals, one obtains the following form of equations (4.6) and (4.7)

$$\begin{aligned} &\xi_1 = 2a\phi + b && \xi_2 = a\psi \\ &\eta_1 = 0 && \eta_2 = -aW_\psi \\ &q_B = c\sqrt[4]{2a\phi + b} \end{aligned} \tag{4.8}$$

where c is an arbitrary constant. Only this infinitesimal generator and the form of q_B, which is a scaling transformation, supplies useful information leading to the similarity solutions. Note that q_B is not a constant but a parabolic function.

5. Similarity solution

In this section, we will derive the similarity transformations and solutions using the infinitesimals given in (4.8). First we transform the equations into a system of ordinary differential one, and solve this system numerically using the Runge-Kutta method with shooting.

Leaving the details of the procedure, thoroughly described by Fosdick amd Rajagopal (1980), Bluman and Kumei (1989), we choose only the scaling transformation $(a = 1, b = 0)$. The characteristic equations are

$$\frac{d\phi}{2\phi} = \frac{d\psi}{\psi} = \frac{dW_\phi}{0} = \frac{dW_\psi}{-W_\psi} \tag{5.1}$$

The similarity variable, similarity functions and q_B are

$$\xi = \frac{\psi}{\sqrt{\phi}} \qquad W_\phi = f(\xi) \qquad W_\psi = \frac{g(\xi)}{\sqrt{\phi}} \qquad q_B = \sqrt[4]{\gamma(\phi)} \tag{5.2}$$

where $\gamma = c\sqrt[4]{2}$. Substituting equation (5.2) and their derivatives into boundary layer equations $(3.8)_{1,2}$, we finally obtain

$$\begin{aligned} &\frac{1}{2}(f^2-1)+2gf'-\xi ff'-2f''-12\kappa f''f'^2=0 \\ &\xi f'-2g'=0 \end{aligned} \tag{5.3}$$

and the boundary conditions take the form

$$f(0)=g(0)=0 \qquad\qquad f(\infty)=1 \tag{5.4}$$

where $\kappa = k^*\gamma^4$.

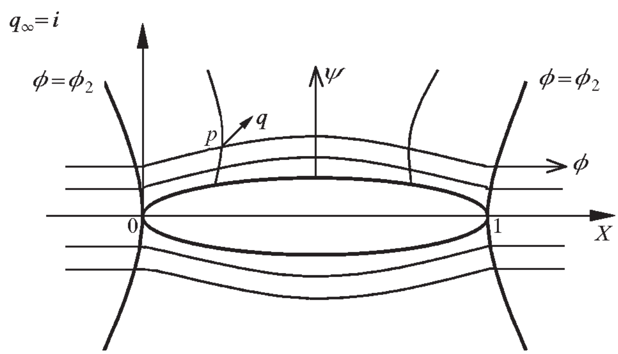

Fig. 1. Orthogonal coordinate system (streamlines of inviscid flow are ϕ coordinates, velocity potential lines are ψ coordinates)

Since the equations are highly nonlinear, a numerical approach towards the solution would be more appropriate. Although the problem is a boundary value problem, it is converted to an initial value problem. We assign a trial value to $f'(0)$, integrate the equations using the Runge-Kutta algorithm and check whether the boundary condition is satisfied at infinity. We repeat the procedure until we find an appropriate of value $f'(0)$. Numerical results for various non-Newtonian coefficients κ are plotted in Fig. 2 - Fig. 4. These figures present the functions f, g and f', respectively, for κ equal to 0, 10 and 30. For $\kappa = 0$ the flow is Newtonian. An increase in κ yields an increase in the non-Newtonian behaviour. From Fig. 2 we conclude that the boundary layer thickness grows as the non-Newtonian effects increase in magnitude. Figure 3 shows the vertical component of the velocity inside the boundary layer. $g(\xi)$ increases when the non-Newtonian effects get intensified.

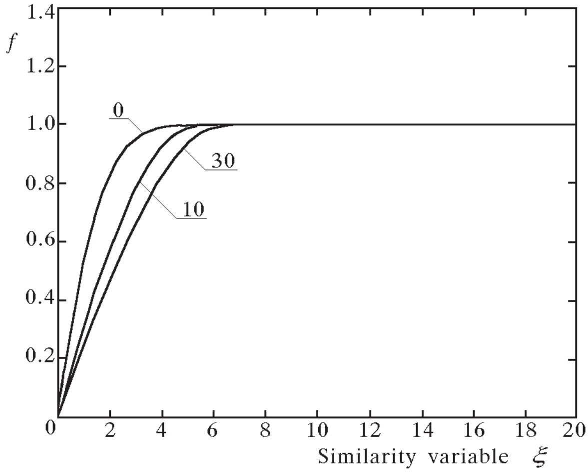

Fig. 2. Function f for various values of κ (as indicated on the curves)

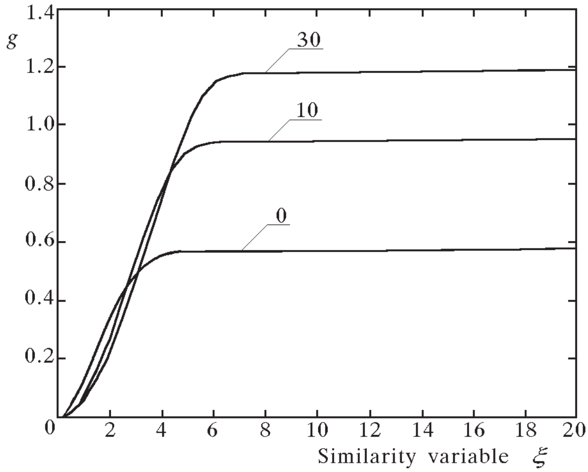

Fig. 3. Function g for various values of κ (as indicated on the curves)

The shear stress at the boundary is calculated from equation (1.1) using the coordinate properties and neglecting the small term. The dimensionless shear stress on the boundary comes out to be

$$t_{\phi\psi} = \frac{1}{\sqrt{\mathrm{Re}}}\Big[q_B^2 \frac{\partial W_\phi}{\partial \psi} + 2kq_B^6 \Big(\frac{\partial W_\phi}{\partial \psi}\Big)^3\Big]_{\psi=0} \tag{5.5}$$

In terms of the similarity variables, the shear stress is

$$t_{\phi\psi} = \frac{\delta^2}{\sqrt{\mathrm{Re}}}\Big[f'(0) + 2\kappa(f'(0))^3\Big] \tag{5.6}$$

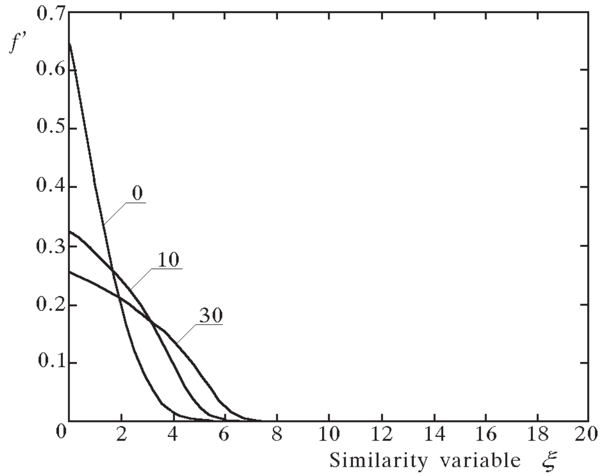

Fig. 4. First derivative of f for various values of κ (as indicated on the curves)

In equation (5.5), $f'(0)$ is to be read from Fig. 4, which gives

$$t_{\phi\psi} \cong \begin{cases} \dfrac{0.65}{\sqrt{\mathrm{Re}}} & \text{for} \quad \kappa = 0 \quad \text{and} \quad \delta = 1 \\ \dfrac{1.00}{\sqrt{\mathrm{Re}}} & \text{for} \quad \kappa = 10 \quad \text{and} \quad \delta = 1 \\ \dfrac{1.25}{\sqrt{\mathrm{Re}}} & \text{for} \quad \kappa = 30 \quad \text{and} \quad \delta = 1 \end{cases}$$

It is evident from the calculations that growing κ increases the shear stress on the boundary.

6. Concluding remarks

A different approach to the boundary layer equations of third grade fluids was presented. The geometry of the profile was included as an arbitrary function in the boundary layer equations which allowed the general ideas to be drawn more easily. The second grade effects were negligible compared to the third grade and viscous effects. By using Lie's group analysis, we first found the general symmetries of the partial differential system. Then we reduced the equations to a system of ordinary differential ones via the similarity transformations. Finally, we solved numerically the resulting ordinary differential equations. It occured that the boundary layer got thicker when the non-Newtonian aspect of the fluid behaviour became more pronounced.

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Rozwiązania podobieństwa równań warstwy przyściennej cieczy nieniutonowskiej trzeciego rzędu w specjalnym układzie współrzędnych

Streszczenie

W pracy przedstawiono dwuwymiarowe równania ruchu dla stacjonarnego przepływu cieczy trzeciego rzędu w specjalnym układzie współrzędnych. Równania wyprowadzono na bazie przepływu potencjalnego cieczy nielekkiej. Przy nielepkim opływie dowolnego obiektu linie prądu tworzą współrzędną ϕ, a linie potencjału prędkości współrzędną ψ. Obydwie generują ortogonalny układ współrzędnych krzywoliniowych. Przy takim opisie postać równań warstwy przyściennej nie zależy od kształtu zanurzonego ciała poddanego opływowi. W pierwszym przybliżeniu założono, że wyrażenia drugiego rzędu są pomijalne w stosunku do członów wiskotycznych i trzeciego rzędu. Człony drugiego rzędu uniemożliwiają transformację skalowania, będącą jedynym przekształceniem prowadzącym do rozwiązań podobieństwa cieczy trzeciego rzędu. W pracy zastosowano metodę opartą na grupie Lie'a w generowaniu równań warstwy przyściennej przy pomocy wyrażeń infinitezymalnych. Równania przekształcono do układu równań różniczkowych zwyczajnych. Numeryczne rozwiązanie równań nieliniowych uzyskano w drodze kombinacji algorytmu Runge-Kutta i techniki trymowania.

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