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DYNAMIC BEHAVIOUR OF AN ELASTIC LAYER WITH UNDULATING BOUNDARIES

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Dynamic behaviour of an elastic layer with periodically undulating boundaries and resting on a rigid base is investigated. To this end a certain simplified averaged 2D-model of the layer is proposed as a tool of the analysis. The obtained model is applied to study some wave propagation problems. It is shown that the periodic shape of the boundaries leads to the dispersion of waves propagating along the unbounded layer.

Key words: dynamics, elastic layer, modelling

1. Introduction

The aim of this paper is to investigate dynamic behaviour of an elastic layer resting on a rigid base and having periodically undulating boundaries. The region Ω occupied by the layer in a physical space referred to the Cartesian coordinates $0x_1x_2x_3$ is given by

$$\Omega = \Big\{(x_1, x_2, x_3) : \quad H_0(\boldsymbol{x}) < x_3 < H(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, x_2) \in \Pi\Big\}$$

where Π is a region on the $0x_1x_2$ plane and $H_0(x)$, $H(x)$ are A-periodic functions with $A = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$; cf. Fig. 1.

The special cases in which either $H_0 =$ const or $H =$ const as well as the case $\Pi = R^2$ can also be taken into account. It is assumed that the diameter $l \equiv \sqrt{l_1^2 + l_2^2}$ of A is sufficiently small when compared with the smallest characteristic length dimension of Π. The layer is made of a linear elastic material and $x_3 =$ const are assumed to be the material symmetry

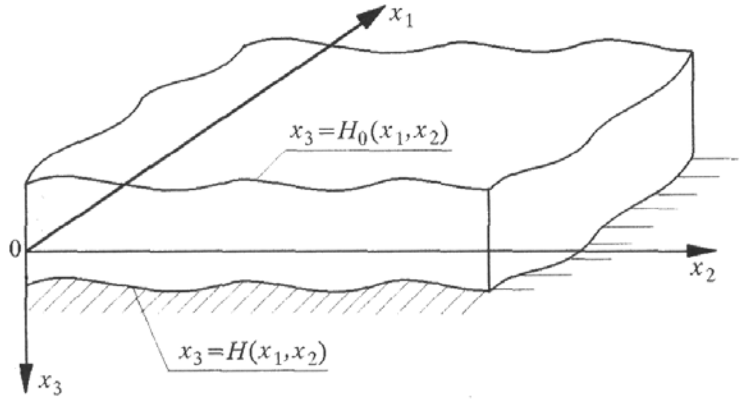


Fig. 1.

planes. Hence the stress-strain relations combined with the strain-displacement ones have the form

$$\sigma_{ij} = C_{ijkl}u_{(k,l)} \qquad C_{\alpha\beta\gamma3} = 0 \qquad C_{\alpha333} = 0 \tag{1.1}$$

here and hereafter the Latin indices $i, j, k, l, ...$ run over 1,2,3 and the Greek indices $\alpha, \beta, \gamma, ...$ run over 1,2; summation convention holds. In general, the elastic moduli C_{ijkl} as well as the mass density ρ can depend on x_3 and be A-periodic functions of $\boldsymbol{x} = (x_1, x_2)$. The displacements u_i of the layer are assumed to be equal zero on the boundary $x_3 = H(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2) \in \Pi$; it means that the layer is resting on the rigid base. The layer is loaded on the boundary $x_3 = H_0$ in the direction of the x_3-axis with the intensity $p = p(\boldsymbol{x}, t)$, t being the time coordinate. The body force b is acting in the x_3-axis direction. Thus, the problem under consideration can be governed by constitutive equations (1.1), the displacement boundary conditions

$$u_i(\boldsymbol{x}, H(\boldsymbol{x}), t) = 0 \qquad \boldsymbol{x} = (x_1, x_2) \in \Pi \tag{1.2}$$

and by the variational equation of motion

$$\int\limits_{\Pi} \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} \sigma_{ij}\overline{u}_{i,j} \; dx_3 d\boldsymbol{x} = \int\limits_{\Pi} \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} \rho(b\overline{u}_3 - \ddot{u}_i\overline{u}_i) \; dx_3 d\boldsymbol{x} + \int\limits_{\Pi} p\overline{u}_3^0 \; d\boldsymbol{x} \tag{1.3}$$

where

$$\overline{u}_3^0 = \overline{u}_3(\boldsymbol{x}, H_0(\boldsymbol{x})) \qquad d\boldsymbol{x} = dx_1 dx_2$$

which holds for every virtual displacement field $\overline{u}_i$ such that $\overline{u}_i(\boldsymbol{x}, H(\boldsymbol{x})) = 0$ for every $\boldsymbol{x} = (x_1, x_2) \in \Pi$. Here, and in the subsequent analysis all functions are assumed to satisfy the required regularity conditions.

As a rule, direct application of (1.1)-(1.3) to analysis of most special dynamic problems, due the periodic and possible discontinuous form of the functions $H(\cdot)$, $H_0(\cdot)$, $C_{ijkl}(\cdot, x_3)$, $\rho(\cdot, x_3)$ is not advisable. Hence, the aims of this contribution are:

(i) To obtain a certain mathematical 2D-model of the layer under consideration represented by equations with constant coefficients;

(ii) To apply this model to investigation of certain dynamic problems.

Aim (i) will be realised in two steps. Firstly, we formulate (see Section 2) the simplified 2D-model of the layer by using the approach similar to that given by Vlasov and Leontiev (1960). That 2D-model will be governed by a partial differential hyperbolic equation with periodic functional coefficients. Secondly, in order to obtain the equation with constant coefficients, we shall apply the tolerance averaging approach detailed by Woźniak and Wierzbicki (2000). To this end we shall recall in Section 3 some basic concepts and assertions related to this approach which will be used in Section 4 to derive of the final form of the model equations. The resulting model will be applied in Section 5 to analysis of some dynamic problems.

2. Formulation of the 2D-model

In the subsequent analysis our attention will be restricted to problems in which the displacement components u_α, $\alpha = 1, 2$, can be neglected as sufficiently small when compared with the displacement u_3. Extending the approach given by Vlasov and Leontiev (1960) we shall assume that

$$u_3(\boldsymbol{x}, x_3, t) = w(\boldsymbol{x}, t)\psi(\boldsymbol{x}, x_3) \qquad u_\alpha(\boldsymbol{x}, x_3, t) = 0 \tag{2.1}$$

where the decay function $\psi(\boldsymbol{x}, \cdot)$ is linear and satisfies for every $\boldsymbol{x} = (x_1, x_2)$ the conditions

$$\psi(\boldsymbol{x}, H_0(\boldsymbol{x})) = 1 \qquad \psi(\boldsymbol{x}, H(\boldsymbol{x})) = 0$$

It follows that conditions (1.2) are satisfied and $w(\boldsymbol{x}, t) = u_3(\boldsymbol{x}, H_0(\boldsymbol{x}), t)$, is the basic kinematic unknown.

Imposing constraints (2.1) on (1.1)-(1.3) and denoting

$$\overline{k} = \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} \Big[C_{3333}(\psi_{,3})^2 + C_{\alpha 3\beta 3}\psi_{,\alpha}\psi_{,\beta}\Big]\,dx_3$$

$$m_{\alpha\beta} = \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} C_{\alpha 3\beta 3}\psi^2\,dx_3 \qquad s_\alpha = \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} C_{\alpha 3\beta 3}\psi_{,\beta}\,dx_3$$

$$r = \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} \rho\psi^2\,dx_3 \qquad f = p + \int\limits_{H_0(\boldsymbol{x})}^{H(\boldsymbol{x})} \rho b\psi\,dx_3 \qquad k = \overline{k} - s_{\alpha,\alpha}$$

after substituting into (1.3) $\overline{u}_\alpha = 0$, $\overline{u}_3 = \overline{w}(\boldsymbol{x})\psi(\boldsymbol{x}, x_3)$, where $\overline{w}(\cdot)$ is an arbitrary test function, we arrive at the equation

$$(m_{\alpha\beta}w_{,\alpha})_{,\beta} - r\ddot{w} - kw + f = 0 \tag{2.2}$$

which holds in Π and has to be satisfied together with the condition

$$(m_{\alpha\beta}w_{,\beta} - s_\alpha w)n_\alpha = 0$$

on $\partial\Pi$, where n_α is the unit normal to $\partial\Pi$. Equation (2.2) represents a 2D-model of the problem under consideration and constitutes a certain generalisation of equation (7.8) p. 56 in the book by Vlasov and Leontiev (1960).

Unfortunately, direct application of (2.2) to analysis of special problems is not advisable due the A-periodic and possibly discontinuous and highly oscillating form of the functional coefficients $m_{\alpha\beta}(\boldsymbol{x})$, $r(\boldsymbol{x})$ and $k(\boldsymbol{x})$. The simplest approximate model of (2.2) can be obtained by means of the homogenization approach, cf. Bakhvalov and Panasenko (1984), Bensoussan et al. (1978), Jikov et al. (1994) and Sanchez-Palencia (1980). Within the framework of this approach (2.2) is "approximated" by a similar equation but with constant coefficients, and the unknown field $w(\cdot)$ can be calculated by means of a certain asymptotic formula. However, the aforementioned homogenized model does not depend on the periods l_1, l_2 and is not able to describe dispersion phenomena which play an important role in the analysis of dynamic problems. That is why in this contribution we shall apply the method of the tolerance averaging of partial differential equations with periodic coefficients to equation (2.2). This method is deprived of the drawbacks typical for homogenization and has been successfully applied to the analysis of special problems in a series of papers by Baron and Jędrysiak (1998), Baron and C.Woźniak (1999), Cielecka et al. (2000), Jędrysiak (2000), Mazur-Śniady (1993), Michalak (2000), Wierzbicki (1995), Wierzbicki et al. (1996), Woźniak C. (1997), Woźniak M. (1996) and other. To make this paper self-consistent, in the subsequent section, following Woźniak and Wierzbicki (2000), we outline the main concepts and assertions of the tolerance averaging.

3. Fundamentals of tolerance averaging

The tolerance averaging of partial differential equations with periodic coefficients is based on the concept of the tolerance space introduced by Zeeman (1965). In the simplest case it is a pair $(\mathbb{R}, \varepsilon)$ where $\mathbb{R}$ is a set of real numbers representing values of a certain physical quantity (e.g. displacement expressed in the known unit) and ε is a positive number, called the tolerance parameter, which determines the accuracy of calculation of these values or the accuracy of a measurement related to the quantity under consideration. For any $a, b \in \mathbb{R}$ we say that a, b are in a tolerance provided that $|a-b| \leqslant \varepsilon$. From the point of view of the performed investigations, we shall not discern between the values a and b. To denote this fact we shall write $a \stackrel{\varepsilon}{\sim} b$ if and only if $|a-b| \leqslant \varepsilon$. By the tolerance system we shall understand the triple $T = (\Phi(\Pi), \varepsilon(\cdot), l)$ where $\Phi(\Pi)$ stands for a set of fields defined in Π which can also depend on time and are unknowns in the problem under consideration, $\varepsilon(\cdot) : \Phi(\Pi) \ni F \to \varepsilon(F) \in \mathbb{R}$ is a mapping which assigns corresponding tolerance parameters to the values of these fields and their derivatives, and l is the diameter of the periodicity cell. In the subsequent analysis we shall use the notation $\varepsilon_F = \varepsilon(F)$.

Let us assume that a certain tolerance system T is known and let $DF(\cdot) \in F(\Pi)$ stand for the function $F(\cdot)$ or its arbitrary derivative (including the time derivative) which appears in the problem under consideration. We say that $F(\cdot)$ is *a slowly varying function*, $F(\cdot) \in SV(T)$, if for every $\boldsymbol{x}', \boldsymbol{x}'' \in Dom\, F$ the condition $\|\boldsymbol{x}' - \boldsymbol{x}''\| \leqslant l$ implies $|DF(\boldsymbol{x}') - DF(\boldsymbol{x}'')| \leqslant \varepsilon_{DF}$ for every $DF \in \Phi(\Pi)$.

Define $A(\boldsymbol{x}) = \boldsymbol{x} + A$ and $\Pi_A = \{\boldsymbol{x} \in \Pi : A(\boldsymbol{x}) \subset \Pi\}$. A function $\varphi(\cdot)$ is called *a periodic-like function*, $\varphi(\cdot) \in PL(T)$, if for every $\boldsymbol{x} \in \Pi_A$ there exists an A-periodic function $\varphi_x(\cdot)$ such that for every $\boldsymbol{y} \in Dom\, \varphi_x$ the condition $\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l$ implies $|\varphi_x(\boldsymbol{y}) - \varphi(\boldsymbol{y})| \leqslant \varepsilon_\varphi$. In this case the function $\varphi_x(\cdot)$ is called *a periodic approximation* of $\varphi(\cdot)$ in $A(\boldsymbol{x})$.

Let $k(\cdot)$ be a certain positive valued A-periodic integrable function. The periodic-like function $\varphi(\cdot)$ will be called *an oscillating function* (with a weight

$k(\cdot))$, $\varphi(\cdot) \in PL^k(T)$ if the condition

$$\int\limits_{A(\boldsymbol{x})} k(\boldsymbol{y})\varphi_x(\boldsymbol{y})\, d\boldsymbol{y} = 0$$

holds for every $\boldsymbol{x} \in \Pi_A$.

It was shown by Woźniak and Wierzbicki (2000) that every periodic-like function can be uniquely represented as a sum of a slowly varying function and a function oscillating with the known weight. Define

$$\langle g\rangle(\boldsymbol{x}) = \frac{1}{area(A)} \int\limits_{A(\boldsymbol{x})} g(\boldsymbol{y})\, d\boldsymbol{y} \qquad \boldsymbol{x} = (x_1, x_2) \in \Pi_A \tag{3.1}$$

where g is an arbitrary integrable function. One can prove that for any $F \in SV(T)$, $\varphi \in PL(T)$ and for arbitrary A-periodic functions $f(\cdot)$, $h(\cdot)$ where $\max\{|h(\boldsymbol{x})| : \boldsymbol{x} \in \overline{A}\} \leqslant l$, the following relations hold for every $\boldsymbol{x} \in \Pi_A$

$$\begin{aligned}
&\langle fF\rangle(\boldsymbol{x}) \overset{\varepsilon}{\sim} \langle f\rangle(\boldsymbol{x})F(\boldsymbol{x}) && \text{for} \quad \varepsilon = \langle |f|\rangle\varepsilon_F \\
&\langle f\varphi\rangle(\boldsymbol{x}) \overset{\varepsilon}{\sim} \langle f\varphi_x\rangle(\boldsymbol{x}) && \text{for} \quad \varepsilon = \langle |f|\rangle\varepsilon_\varphi \\
&\langle f(hF)_{,\alpha}\rangle(\boldsymbol{x}) \overset{\varepsilon}{\sim} \langle fFh_{,\alpha}\rangle(\boldsymbol{x}) && \text{for} \quad \varepsilon = \langle |f|\rangle(\varepsilon_F + l\varepsilon_{\nabla F}) \\
&\langle h(f\varphi)_{,\alpha}\rangle(\boldsymbol{x}) \overset{\varepsilon}{\sim} -\langle f\varphi h_{,\alpha}\rangle(\boldsymbol{x}) && \text{for} \quad \varepsilon = \varepsilon_G + l\varepsilon_{\nabla G}
\end{aligned} \tag{3.2}$$

where $G = \langle hf\varphi\rangle l^{-1}$; it is assumed that all aforementioned functions satisfy the required regularity conditions, cf. Woźniak and Wierzbicki (2000).

The tolerance averaging of partial differential equations with periodic functional coefficients is based on two assumptions. Firstly, the conformability assumption (CA) states that the averaging can be carried out only if the unknown functions in the problem under consideration are periodic-like with respect to a certain tolerance system. Secondly, the tolerance averaging approximation (TA) makes it possible to approximate the left-hand sides of relations (4.2) by their right-hand sides. For a more detailed analysis of the tolerance averaging method and its applications to thermoelasticity the reader is referred to Woźniak and Wierzbicki (2000).

4. Averaged 2D-model

Assuming that a certain tolerance system T is known and applying (CA) to (2.2) we obtain $w(\cdot,t) \in PL(T)$ and hence $w(\cdot,t) = W(\cdot,t) + V(\cdot,t)$, where $W(\cdot,t) \in SV(T)$ and $V(\cdot,t) \in PL^k(T)$ for every time t. It means that $W(\boldsymbol{x},t) = \langle kw\rangle(\boldsymbol{x})\langle k\rangle^{-1}$ is an averaged displacement at the point $\boldsymbol{x} = (x_1,x_2)$ of the boundary $x_3 = H_0(\boldsymbol{x})$, $\boldsymbol{x} = (x_1,x_2) \in \Pi$. Averaging (2.2) by means of (3.1), after applying (TA), we obtain

$$\langle m_{\alpha\beta}\rangle W_{,\alpha\beta} - \langle r\rangle \ddot{W} - \langle k\rangle W + \langle mV_{,\alpha}\rangle_{,\alpha} - \langle r\ddot{V}\rangle + \langle f\rangle = 0 \tag{4.1}$$

From (2.2) by using (TA) we also derive the periodic variational cell problem (on an arbitrary cell $A(\boldsymbol{x})$, $\boldsymbol{x} = (x_1,x_2) \in \Pi_A$) for the A-periodic approximation V_x of V

$$\begin{aligned}&\langle \overline{V}_{,\beta}V_{x,\alpha}m_{\alpha\beta}\rangle(\boldsymbol{x}) + \underline{\langle \overline{V}V_x k\rangle(\boldsymbol{x})} + \underline{\langle \overline{V}\ddot{V}_x r\rangle(\boldsymbol{x})} = \\ &= -\langle \overline{V}_{,\beta}m_{\alpha\beta}\rangle(\boldsymbol{x})W_{,\alpha}(\boldsymbol{x},t) - \underline{\langle \overline{V}r\rangle(\boldsymbol{x},t)\ddot{W}(\boldsymbol{x},t)} - \underline{\langle \overline{V}f\rangle(\boldsymbol{x},t)}\end{aligned} \tag{4.2}$$

which holds for every A-periodic test function $\overline{V}$ such that $\langle \overline{V}k\rangle = 0$, and where V_x has to satisfy the condition $\langle V_x k\rangle(\boldsymbol{x}) = 0$. After neglecting the underlined terms in (4.2) we obtain the known periodic cell problem for the homogenized 2D-model of the layer. Hence, the applied approach constitutes a certain generalisation of the homogenization approach.

The A-periodic solution $V_x(\cdot)$ to cell problem (4.2), which satisfies the condition $\langle V_x k\rangle(\boldsymbol{x}) = 0$ will be obtained using Galerkin's approximation. To this end we shall introduce the system $h^A(\cdot)$, $A = 1,...,N$ of linearly independent A-periodic shape functions such that $\langle kh^A\rangle = 0$, $h^A(\boldsymbol{x}) \in O(l)$ and $lh^A_{,\alpha}(\boldsymbol{x}) \in O(l)$. The functions $h^A(\cdot)$ can be taken as the mode shape functions related to a certain eigenvalue periodic cell problem related to (4.2), which was formulated by Woźniak and Wierzbicki (2000), or should approximate the expected modes of free periodic vibrations of the cell A in the problem under consideration. Then the approximate solution to (4.2) will be taken in the form (summation over $A = 1,...,N$ holds)

$$V_x(\boldsymbol{y},t) \approx h^A(\boldsymbol{y})V^A(\boldsymbol{x},t) \qquad \boldsymbol{y} \in A(\boldsymbol{x}) \qquad \boldsymbol{x} = (x_1,x_2) \in \Pi_A \tag{4.3}$$

where $V^A(\boldsymbol{x},t)$ are the new unknowns. It was shown by Woźniak and Wierzbicki (2000) that $V^A(\cdot,t)$ have to be slowly varying functions. By means of (4.3) we obtain from (4.1), (4.2), after some transformations and assuming that $\overline{V} = h^A$, $A = 1,...,N$, the following system of $N+1$ equations with constant coefficients

$$\begin{aligned}&\langle m_{\alpha\beta}\rangle W_{,\alpha\beta} - \langle r\rangle \ddot{W} - \langle k\rangle W + \langle m_{\alpha\beta}h^A_{,\beta}\rangle V^A_{,\alpha} - \langle rh^A\rangle \ddot{V}^A + \langle f\rangle = 0 \\ &\underline{\langle rh^Ah^B\rangle}\ddot{V}^B + \langle m_{\alpha\beta}h^A_{,\alpha}h^B_{,\beta} + \underline{kh^Ah^B}\rangle V^B + \langle m_{\alpha\beta}h^A_{,\beta}\rangle W_{,\alpha} + \\ &+\underline{\langle rh^A\rangle}\ddot{W} + \underline{\langle fh^A\rangle} = 0\end{aligned} \tag{4.4}$$

for $N+1$ unknowns W, V^A. It has to be emphasised that solutions to the above system have a physical sense if only the conditions

$$W(\cdot,t) \in SV(T) \qquad V^A(\cdot,t) \in SV(T) \tag{4.5}$$

hold for every t. Under these conditions we obtain, for the displacement field $w(\cdot)$, the approximate formula

$$w(\boldsymbol{x},t) \approx W(\boldsymbol{x},t) + h^A(\boldsymbol{x})V^A(\boldsymbol{x},t) \qquad \boldsymbol{x} = (x_1,x_2) \in \Pi_A \tag{4.6}$$

where the approximation $\approx$ depends on the number of terms in (4.3) as well as on the tolerance parameters in the tolerance system T. Formulae (4.4)-(4.6) represent the averaged 2D-model of the layer under consideration. In contrast with the homogenized 2D-model, the above 2D- model describes the effect of the cell size on the global behaviour of the layer and includes *a posteriori* reability conditions (4.5). The cell size effect is due to the underlined terms in (4.4) depending on the shape functions $h^A(\cdot)$, the values of which are of order of $O(l)$.

5. Applications

In order to study the effect of boundary undulating on the dynamic behaviour of the layer we shall consider the case in which the functions $H_0(\cdot)$, $H(\cdot)$ depend only on x_1 and have the period l. For the sake of simplicity we shall also introduce only one shape function $h = h(x_1)$ with the period l. Denoting $m = m_{11}$, $s = (h')^2m + h^2k$, $g = hl^{-1}$ we obtain from (4.4) the following system of equations for $W = W(x_1,t)$ and $V = V(x_1,t)$

$$\begin{aligned} &\langle m\rangle W_{,11} - \langle r\rangle \ddot{W} - \langle k\rangle W + \langle mh_{,1}\rangle V_{,1} - l\langle rg\rangle \ddot{V} + \langle f\rangle = 0 \\ &l^2\langle rg^2\rangle \ddot{V} + \langle s\rangle V + \langle mh_{,1}\rangle W_{,1} + l\langle rg\rangle \ddot{W} + l\langle fg\rangle = 0 \end{aligned} \tag{5.1}$$

where the dependence of the coefficients on the period l is given in an explicit form.

The form of the shape function $h(\cdot)$ in (5.1) depends on the periodic material heterogeneity of the layer and on the periodic functions $x_3 = H_0(x_1)$, $x_3 = H(x_1)$ describing the waveness of the layer boundaries. To obtain an illustrative example of (5.1) let us assume that the layer is homogeneous and isotropic with the Lamé moduli λ, μ and $H_0(x_1) = 0$ for every x_1. In this case $\psi(x_3) = 1 - x_3/H(x_1)$ and

$$m = \frac{\mu}{3}H(x_1) \qquad\qquad r = \frac{\rho}{3}H(x_1)$$

$$k = \Big[\lambda + 2\mu\frac{\mu}{3}(H'(x_1))^2 - \frac{\mu}{2}H''(x_1)H(x_1)\Big]\frac{1}{H(x_1)}$$

where $H(x_1)$ is now the thickness of the layer at x_1. Let us assume that $H(\cdot)$ is the piecewise constant function such that $H(x_1) = H'$ for $x_1 \in (-l', 0)$ and $H(x_1) = H''$ for $x_1 \in (0, l'')$ where $l' + l'' = l$. In order to satisfy the condition $\langle kh \rangle = 0$ we shall assume $h(\cdot)$ as the continuous function with the period l, linear in $(-l', 0)$ and $(0, l'')$ such that $h(-l') = h(l'') = -h(0) = \sqrt{3}\, l/6$ (cf. Woźniak and Wierzbicki (2000), Section 6). Setting $\nu' = l'/l$, $\nu'' = l''/l$ we obtain $\langle H \rangle = \nu' H' + \nu'' H''$ and the coefficients in (5.1) will take the form

$$\begin{aligned}
&\langle m \rangle = \frac{\mu}{3}\langle H \rangle && \langle r \rangle = \frac{\rho}{3}\langle H \rangle \\
&\langle k \rangle = (\lambda + 2\mu)\Big\langle \frac{1}{H} \Big\rangle && \langle mh_{,1} \rangle = \frac{2\sqrt{3}\mu}{3}(H' - H'') \\
&\langle rg^2 \rangle = \frac{\rho}{3}\langle H \rangle && \langle rg \rangle = 0 \\
&\langle s \rangle = 4\mu\Big(\frac{H'}{\nu'} + \frac{H''}{\nu''}\Big) + l^2\frac{\mu}{3}\langle H \rangle
\end{aligned}$$

It can be seen that for $H' = H''$ and after neglecting inertial forces the first from equations (5.1) reduces to the form given by Vlasov and Leontiev (1960), and the second one yields $V = 0$.

In order to study the boundary undulating effect on the dynamic behaviour of the layer we shall investigate the wave propagation and free vibration problems for the unbounded layer, $\Pi = \mathbb{R}^2$ using equations (5.1) with $f = 0$.

5.1. Wave propagation analysis

Let L be the wave-length of the functions W and V in Eqs. (5.1). First, we consider *the short waves* in which the ratio L/H is assumed to be small compared with 1, i.e. $L/H \ll 1$. In the asymtotic analysis of equations (5.1) the terms of order $O((L/H)^2)$ will be neglected compared to 1. After some calculations the above equations yield

$$\begin{aligned}
&\langle r \rangle \ddot{W} - \langle m \rangle W_{,11} - \langle mh_{,1} \rangle V_{,1} = 0 \\
&l^2 \langle rg^2 \rangle \ddot{V} + \langle m(h_{,1})^2 \rangle V + \langle mh_{,1} \rangle W_{,1} = 0
\end{aligned} \tag{5.2}$$

For the investigation of wave propagation in the x_1-direction the functions W and V can be assumed in the form

$$W = W(x_1 - ct) \qquad\qquad V = V(x_1 - ct) \tag{5.3}$$

Substituting (5.3) into equations (5.2) we obtain

$$\begin{aligned} &(\langle r\rangle c^2 - \langle m\rangle)W_{,11} - \langle mh_{,1}\rangle V_{,1} = 0 \\ &c^2 l^2 \langle rg^2\rangle V_{,11} + \langle m(h_{,1})^2\rangle V + \langle mh_{,1}\rangle W_{,1} = 0 \end{aligned} \tag{5.4}$$

At the begining, let us consider two special cases.

Case 1. Suppose that the boundary is plane and the layer is homogeneous. These assumptions yield

$$\langle mh_{,1}\rangle = m\langle h_{,1}\rangle = 0$$

and

$$(\langle r\rangle c^2 - \langle m\rangle)W_{,11} = 0$$

Assuming that $W_{,11} \neq 0$ we obtain the value c_0 of the wave propagation speed as

$$c_0^2 = \frac{\langle m\rangle}{\langle r\rangle} \tag{5.5}$$

Case 2. Assume that the length-scale effect on the layer behaviour is neglected. From $(5.4)_2$ after neglecting the term $\langle rh^2\rangle$, we obtain

$$\langle m(h_{,1})^2\rangle V + \langle mh_{,1}\rangle W_{,1} = 0$$

and $(5.4)_1$ yields

$$\Big(\langle r\rangle c^2 - \langle m\rangle + \frac{\langle mh_{,1}\rangle^2}{\langle m(h_{,1})^2\rangle}\Big)W_{,11} = 0$$

The wave propagation speed in this case is given by

$$\widetilde{c}^2 = \frac{\langle m\rangle}{\langle r\rangle} - \frac{\langle mh_{,1}\rangle^2}{\langle r\rangle\langle m(h_{,1})^2\rangle} \tag{5.6}$$

It can be proved that $\widetilde{c}^2 > 0$.

Now. let us pass to *the general case*. Under denotations

$$\rho_0 \equiv \langle r \rangle \qquad \rho_2 \equiv \langle rg^2 \rangle$$

$$\mu_0 \equiv \langle m \rangle \qquad \mu_1 \equiv \langle mh_{,1} \rangle \qquad \mu_2 \equiv \langle m(h_{,1})^2 \rangle$$

equations (5.4) can be transformed to the form

$$\begin{aligned}
&(\mu_0^2 - c^2\rho_0)W_{,11} + \mu_1 V_{,1} = 0 \\
&c^2 l^2 \rho_2 V_{,11} + \mu_2 V + \mu_1 W_{,1} = 0
\end{aligned} \tag{5.7}$$

Assuming that $W_{,11} \neq 0$ and $\mu_1 \neq 0$, from $(5.7)_1$ we obtain

$$V = -\frac{\rho(c_0^2 - c^2)}{\mu_1} W_{,1} \tag{5.8}$$

where, in accordance with (5.5), we have used the denotation

$$c_0^2 = \frac{\mu_0}{\rho_0}$$

The relation $(5.7)_2$ together with (5.8) leads to the equation

$$c^2(c_0^2 - c^2) l^2 \rho_2 W_{,11} + \mu_2(\widetilde{c}^2 - c^2)W = 0 \tag{5.9}$$

where

$$\widetilde{c}^2 = \frac{1}{\rho_0}\Big(\mu_0 - \frac{(\mu_1)^2}{\mu_2}\Big)$$

is the wave propagation speed in special case 2, defined by (5.6).

It can be shown that the three types of waves can propagate in the layer under consideration.

(i) If $c < \widetilde{c}$ or $c > c_0$ then sinusoidal waves exist

$$U = A \sin k_c(x_1 - ct) + B \cos k_c(x_1 - ct) \tag{5.10}$$

where

$$k_c^2 \equiv \frac{\mu_2}{l^2 \rho_2} \cdot \frac{\widetilde{c}^2 - c^2}{c^2(c_0^2 - c^2)} > 0$$

(ii) If $\widetilde{c} < c < c_0$ then there exist exponential waves

$$U = A \exp[-\kappa_c(x_1 - ct)] + B \exp[\kappa_c(x_1 - ct)] \tag{5.11}$$

where

$$\kappa_c^2 \equiv -\frac{\mu_2}{l^2 \rho_2} \frac{\widetilde{c}^2 - c^2}{c^2(c_0^2 - c^2)} > 0$$

(iii) If $c = \widetilde{c}$ then $W_{,11} = 0$, $V_{,1} = 0$, and this degenerate case, which is situated between the sinusoidal and exponential waves, takes place.

Note that the speed $c = c_0$ is not allowable in the class of problems under consideration.

Let us consider propagation of *an arbitrary transverse* wave. Substituting (5.3) into the system (5.1) we obtain

$$\begin{aligned} &(\langle r\rangle c^2 - \langle m\rangle)W_{,11} - \langle mh_{,1}\rangle V_{,1} + \langle k\rangle W = 0 \\ &c^2 l^2 \langle rg^2\rangle V_{,11} + \langle m(h_{,1})^2\rangle V + \langle mh_{,1}\rangle W_{,1} = 0 \end{aligned} \tag{5.12}$$

Three special cases of (5.12) will be now considered.

Case 1. If the boundary is plane and the layer is homogeneous then system (5.12) yields

$$(c^2 - c_0^2)W_{,11} + \frac{\langle k\rangle}{\rho_0}W = 0 \qquad c \neq c_0 \tag{5.13}$$

If $c < c_0$ then we deal with the exponential waves and if $c > c_0$ then the sinusoidal waves propagate.

Case 2. If the length-scale effect is neglected then system (5.12) can be reduced to the equation

$$(c^2 - \widetilde{c}_0^2)W_{,11} + \frac{\langle k\rangle}{\rho_0}W = 0 \qquad c \neq \widetilde{c} \tag{5.14}$$

If $c < \widetilde{c}$ then there exist the exponential waves, and if $c > \widetilde{c}$ then the sinusoidal waves propagate.

Let us pass to the analysis of the *general case*. Let system (5.12) be transformed to the form

$$\begin{aligned} &(c_0^2 - c^2)W_{,11} + \frac{\langle k\rangle}{\rho_0}W + \frac{\mu_1}{\rho_0}V_{,1} = 0 \\ &c^2 l^2 \rho_2 V_{,11} + \mu_2 V + \mu_1 W_{,1} = 0 \end{aligned} \tag{5.15}$$

Assuming that $\mu_1 \neq 0$ from $(5.15)_1$ we obtain

$$\mu_1 W_{,1} = -\rho_0(c_0^2 - c^2)W_{,11} - \langle k\rangle W \tag{5.16}$$

Equation $(5.15)_2$ together with (5.16), under denotations

$$e^2 \equiv \frac{\mu_2}{\rho_2} \qquad \omega_0^2 \equiv \frac{\langle k \rangle}{\rho_0}$$

yields the following equation for W

$$c^2(c_0^2 - c^2)l^2 W_{,1111} + e_2(\widetilde{c}^2 - c^2)W_{,11} + e^2\omega_0^2 W = 0 \tag{5.17}$$

It is easy to show that the three types of waves can be considered here:

(i) if $c \leqslant \widetilde{c}$ then only the sinusoidal waves exist;

(ii) if $\widetilde{c} < c \leqslant c_0$ then only the exponential waves can propagate;

(iii) if $c > c_0$ then both the sinusoidal and exponential waves can propagate.

In the numerical analysis the function $H(\cdot)$, which describes the boundary undulations, was chosen in the form

$$H(x_1) = l \sin \frac{2\pi x_1}{l} \tag{5.18}$$

and the decay function was considered as linear.

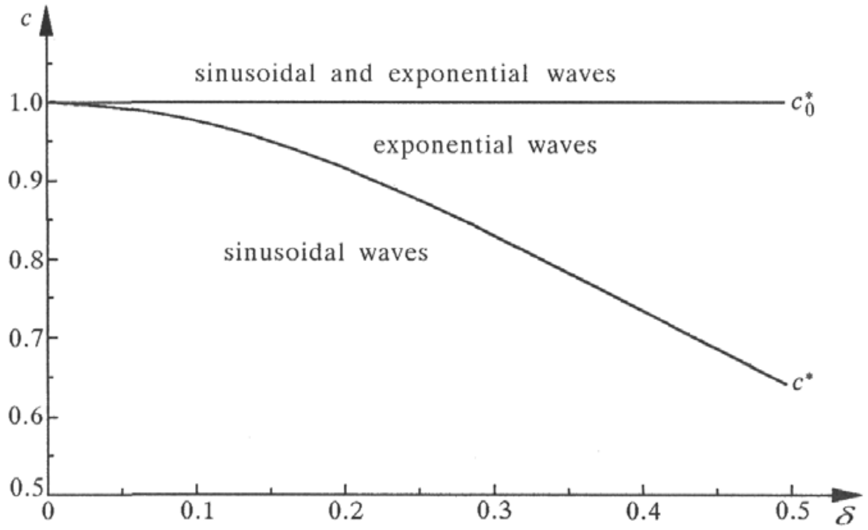


Fig. 2.

In Fig. 2 the dimensionless speeds

$$c_0^* = c_0\sqrt{\frac{\rho}{\mu}} \qquad \widetilde{c}^* = \widetilde{c}\sqrt{\frac{\rho}{\mu}}$$

versus the dimensionless parameter $\delta = l/H$ are presented. Three regions shown in this figure display the three types of the waves mentionted above.

5.2. Free vibration analysis

Substituting the right-hand sides of

$$W = \overline{W}(x_1)\cos\omega t \qquad\qquad V = \overline{V}(x_1)\cos\omega t$$

into system (5.1) we obtain

$$\begin{aligned}&\langle m\rangle \overline{W}_{,11} + \Big(\omega^2\langle r\rangle - \langle k\rangle\Big)\overline{W} + \langle mh_{,1}\rangle V_{,1} = 0\\ &\Big(\omega^2 l^2\langle rg^2\rangle - \langle m(h_{,1})^2 - l^2kg^2\rangle\Big)\overline{V} - \langle mh_{,1}\rangle \overline{W}_{,1} = 0\end{aligned} \tag{5.19}$$

In order to simplify the subsequent analysis the term $\langle l^2kg^2\rangle$ will be neglected as small when compared to $\langle m(h_{,1})^2\rangle$. Under the denotations

$$\omega_0^2 \equiv \frac{\langle \kappa a_{33}\rangle}{\rho_0} \qquad\qquad \omega_1^2 \equiv \frac{\mu_2}{l^2\rho_2}$$

$$\widetilde{\omega}_1^2 \equiv \frac{1}{l^2\rho_2}\Big(\mu_2 - \frac{(\mu_1)^2}{\mu_0}\Big)$$

and assuming that $\mu_1 \neq 0$, system (5.19) yields

$$\frac{c_0^2(\widetilde{\omega}_1^2 - \omega^2)}{\omega_1^2 - \omega^2}\overline{W}_{,11} + (\omega^2 - \omega_0^2)\overline{W} = 0 \tag{5.20}$$

Let us introduce the function $k = k(\omega)$ given by

$$k^2 \equiv \frac{(\omega^2 - \omega_0^2)(\omega_1^2 - \omega^2)}{c_0^2(\widetilde{\omega}_1^2 - \omega^2)}$$

If $k^2 > 0$ (i.e. $\omega_0 < \omega < \widetilde{\omega}_1$ or $\omega > \omega_1$) then the sinusoidal vibrations with the wave number $k = 2\pi/L$ take place.

Setting

$$e^2 \equiv \frac{\mu_2}{\rho_2} \qquad\qquad \widetilde{e}^2 \equiv \frac{1}{\rho_2}\Big(\mu_2 - \frac{(\mu_1)^2}{\mu_0}\Big) \qquad\qquad 0 < \widetilde{e} < e$$

we obtain

$$\omega_1^2 = \frac{e^2}{l^2} \qquad\qquad \widetilde{\omega}_1^2 = \frac{\widetilde{e}^2}{l^2}$$

and from equation (5.20) the following dispersion relation takes place

$$l^2\omega^4 - [e^2 + l^2(\omega_0^2 + k^2c_0^2)]\omega^2 + \omega_0^2e^2 + k^2c_0^2\widetilde{e}^2 = 0 \tag{5.21}$$

Setting $\varepsilon \equiv kl = 2\pi l/L$ equation (5.21) can be transformed to the form

$$\varepsilon^2\omega^4 - [(ke)^2 + \varepsilon^2(\omega_0^2 + k^2c_0^2)]\omega^2 + k^2[(\omega_0 e)^2 + (kc_0\widetilde{e})^2] = 0 \tag{5.22}$$

Taking into account that $\varepsilon \ll 1$ the following asymptotic formulae for the free vibration frequencies ω can be obtained as the solutions to equation (5.22)

$$\omega^2 = \begin{cases} \omega_0^2 + k^2c_0^2\Big(1 - \dfrac{(\mu_1)^2}{\mu_0\mu_2}\Big) - (\omega_0^2 + k^2\widetilde{c})\dfrac{\rho_2}{\rho_0}\Big(\dfrac{\mu_1}{\mu_2}\Big)^2\varepsilon^2 + O(\varepsilon^4) \\ \omega_1^2 + k^2c_0^2\dfrac{(\mu_1)^2}{\mu_0\mu_2} + O(\varepsilon^2) \end{cases} \tag{5.23}$$

The frequency ω_l given by

$$\omega_l^2 = \omega_0^2 + k^2c_0^2\Big(1 - \frac{(\mu_1)^2}{\mu_0\mu_2}\Big) - (\omega_0^2 + k^2\widetilde{c})\frac{\rho_2}{\rho_0}\Big(\frac{\mu_1}{\mu_2}\Big)^2\varepsilon^2 \tag{5.24}$$

can be treated as the lower free vibration frequency and

$$\omega_h^2 = \omega_1^2 + k^2c_0^2\frac{(\mu_1)^2}{\mu_0\mu_2} \tag{5.25}$$

represents the higher free vibration frequency.

In the numerical analysis the function $H(x_1)$, which describes the boundary undulations, was chosen in the form of (5.18), and the decay function was taken as linear. The dimensionless frequencies introduced by the formulae

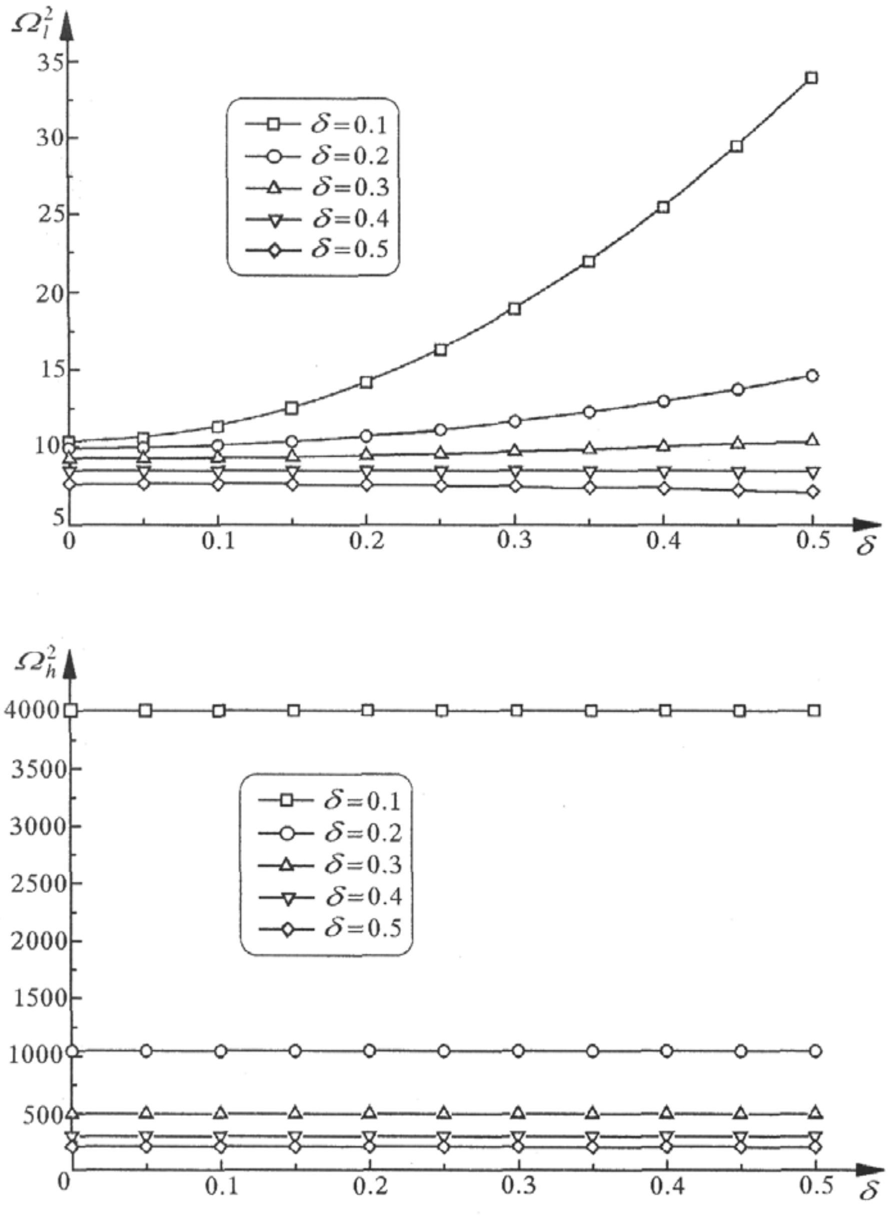
$$\Omega_l^2 = \frac{\rho H^2}{\mu}\omega_l^2 \qquad\qquad \Omega_h^2 = \frac{\rho H^2}{\mu}\omega_h^2 \tag{5.26}$$

versus the dimensionless wave number ε are presented in Fig. 3, where the non-dimensional coefficient $\delta = l/H$ is used as a parameter.

6. Conclusions

We close the paper with some conclusions and comments concerning the new results obtained in the field of dynamics of an elastic layer resting on a rigid base.

- The simplified 2D-model of an elastic layer introduced by Vlasov and Leontiev (1960) was extended to the case of a layer with periodically undulating boundaries; to this end a new form of the decay function ψ was proposed in Section 2.



Ω_l^2
35
30
25
20
15
10
5
0
0.1
0.2
0.3
0.4
0.5
δ
δ=0.1
δ=0.2
δ=0.3
δ=0.4
δ=0.5
Ω_h^2
4000
3500
3000
2500
2000
1500
1000
500
0
0.1
0.2
0.3
0.4
0.5
δ
δ=0.1
δ=0.2
δ=0.3
δ=0.4
δ=0.5

Fig. 3.

- The 2D-model of the elastic layer obtained by the Vlasov and Leontiev approach is described by equation (2.2), which has functional periodic coefficients, and hence is not advisable for the analysis of special problems. That is why the method of tolerance averaging, described by Woźniak and Wierzbicki (2000), was applied in order to derive new model equations (4.4) with constant coefficients.

- The derived averaged 2D-model, described by (4.4)-(4.6), takes into account the effect of the boundary periodic undulation size on the dynamic layer behaviour. This effect is neglected if the known homogenization method is applied to the averaging of equation (2.2). In this case the underlined terms drop out from (4.5), and after eliminating the unknowns V^A we obtain from (4.4) an approximate form of the homogenized equation for W.

- Derived model equations (4.4) consist of the partial differential equation for the displacement W coupled with the system of N ordinary differential equations for V^A, $A = 1, ..., N$. Hence, V^A are independent of the boundary conditions, and that is why they were referred to as the inertial kinematic variables in Woźniak (1997). Following Woźniak and Wierzbicki (2000), equations (4.4) can be also supplemented by the boundary effect equations in order to satisfy the boundary conditions not only for W but also for w, cf. (4.6).

- The derived model includes *a posteriori* conditions (4.5), on the basis of which the degree of the approximation of the obtained solutions can be estimated.

- The illustrative example in Section 5 shows that the derived model can be successfully applied to the analysis of wave propagation which takes into account dispersion effects.

- The proposed model makes it possible to describe the behaviour of the elastic layer under consideration on different levels of accuracy depending on the number N of terms in finite sum (4.4). The main drawback of the model lies in the specification of the mode-shape functions h^A and in the large number of the unknowns V^A, which – in most cases – have to be taken into account in approximation (4.4) to cell problem (4.2).

At the end of this paper it has to be emphasised that the obtained model can be applied only if constraint conditions (2.2) are well motivated from the physical wievpoint in the problem considered cf. Vlasov and Leontiev (1960).

More general constraints leading to 2D-models, in which the displacements u_α are not neglected, will be investigated in a separate paper.

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O dynamice warstwy sprężystej z pofałdowanymi brzegami

Streszczenie

Praca dotyczy analizy dynamicznej warstwy sprężystej z periodycznie pofałdowanymi brzegami. W tym celu zaproponowano jako narzędzie analizy nowy uśredniony 2D-model warstwy. Wykazano, że periodyczne pofałdowanie brzegów powoduje efekty dyspersyjne dla fal propagujących się wzdłuż warstwy.

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