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OPTIMAL CONTROL OF A DUFFING OSCILLATOR UNDER PARAMETRIC AND EXTERNAL EXCITATIONS

Artur Nowoświat

Department of Building Processes, Technical University of Silesia, Gliwice
e-mail: now2@wp.pl

The present paper deals with investigations of a stochastic model of the Duffing oscillator operating continuously. The standard theory of the optimal stochastic control described in Fleming and Richel (1975) is presented, as well as the application of these concept to the perturbation control techniques suggested by Suhardio et al. (1992). The method presented in this paper is, therefore, a generalization of the method of the development of nonlinear control units into a series, as described for Duffing's deterministic oscillator involving stochastic systems derived by the author and applied in his master thesis (Nowoświat, 1999). Detailed analysis and numerical calculations have been done using the Runge-Kutta procedures.

Key words: perturbation control techniques, Duffing's oscillator

1. Introduction

The problem of optimal control of stochastic parametrically and externally excited nonlinear systems was recently studied by Beaman (1984), Yoshida (1984), Young and Chang (1988). By the combined use of Gaussian statistical linearization and linear quadratic Gaussian (LQG) theory a suboptimal linear Beaman (1984), Yoshida (1984) or nonlinear Young and Chang (1988) state feedback controller is synthesized. At the same time, a perturbation technique was applied to the determination of optimal control in the deterministic model of nonlinear Duffing's oscillator in Spencer et al. (1966), Suhardio et al. (1992). The effect of higher-order feedback corrections based upon series expansions of the optimal cost function and the optimal control function in the Hamilton-Jacobi-Bellman approach were shown. The application of the perturbation method to the determination of optimal control of a nonlinear discrete-time system was given in Shefer and Breakwell (1987).

The study of optimal control of nonlinear stochastic systems has been given considerable attention in recent years. In involved stochastic systems which are described by the Itô stochastic differential equations with the Gaussian parametric and external noise excitations.

The success obtained due to the applied approach indicates that it is advisable to include external excitations in the Duffing deterministic oscillator. It has been achieved by adding a model of the external constraint to the classical formulation of Duffing's oscillator. In effect, it led to analysis of stochastic differential equations known in the literature as the Itô equations. These equations are the most popular mathematical means used when considering this problem. This work contains only theoretical considerations involving examination of dynamic systems. The mentioned considerations, as well as the obtained results can be applied to technology, e.g. while analysing suspension systems of wheeled vehicles as well as other problems related with Duffing's oscillator.

This paper is concerned with the study of a stochastic continuous time model of Duffing's oscillator. Using the concepts of standard methods of stochastic optimal control, see e.g. Fleming and Richel (1975), and combining them with the concept by Suhardio et al. (1992) on the application of a perturbation technique to the control problems, we derive results on the optimal control of Duffing's oscillator under parametric and external excitations. Then, a second-order stochastic parametrically and externally excited Duffing's type system is selected to illustrate the application of the 'development into a series' technique to the optimal control of nonlinear stochastic systems and to compare the procedure with the other methods.

2. Perturbation method for a non linear Duffing's oscillator

Consider a nonlinear stochastic model of a dynamic system described by the Itô vector differential equation

$$d\boldsymbol{x}(t) = [f(\boldsymbol{x},t) + \mathbf{B}\boldsymbol{u}]\,dt + \sum_{k=0}^{M} \boldsymbol{G}_k\,d\xi_k \tag{2.1}$$

where

$\boldsymbol{x}$ – state vector, $\boldsymbol{x} = [x^1, ..., x^n]^\top$

$\boldsymbol{u}$ – controlling vector, $\boldsymbol{u} = [u^1, ..., u^n]^\top$

B	–	matrix with constant elements with the dimensions $n \times l$, $\mathsf{B} = [B^i_j]$
$\boldsymbol{G}_k$	–	deterministic vectors, $\boldsymbol{G}_k = [G^1_k, ..., G^n_k]^\top$
ξ_k	–	independent standard Wiener processes.

Let us assume nonlinear vector functions $f: \ R^n \to R^n$ in the form of polynomials

$$f^i(\boldsymbol{x}) = \sum_{j=1}^{n} A^i_j x^j + \sum_{j,k=1}^{n} A^i_{jk} x^{jk} + ... \tag{2.2}$$

where A^i_j, A^i_{jk}, ... are constant coefficients and $x^{jk} = x^j x^k$, and $x^{jkl} = x^j x^k x^l$.

Assume also that the control $\boldsymbol{u}$ takes the form of a polynomial

$$u^j = K^j_0 + \sum_{k=1}^{n} K^j_k x^k + \sum_{k,l=1}^{n} K^j_{kl} x^{kl} + ... \tag{2.3}$$

where K^j_0, K^j_k, K^i_{kl} are constant coefficients.

According to the model presented by Suharido et al. (1992) and Yoshida (1984) we minimize the following cost function

$$\begin{aligned} \boldsymbol{I} \ &= \ \boldsymbol{E}\Big[V_0(t_1) + \boldsymbol{P}\boldsymbol{x}(t_1) + \boldsymbol{x}^\top(t_1)\mathsf{M}\boldsymbol{x}(t_1) + \\ &+ \int\limits_{t_0}^{t_1} \Big(\boldsymbol{x}^\top \mathsf{Q}\boldsymbol{x} + \boldsymbol{u}^\top \mathsf{R}\boldsymbol{u} + \sum_{j,k,l=1}^{n} Q_{jkl} x^{jkl} + ...\Big)\, dt\Big] \end{aligned} \tag{2.4}$$

where

$\boldsymbol{E}[\cdot]$	–	moment vector
$\boldsymbol{P}$	–	constant vector
M, Q	–	symmetric semi-positively determined matrices
R	–	positively determined matrix
$Q_{jkl}, ...$	–	selected symmetric tensors.

The elements K^j_0, K^j_k, K^j_{kl}, ... occurring in equation (2.3), can be calculated by means of Bellmann's equation (Fleming and Richel, 1975), which in our case takes the form

$$\frac{\partial V}{\partial t} + \min_u [\mathcal{L}^*(V) + L(\boldsymbol{x}, \boldsymbol{u}, t)] = 0 \tag{2.5}$$

where

$$
\begin{aligned}
L(\boldsymbol{x},\boldsymbol{u},t) &= \boldsymbol{x}^\top \mathbf{Q}\boldsymbol{x} + \boldsymbol{u}^\top \mathbf{R}\boldsymbol{u} + \sum_{j,k,l=1}^{n} Q_{jkl}x^{jkl} + \dots \\
V &= V_0(t) + \boldsymbol{P}\boldsymbol{x}(t_1) + \boldsymbol{x}^\top(t_1)\mathbf{M}\boldsymbol{x}(t_1) + \int\limits_t^{t_1} L(\boldsymbol{x},\boldsymbol{u},t)\; dt
\end{aligned}
\tag{2.6}
$$

whereas $\mathcal{L}$ is an operator, which, according to $(2.6)_1$, can be defined in the following way

$$
\mathcal{L}^*(\cdot) = \frac{\partial(\cdot)}{\partial t} + \sum_{i=1}^{n}\Big(f^i + \sum_{j=1}^{m} B^i_j u^j\Big)\frac{\partial(\cdot)}{\partial x^i} + \frac{1}{2}\sum_{k=1}^{M}\Big(\boldsymbol{G}_k, \frac{\partial}{\partial x}\Big)^2(\cdot) \tag{2.7}
$$

or

$$
\mathcal{L}^*(\cdot) = \frac{\partial(\cdot)}{\partial t} + \sum_{i=1}^{n}\Big(f^i + \sum_{j=1}^{m} B^i_j u^j\Big)\frac{\partial(\cdot)}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{M}\frac{\partial^2(\cdot)}{\partial x^i \partial x^j}a_{ij} \tag{2.8}
$$

where

$$
\mathbf{a} = [a_{ij}] = \sum_{k=0}^{M}\boldsymbol{G}_k\boldsymbol{G}_k^\top
$$

The function V will be considered in the form of Maclaurin's series

$$
V = V_0(t) + \sum_{i=1}^{n} V_i(t)x^i + \sum_{i,j=1}^{n} V_{ij}(t)x^{ij} + \sum_{i,j,k=1}^{n} V_{ijk}(t)x^{ijk} + \dots \tag{2.9}
$$

where V_{ij}, V_{ijk}, ... are tensors symmetric with respect to their indices.

In addition, we consider the quadratic terms in the cost function given in equation (2.4), that is $Q_{ijk} = \dots = 0$. Furthermore, we set $A^i_{jk} = A^i_{jklm} = \dots = 0$ and $V_i = V_{ijk} = \dots = 0$.

Similarly, the elements of the equal order of control (2.3) will be equated to zero

$$
K^i_0 = K^i_{jk} = \dots = 0 \tag{2.10}
$$

The remaining coefficients of Lapunov's function (2.10) and of control function (2.3) can be found by solving the equations for V_{ij}, K_{ji}, V_{ijk}, K_{jkli}, ...

$$
\min_u\Big\{\Big(\dot{V}_0 + \sum_{i,j=1}^{n}\dot{V}_{ij}x^{ij} + \sum_{i,j,k,l=1}^{n}\dot{V}_{ijkl}x^{ijkl} + \dots\Big) +
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\Big[\Big(2\sum_{j=1}^{n} V_{ij}x^{j} + 4\sum_{j,k,l=1}^{n} V_{ijkl}x^{jkl} + ...\Big)\cdot \\
&\cdot\Big(\sum_{j=1}^{n} A^{i}_{j}x^{j} + \sum_{j,k,l=1}^{n} A^{i}_{jkl}x^{jkl} + ... + \sum_{j=1}^{m} B^{i}_{j}u^{j}\Big)\Big] + \\
&+\frac{1}{2}\sum_{i,j=1}^{n}\Big(2V_{ij} + 12\sum_{k,l=1}^{n} V_{ijkl}x^{kl} + ...\Big)a_{ij} + \\
&+\sum_{i,j=1}^{m} u^{i}R_{ij}u^{j} + \sum_{i,j=1}^{n} x^{i}Q_{ij}x^{j} + \sum_{i,j,k,l=1}^{n} Q_{ijkl}x^{ijkl}\Big\} = 0
\end{aligned}
\tag{2.11}
$$

Taking the differential coefficients an the left-hand side of equation (2.11) in relation to u, and taking into account that the matrix $\mathbf{R}$ is symmetric, we get

$$
2\sum_{i,j=1}^{n} V_{ij}x^{j}B^{i}_{p} + 4\sum_{i,j,k,l=1}^{n} V_{ijkl}x^{jkl}B^{i}_{p} + ... + 2\sum_{j=1}^{n} R_{pj}u^{j} = 0 \tag{2.12}
$$

for $p = 1, 2, ..., n$.

Substituting equation (2.3) into equations (2.11) and (2.12) we get an equation for u^{i}, this control being the optimal one u^{i*}, with quantities V and $\boldsymbol{x}$ corresponding to the quantities V^{*} and $\boldsymbol{x}^{*}$ for the control $\boldsymbol{u}^{*}$ (for convenience the asterisk will be omitted in the following study).

Thus, we get

$$
\begin{aligned}
&\Big\{\Big(\dot{V}_{0} + \sum_{i,j=1}^{n} \dot{V}_{ij}x^{ij} + \sum_{i,j,k,l=1}^{n} \dot{V}_{ijkl}x^{ijkl} + ...\Big) + \\
&+\sum_{i=1}^{n}\Big[\Big(2\sum_{j=1}^{n} V_{ij}x^{j} + 4\sum_{j,k,l=1}^{n} V_{ijkl}x^{jkl} + ...\Big)\cdot \\
&\cdot\Big(\sum_{j=1}^{n} A^{i}_{j}x^{j} + \sum_{j,k,l=1}^{n} A^{i}_{jkl}x^{jkl} + ... + \\
&+\sum_{j=1}^{m} B^{i}_{j}\Big(\sum_{l=1}^{n} K^{j}_{l}x^{l} + \sum_{l,p,q=1}^{n} K^{j}_{lpq}x^{lpq} + ...\Big)\Big)\Big] + \\
&+\frac{1}{2}\sum_{i,j=1}^{n}\Big(2V_{ij} + 12\sum_{k,l=1}^{n} V_{ijkl}x^{kl} + ...\Big)a_{ij} +
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{n}\Big(\sum_{l=1}^{n}K_l^j x^l+\sum_{l,p,q=1}^{n}K_{lpq}^j x^{lpq}+\dots\Big)R_{ij}\cdot\\
&\cdot\Big(\sum_{l=1}^{n}K_l^j x^l+\sum_{l,p,q=1}^{n}K_{lpq}^j x^{lpq}+\dots\Big)+\\
&+\sum_{i,j=1}^{n}Q_{ij}x^{ij}+\sum_{i,j,k,l=1}^{n}Q_{ijkl}x^{ijkl}+\dots\Big\}=0
\end{aligned}
$$

Substituting equation (2.3) into (2.12) we have

$$
\begin{aligned}
&2\sum_{i,j=1}^{n}V_{ij}x^j B_p^i+4\sum_{i,j,k,l=1}^{n}V_{ijkl}x^{jkl}B_p^i+\dots+\\
&+2\sum_{j=1}^{n}R_{pj}\Big(\sum_{l=1}^{n}K_l^i x^l+\sum_{l,p,q=1}^{n}K_{lpq}^i x^{lpq}+\dots\Big)=0
\end{aligned}
\tag{2.14}
$$

for $p=1,2,...,n$.

When we take into account the symmetric operator $sym[\cdot]$ and when we repeat such a procedure, we can obtain the required conditions to ensure the optimality in equation (2.4). These conditions, expressed by a set of differential equations concerning the coefficients $V_{ij}, V_{ijk}, \dots$ are to be found by subsequent equating of expressions of the zero order, the second order, the fourth order and so on, and by taking into account the fact that $x^{ji}=x^{ij}$

$$
\begin{aligned}
&\dot V_0+\sum_{i,j=1}^{n}\dot V_{ij}a_{ij}=0\\
&sym\Big[\dot V_{ij}+2\sum_{l=1}^{n}V_{li}A_j^l+6\sum_{k,l=1}^{n}V_{ijkl}a_{kl}+Q_{ij}+\sum_{k,l=1}^{n}R_{kl}K_i^k K_j^l+\\
&\qquad+2\sum_{k,l=1}^{n}V_{kl}B_l^k K_j^l\Big]=0\\
&sym\Big[\dot V_{ijkl}+2\sum_{p=1}^{n}V_{pi}A_{jkl}^p+4\sum_{p=1}^{n}V_{pijk}\Big(A_l^p+\sum_{q=1}^{n}B_q^p K_l^q\Big)+\\
&\qquad+15\sum_{p,q=1}^{n}V_{pqijkl}a_{pq}+\sum_{p,q=1}^{n}K_{ij}^p R_{pq}K_{kl}^q+Q_{ijkl}\Big]=0
\end{aligned}
\tag{2.15}
$$

$$sym\Big[\dot{V}_{ij} + 2\sum_{l=1}^{n} V_{li}A_j^l + 6\sum_{k,l=1}^{n} V_{ijkl}a_{kl} + Q_{ij} +$$
$$+\sum_{k,l=1}^{n} R_{kl}\Big(\sum_{p_1,q_1=1}^{n} R_{kp_1}^{-1}V_{q_1i}B_{p_1}^{q_1}\Big)\Big(\sum_{p_2,q_2=1}^{n} R_{lp_2}^{-1}V_{q_2i}B_{p_2}^{q_2}\Big) -$$
$$-2\sum_{k,l=1}^{n} V_{kl}B_l^k\Big(\sum_{p,r=1}^{n} R_{lp}^{-1}V_{rj}B_p^r\Big)\Big] = 0$$

3. Duffing's oscillator

We consider a stochastic model of Duffing's oscillator described by the Itô vector stochastic differential equation

$$\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= (-\omega_0^2 x_1 - 2hx_2 - \varepsilon x_1^3 - bu)\,dt + \\ &+ (-\omega_0^2 x_1 - \varepsilon x_1^3)\delta_1\,d\xi_1 + \delta_2 x_2 d\xi_2 + \delta_0 d\xi_0 \end{aligned} \tag{3.1}$$

where ω_0^2, h, ε, b and σ are constant parameters, ξ_1, ξ_2, ξ_3 are mutually independent standard Wiener processes.

Assuming that the control is a state feedback in the form of odd order polynomials i.e.

$$u = \sum_{q=1,3,5,\dots}\sum_{i=0}^{q}\binom{q}{i} k_{q-i\;i}x_1^{q-i}x_2^i \tag{3.2}$$

where k_{ij} are constant coefficients which are designed to minimize the quadratic cost function

$$I = \lim_{T\to\infty}\frac{1}{T}E\Big[\int_0^\infty \Big(q_1x_1^2(s) + q_2x_2^2(s) + ru^2(s)\Big)\,ds\Big] \tag{3.3}$$

where q_1, q_2 and r are weight coefficients.

3.1. Main results

To obtain the control force in equation (3.2), which minimizes the cost function in equation (3.3) constrained by the system of equations (3.1), one has to solve the following Bellman's equation (Fleming and Richel, 1975)

$$\frac{\partial V}{\partial t} + \min_u [\mathcal{L}^*(V) + L(\boldsymbol{x}, u)] = 0 \tag{3.4}$$

where $V = V(\boldsymbol{x}, t)$ is the Lyapunov function proposed in the form

$$\begin{aligned} V(\boldsymbol{x}, t) &= \alpha_{00} + \sum_{p=2,4,6,\ldots} \sum_{i=0}^{p} \binom{p}{i} \alpha_{p-i\,i} x_1^{p-i} x_2^i \\ L(\boldsymbol{x}, t) &= q_1 x_1^2 + q_2 x_2^2 + r u^2 \end{aligned} \tag{3.5}$$

$\mathcal{L}^*(\cdot)$ is an infinitesimal operator defined by

$$\begin{aligned} \mathcal{L}^*(\cdot) &= x_2 \frac{\partial(\cdot)}{\partial x_1} + (-\omega_0^2 x_1 - 2h x_2 - \varepsilon x_1^3 - bu) \frac{\partial(\cdot)}{\partial x_2} + \\ &+ \frac{1}{2}\Big[\delta_0^2 + \delta_1^2(-\omega_0^2 x_1 + \varepsilon x_1^3)^2 + \delta_2^2 x_2^2\Big] \frac{\partial^2(\cdot)}{\partial x_2^2} \end{aligned} \tag{3.6}$$

For simplicity, we consider three cases: when $n = 1$ and $p = 2$, $n = 3$ and $p = 4$, $n = 5$ and $p = 6$. Substituting quantities (3.5) and (3.6) into equation (3.4) we minimize the obtained equation with respect to u. Then, we obtain the linear algebraic relationships between coefficients k_{ij} and α_{ij}

$$\begin{array}{lll} k_{10} = a_1 \alpha_{11} & k_{01} = a_1 \alpha_{02} & k_{30} = 2a_1 \alpha_{31} \\ k_{21} = 2a_1 \alpha_{22} & k_{12} = 2a_1 \alpha_{13} & k_{03} = 2a_1 \alpha_{04} \\ k_{50} = 3a_1 \alpha_{51} & k_{41} = 3a_1 \alpha_{42} & k_{32} = 3a_1 \alpha_{33} \\ k_{23} = 3a_1 \alpha_{24} & k_{14} = 3a_1 \alpha_{15} & k_{05} = 3a_1 \alpha_{06} \end{array} \tag{3.7}$$

where $a_1 = b/r$.

Substituting quantities (3.5)-(3.7) into equation

$$\frac{\partial V}{\partial t} + \mathcal{L}^*(V) + L(\boldsymbol{x}, u) = 0 \tag{3.8}$$

and averaging with respect to the stationary measure generated by the output process (solution to equation (3.1)) and equating the coefficients of even order polynomials to zero we find the nonlinear differential equations for the coefficients α_{ij}, $i+j=p$, $i,j=0,1,...,p$, $p=2,4,6,...$

$$\dot{\alpha}_{ij}(t) = F_{ij}(\alpha_{00}, \alpha_{10}, ..., \alpha_{06}, \omega_0^2, h, \varepsilon, b, \sigma, q_1, q_2, r) \tag{3.9}$$

In a general case, when both external and parametric excitations are considered i.e. $\delta_0 \neq 0$, $\delta_1 \neq 0$ and $\delta_2 \neq 0$ the stationary solutions to equations (3.9) $\alpha_{ij}(\infty)$ determine the optimal stationary coefficients $k_{ij} = k_{ij}(\infty)$, $i+j=q$, $i,j=0,1,...,q$, $q=1,3,5,...$ by relations (3.7). To find the optimal costs we have to substitute the feedback control u defined by (3.3), (3.7) and $\alpha_{ij}(\infty)$ into the cost function I in equation (3.3). After averaging we find that the cost function I depends on the stationary coefficients k_{ij} and stationary moments of the output process $\boldsymbol{x} = [x_1, x_2]^\top$ i.e. the quantities $E[x_1^{p_1} x_2^{p_2}]$ where $p_1 + p_2 = 2,4,6,...$. These stationary moments can be found (approximately) from moment equations for the original system (3.1) where u is defined by (3.3), (3.7) and $\alpha_{ij}(\infty)$ where, for instance, the cumulant closure technique is applied. This can be treated as a general procedure. In a particular case when only linear parametric excitations are considered i.e. $\delta_0 \neq 0$, $\delta_1 = 0$ and $\delta_2 = 0$, then the general procedure of the determination of the optimal control and minimal cost function reduces significantly. It has the form described below.

Simplified procedure:

Step 1. We find the stationary solutions for α_{ij}, $i+j=2$, $i,j=0,1,2$ from the system of equations (3.9) which is closed.

Step 2. From equalities (3.7) we find the coefficients k_{ij}, $i+j=1$, $i,j=0,1$ and we substitute them into the moment equations.

Step 3. We find the stationary moments $E[x_i x_j]$, $i+j=2$, $i,j=0,1,2$.

Step 4. Using the stationary solutions α_{ij}, k_{ij} and $E[x_i x_j]$ obtained in steps 1 to 3, similarly we find the stationary solutions for α_{ij}, $i+j=4$, $i,j=0,1,2,3,4$ from system (3.9), k_{ij}, $i+j=3$, $i,j=0,1,2,3$ from system (3.7) and $E[x_i x_j x_k x_l]$, $i+j+l+k=4$, $i,j,l,k=0,1,2,3,4$ from the moment equations.

Step 5. Using the stationary solutions α_{ij}, k_{ij}, $E[x_i x_j]$ and $E[x_i x_j x_k x_l]$ obtained in steps 1 to 4, similarly we find the stationary solutions for α_{ij}, $i+j=6$, $i,j=0,1,...,6$ from system (3.9), k_{ij}, $i+j=5$, $i,j=0,1,...,5$ from system (3.7) and $E[x_i x_j x_l x_k x_p x_q]$, $i+j+l+k+p+q=6$, $i,j,l,k,p,q=0,1,...,6$ from the moment equations.

Step 6. We calculate the stationary value of criterion (3.3) i.e. the integral given by (2.4). The advantage of this procedure, when compared with the general one, lies is in finding the stationary solutions to system of equations (3.9) and moment equations. In the general procedure these equations are coupled and have to be solved together while in the simplified procedure they can be solved separately.

We illustrate this approach by the following example.

3.2. Example

We consider the Duffing oscillator for the following parameters $\omega_0^2 = 1$, $h = 0.01$, $\delta_0 \neq 0$, $\delta_1 = 0$, $\delta_2 = 0$, $q_1 = 100$, $q_2 = 10$, $r = 1$.

The plot of I versus parameters t, ε and δ_0 is given in Fig. 1, Fig. 2 and Fig. 3, respectively. The diagrams enable one to compare the first-order and higher order controls.

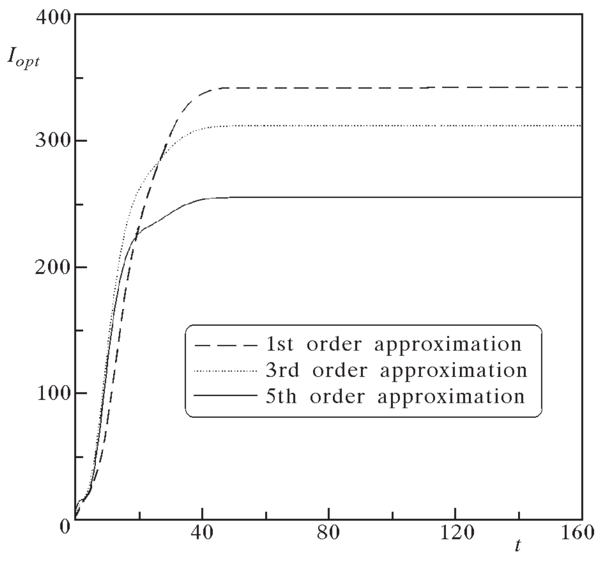

Fig. 1. Cost function I versus coefficient t

3.3. Final conclusions

The minimization of the cost function I, (3.3), has been analysed for three approximations of the control function, viz. approximation of the 1st order (linear approximation), 3rd order and 5th order. The obtained numerical results

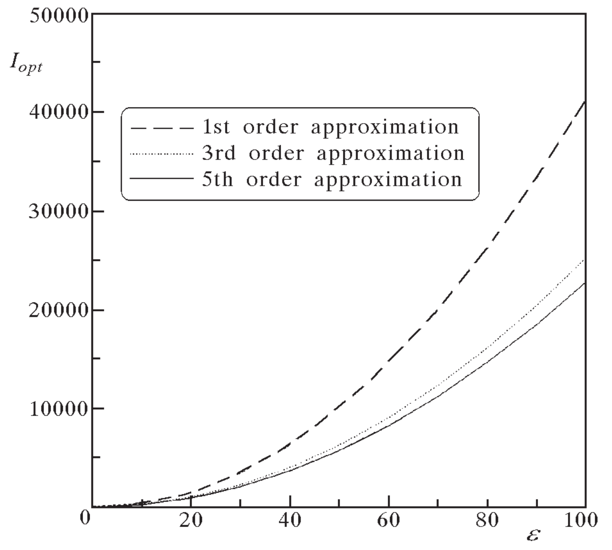

Fig. 2. Cost function I versus coefficient α

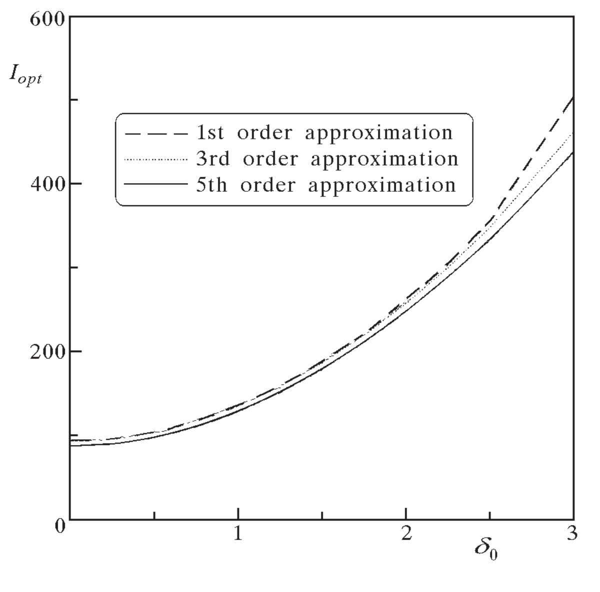

Fig. 3. Cost function I versus coefficient δ_0

indicate that the approximation of the 5th order is the closest to the solutions of statistical linearization described in Socha (2000).

The method of series involving systems with one degree of freedom presented in this paper provides good and interesting results at a rather low expenditure of labour, but in the case of systems with a larger number of degrees of freedom this method proves to be ineffective due to a considerable number of calculations to be done. Yet, the numerical results remain satisfactory when compared with other methods of the statistical linearization, as indicated in the master thesis by Nowoświat (1999).

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Optymalne sterownie parametrycznym i zewnętrznie wzbudzonym oscylatorem Duffinga

Streszczenie

W pracy zaproponowano metodę perturbacji dla sterowania układów dynamicznych z wymuszeniami o charakterze białych szumów Gaussowskich i kryteriami w przestrzeni funkcji gęstości prawdopodobieństw. Przy wyznaczaniu współczynników sterowania, jak również współczynników funkcji Lapunowa, korzysta się z równania Bellmana. Tak wyznaczone współczynniki niezbędne są do minimalizacji funkcji kosztów I. Szczegółową analizę i obliczenia numeryczne wykonano za pomocą procedur Rungego-Kutty.

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