JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
40, 4, pp. 1001-1019, Warsaw 2002

A STOCHASTIC EIGENPROBLEM OF BEAMS WITH CRACKED CROSS-SECTIONS

JAN DREWKO

Faculty of Marine Technology, Technical University of Szczecin, Poland
e-mail: drewko@shiptech.tuniv.szczecin.pl

TRAN DUONG HIEN

Department of Civil Engineering, Technical University of Szczecin, Poland
e-mail: tdhien@man.szczecin.pl

A nonstatistical methodology is introduced for eigenproblems of cracked cross-sectional beam systems with random parameters. The formalism is based on a combination of the second-order perturbation technique and mean-centered second moment analysis. The system random parameters are defined by their first two probabilistic moments. Hierarchical equations are obtained and solved for the first two probabilistic moments for the eigenvalue field. As the system matrix is nonsymmetric, a procedure for the exact solution of the sensitivity equations is proposed with each eigenvalue derivative solved for separatively. Analytical and numerical aspects of the problem are discussed and illustrative results are given. The approach presented is general and may be employed for a wide class of problems of fracture mechanics.

Key words: fracture mechanics, effective stiffness, stochastic eigenpair, second-order perturbation

1. Introduction

In structural analysis damages of beams and frames are frequently considered in terms of the system parameter behaviour. Useful information for improving the strength of a beam can be obtained on the basis of the eigenproblem sensitivity by evaluating the change of a natural frequency to variations of system parameters, the size and position of a cross-sectional crack for instance. This issue has been discussed extensively in the literature (Kobayashi, 1971; Nowacki, 1972; Tada et al., 1973; Knott, 1976; Drewko and Sperski, 1991; Drewko, 1999a,b, [9]). In most of those formulations, however, the material and local geometric parameters such as Young's modulus, Poisson's ratio, cross-sectional area, crack length, etc., are assumed to be deterministic quantities. Since the natural character of the system parameters are irregularity and uncertainty, a problem formulated in the framework of simplified, deterministic modelling may in some cases be misleading in interpretation of the given system. A model regarding the randomness of the system parameters should be adequate in this context.

Recently, an approach based on the mean-based second-moment analysis in combination with a first- or second-order perturbation technique has been investigated, verified on a number of engineering systems with random parameters as in Hishada and Nakagiri (1981), Vanmarcke and Grigoriu (1983), Liu et al. (1986), for instance, and proved to be much more efficient than statistical techniques typical of Monte Carlo simulation (Tocher, 1968). The main advantage of this non-statistical methodology is that only the first two probabilistic moments of random parameters, i.e. spatial expectations and cross-covariances or cross-correlations, are required on input, while in the statistical approach the whole probabilistic structure, i.e. probability density or probability distribution functions, and a large number of samples generated randomly are needed. Shortcomings of the approach are typical of any perturbation-based techniques – the method is effective for systems with relatively small fluctuation in random parameters and described by unique and smooth solutions.

Various classes of uncertainties have been incorporated into a system description in a natural manner by means of stochastic variational principles. We may mention the stochastic version of the minimum potential energy principle (Liu et al., 1986), multi-field principles (liu et al., 1988; Kleiber and Hien, 1992), Hamilton's principle (Hien and Kleiber, 1990), virtual temperature principle (Hien and Klieber, 1997, 1998), for instance. The most important feature of these developments is that the random parameter problems are formulated so that the differential operators acting on the left-hand sides of the resulting hierarchical equations are the same, all the probabilistic features being translated into the right-hand sides as easily computable functions. On the basis of these stochastic statements a large number of engineering problems, structural statics and dynamics in particular, have been formulated and many numerical algorithms have been implemented into computer codes. In the literature such non-statistical stochastic formulations related to eigenproblems are still rather scarce (Kleiber and Hien, 1992, 1997; Collins and Thomson, 1969), while for a wide class of problems of fracture mechanics are seemingly nonexisting.

Expanding the above-mentioned developments in this paper, an analytical methodology is put forward for eigenproblems of cracked cross-sectional beam systems with random parameters. A second-order perturbation second-centered moment formulation is employed and a hierarchical equation system is obtained, Section 2. On account of the unsymmetrical character of the system matrix a procedure for the exact solution of the sensitivity equations is introduced and each eigenvalue derivative with respect to a random variable is solved for separatively. Standard algorithms worked out and implemented for beam systems with crack length defined as random variables are described in Section 3. Section 4 illustrates the proposed formalism via two numerical examples. Concluding remarks are given in the last section.

2. Problem statement. Stochastic formulation

It is well-known (Nowacki, 1972) that the equation of motion of a prismatic beam with the rotation inertia and shear deformation effects taken into account can be expressed in terms of the transverse vibration $w(z,t)$ at the abscissa z defined along the beam axis and at time t in the form

$$EJ\frac{\partial^4 w}{\partial z^4} + \varrho A\frac{\partial^2 w}{\partial t^2} - \varrho J\Big(1+\frac{E}{\widehat{k}G}\Big)\frac{\partial^4 w}{\partial z^2 \partial t^2} + \frac{J\varrho^2}{\widehat{k}G}\frac{\partial^4 w}{\partial t^4} = 0 \tag{2.1}$$

which, by using the variable separation $w(z,t) = y(z)Z(t)$ with

$$\frac{\partial^4 y}{\partial z^4} + \frac{\varrho A\omega^2\kappa^2}{EJ}\Big(1+\frac{E}{\widehat{k}G}\Big)\frac{\partial^2 y}{\partial z^2} - \frac{\varrho A\omega^2}{EJ}\Big(1-\frac{\varrho A\omega^2\kappa^2}{\widehat{k}GA}\Big)y = 0 \tag{2.2}$$

can be integrated for the general solution written in terms of the deflection amplitude $y(z)$ as

$$y(z) = \varphi_1\cosh k_1 z + \varphi_2\sinh k_1 z + \varphi_3\cos k_2 z + \varphi_4\sin k_2 z \tag{2.3}$$

with

$$k_{\frac{1}{2}} = \sqrt{\mp\frac{\varrho A\omega^2\kappa^2}{2EJ}\Big(1+\frac{E}{\widehat{k}G}\Big) + \sqrt{\Big[\frac{\varrho A\omega^2\kappa^2}{2EJ}\Big(1+\frac{E}{\widehat{k}G}\Big)\Big]^2 + \frac{\varrho A\omega^2}{EJ}\Big(1-\frac{\varrho A\omega^2\kappa^2}{\widehat{k}GA}\Big)}} \tag{2.4}$$

$$\kappa^2 = \frac{J}{A} \qquad \widehat{k} = \frac{\kappa^2 b}{S}$$

where the symbols φ_α, $\alpha = \overline{1,4}$, denote the system coefficients to be determined, ω – frequency, ϱ – mass density, E and G – Young's and shear moduli, S – static moment and of the upper (or lower) part of the cross-sectional area with respect to the cross-section neutral axis, J – moment of inertia of the cross-sectional area with respect to the neutral axis, b – length of the part of the neutral axis on the cross-section, A – cross-sectional area. It is noted that the above four integration constants φ_α take various values for each beam section between the cracks.

If the beam is assumed to be weakened by cross-sectional cracks, a crack of the length a of the cross-section at x can effectively be modelled by employing an elastic hinge of stiffness being a decreasing function of the crack length, $K = K(a)$ (Drewko, 1999a,b, 2000). In this case, the system at hand requires additional conditions to be satisfied at the crack coordinate $z = x$ as follows

— equilibrium condition

$$[EJy'']_x = K(a)(y'|_{x+} - y'|_{x-}) \tag{2.5}$$

— deflection continuity condition

$$y|_{x-} = y|_{x+} \tag{2.6}$$

— bending moment continuity condition

$$[EJy'']_{x-} = [EJy'']_{x+} \tag{2.7}$$

— shear force continuity condition

$$[EJy''']_{x-} = [EJy''']_{x+} \tag{2.8}$$

By employing the initial parameter technique and on account of the specific boundary constraints, Eq. (2.3) becomes (cf. Drewko and Sperski, 1991)

$$\boldsymbol{\Omega}(\omega)\boldsymbol{\phi} = \mathbf{0} \tag{2.9}$$

where $\boldsymbol{\phi} = \{\varphi_1, \varphi_2, \dots, \varphi_{4(N+1)}\}$, with N being the number of the cracks, cf. Eq. (2.3). Equation (2.9), typical of the Sturm-Liouville problem, is solved for the natural eigenpair(s) (ω^2, ϕ), each unknown ω being an argument of the matrix functions $\boldsymbol{\Omega}(\omega)$ entered this nonsymmetric matrix.

By employing a combination of the second-order perturbation technique and second moment analysis, Eq. (2.9) serves as the basis for the stochastic formulation discussed below.

Let us assume $\boldsymbol{q} = \{q_r(x)\}$, $r = 1, 2, \ldots, R$, to be a set of R random variables (parameters) which can represent spatial randomness in the cross-sectional area or its dimensions, static moment and moment of inertia, Young's and shear moduli, system mass, crack length, x-abscissa, etc. Clearly, the unknowns (ω^2, ϕ) are implicit functions of the random variables. The first two probabilistic moments for the random variable field q_r are defined as

$$\begin{aligned} \mathrm{E}[q_r] = q_r^0 &= \underbrace{\int_{-\infty}^{+\infty}}_{R-\text{fold}} q_r p_R \, dq_1 dq_2 \ldots dq_R \\ \mathrm{Cov}\,(q_r, q_s) &= \underbrace{\int_{-\infty}^{+\infty}}_{R-\text{fold}} (q_r - q_r^0)(q_s - q_s^0) p_R dq_1 dq_2 \ldots dq_R \end{aligned} \tag{2.10}$$

The latter definition corresponds to (from now on an indicial notation is used, and twice-repeated indices imply summation)

$$\mathrm{Cov}\,(q_r, q_s) = \alpha_r \alpha_s q_r^0 q_s^0 \mu_{rs} \tag{2.11}$$

with

$$\begin{aligned} \alpha_r &= \frac{\sqrt{\mathrm{Var}\,(q_r)}}{\mathrm{abs}(q_r^0)} \\ \mu_{rs} &= \underbrace{\int_{-\infty}^{+\infty}}_{R-\text{fold}} q_r q_s p_R \, dq_1 dq_2 \ldots dq_R \qquad \text{(no sum on } R) \end{aligned} \tag{2.12}$$

where $\mathrm{E}[q_r]$, $\mathrm{Cov}\,(q_r, q_s)$, $\mathrm{Var}\,(q_r)$, μ_{rs}, α_r and $p_R = p_R(q_1, q_2, \ldots, q_R)$ are the spatial expectations, cross-covariances, variances, cross-correlation functions, coefficients of variation and R-variate probability density function, respectively.

The nonstatistical formalism embodies the probabilistic distributions, as reflected in the first two probabilistic moments for the random field variables $q_r(x)$ to obtain the first two moments for the natural frequency $\omega(q(x), x)$ and eigenvector $\varphi_\alpha(q(x), x)$. The basic idea behind the second moment analysis, when combined with the second-order perturbation technique, involves expanding all the functions of the random variables $q_r(x)$ included in Eq. (2.9), i.e. the ones explicit in $q(x)$, i.e. cross-sectional area $A(q(x),x)$ or its dimensions $b(q(x),x)$ and $h(q(x),x)$, static moment $S(q(x),x)$ and moment of inertia $J(q(x),x)$, Young's and shear moduli $E(q(x),x)$ and $G(q(x),x)$, Poisson's ratio $\nu(q(x),x)$, mass $m(q(x),x)$, crack length $a(q(x),x)$, etc., and two ones implicit in $q(x)$, i.e. $\omega(q(x),x)$ and $\phi(q(x),x)$ around the spatial expectations of $\boldsymbol{q} = \{q_r(x)\}$, i.e. $\boldsymbol{q}^0 = \{q_r^0\} = \{q_r^0(x)\}$, via Taylor's series and retaining terms up to the second order. These expansions are written symbolically in the variational form as

$$(\cdot) = (\cdot)^0 + (\cdot)^{,r}\Big|_{\boldsymbol{q}=\boldsymbol{q}^0}\delta q_r + \frac{1}{2}(\cdot)^{,rs}\Big|_{\boldsymbol{q}=\boldsymbol{q}^0}\delta q_r\delta q_s \tag{2.13}$$

where

$$\delta q_r = \epsilon\Delta q_r = \epsilon(q_r - q_r^0) \tag{2.14}$$

is the first variation of q_r about q_r^0, while

$$\delta q_r\delta q_s = \epsilon^2\Delta q_r\Delta q_s \tag{2.15}$$

is the mixed second variation of q_r and q_s about q_r^0, and q_s^0 and ϵ is a given small parameter. The symbol $(\cdot)^0$ represents the value of the functions taken at the expectations q_r^0 while $(\cdot)^{,r}$ and $(\cdot)^{,rs}$ denote the first and mixed second partial absolute derivatives with respect to q_r evaluated at q_r^0, respectively.

In accordance with the philosophy of the perturbation approach, expansions (2.13) are now substituted into Sturm-Liouville system (2.9). Since the first and second variations are arbitrary and mutually independent, equating the variational terms of equal orders in the resulting expression yields the following hierarchical equation system:

— zeroth-order (one equation)

$$\Omega^0_{\alpha\beta}(\omega^0)\phi^0_\beta = 0 \tag{2.16}$$

— first-order (R equations, $r = 1,2,\dots,R$)

$$\Omega^0_{\alpha\beta}(\omega^0)\phi^{,r}_\beta\Big|_{\boldsymbol{q}=\boldsymbol{q}^0} = -\Omega^{,r}_{\alpha\beta}(\omega^0,\omega^{,s})\Big|_{\boldsymbol{q}=\boldsymbol{q}^0}\phi^0_\beta \tag{2.17}$$

— second-order (one equation)

$$\begin{aligned}\Omega^0_{\alpha\beta}(\omega^0)\phi^{(2)}_\beta\Big|_{\boldsymbol{q}=\boldsymbol{q}^0} &= -\Big\{\Omega^{(2)}_{\alpha\beta}(\omega^0,\omega^{,t},\omega^{(2)})\Big|_{\boldsymbol{q}=\boldsymbol{q}^0}\phi^0_\beta + \\ &+2[\Omega^{,r}_{\alpha\beta}(\omega^0,\omega^{,t})\phi^{,s}_\beta]_{\boldsymbol{q}=\boldsymbol{q}^0}\,\mathrm{Cov}\,(q_r,q_s)\Big\}\end{aligned} \tag{2.18}$$

where $\alpha, \beta = 1, 2, \dots, 4(N+1)$ and the symbol $(\cdot)^{(2)}$ denotes the double sum $(\cdot)^{,rs}\,\mathrm{Cov}\,(q_r, q_s)$ with $r, s, t = 1, 2, \dots, R$.

We observe that (i) all the quantities included in Eqs (2.16)-(2.18) are deterministic functions – evaluated at $q_r = q_r^0$, and (ii), except for the unknowns ϕ^0, $\phi^{,r}$ and $\phi^{(2)}$, all the entries on the left-hand side of Eqs (2.16)-(2.18) are identical, i.e. the same operator Ω^0 acts on the left-hand sides of all the equations, all the probabilistic characteristics of the problem being translated entirely into the effective loads on the right-hand sides.

An aspect, particularly important in terms of computational implementation, should be made here that by employing the second moment analysis we have to deal with only one second-order equation, instead of R^2 equations as required traditionally in the 'deterministic' perturbation approach. (In fact, the total number of the second-order equations to be solved is $R \times (R+1)/2$ since the second derivatives are symmetric with respect to q_r and q_s, i.e. $(\cdot)^{,rs} = (\cdot)^{,sr}$.) It is because Eq. (2.18) is obtained by averaging the second-order variation terms. More specificly, the second-order term involving the mixed derivatives of the eigenvector ϕ, for instance, is averaged as, cf. Eqs (2.13)-(2.15)

$$
\begin{aligned}
&\mathrm{E}[\phi^{,rs}\delta q_r \delta q_s] = \epsilon^2 \phi^{,rs}\mathrm{E}[(q_r - q_r^0)(q_s - q_s^0)] = \\
&= \epsilon^2 \phi^{,rs}\,\mathrm{Cov}\,(q_r, q_s) = \epsilon^2 \phi^{(2)}
\end{aligned}
\tag{2.19}
$$

Zeroth-order equation (2.18) is seen to be in aform identical to governing equation (2.9) and as such it can serve as the basis to obtain the zeroth-order natural frequencies $\omega^0 = \omega_i^0$ and shapes $\phi^0 = \phi_i^0$, $i = 1, 2, \dots, \infty$. Once (ω^0, ϕ^0) are known, the higher-order pairs $(\omega^{,r}, \phi^{,r})$ and $(\omega^{(2)}, \phi^{(2)})$ can be solved for from R independent first-order equations (2.17) and single second-order equation (2.18), in succession. Since the system matrix $\Omega_{\alpha\beta}$ is singular, the first- and second-order nonhomogeneous equations are generally solved for $(\omega^{,r}, \phi^{,r})$ and $(\omega^{(2)}, \phi^{(2)})$ by using an approximate procedure. As an exceptional case, however, the exact values of the first-order natural frequencies $\omega^{,r}$ can be calculated in a simple way. This issue will be discussed in detail in the next section.

Having solved the equation system (2.16)-(2.18) for (ω^0, ϕ^0), $(\omega^{,r}, \phi^{,r})$ and $(\omega^{(2)}, \phi^{(2)})$ probabilistic distributions of the random fields ω and ϕ may, for a given ϵ, be computed (setting $\epsilon = 0$ yields the deterministic solution). In our case, the formal solution is obtained by setting $\epsilon = 1$ which, of course, stipulates that the fluctuation of the random field variables q_r is small. Thus, by averaging the expanded equation for ω, cf. Eq. (2.13)

$$\omega = \omega^0 + \omega^{,r}\Big|_{\boldsymbol{q}=\boldsymbol{q}^0} \Delta q_r + \frac{1}{2}\omega^{,rs}\Big|_{\boldsymbol{q}=\boldsymbol{q}^0} \Delta q_r \Delta q_s \tag{2.20}$$

the second-order accurate expectations for the natural frequencies read

$$\begin{aligned} \mathrm{E}[\omega] &= \omega^0 + \omega^{,r}\mathrm{E}[q_r - q_r^0] + \frac{1}{2}\omega^{,rs}\mathrm{E}[(q_r - q_r^0)(q_s - q_s^0)] = \\ &= \omega^0 + 0 + \frac{1}{2}\omega^{,rs}\,\mathrm{Cov}\,(q_r, q_s) = \omega^0 + \frac{1}{2}\omega^{(2)} \end{aligned} \tag{2.21}$$

Clearly, if only the first-order accuracy is losked for Eq. (2.21) reduces to

$$\mathrm{E}[\omega] = \omega^0 \tag{2.22}$$

In order to determine the second probabilistic moments for the natural frequencies, expansion (2.20) is used without the second-order term, i.e.

$$\omega_i(x) = \omega_i^0(x) + \omega_i^{;r}(x)\Big|_{\boldsymbol{q}=\boldsymbol{q}^0} \Delta q_r \qquad i = 1, 2, \ldots \tag{2.23}$$

so that the first-order accurate cross-covariances for $(\omega_i(x_1), \omega_j(x_2))$ take the form, cf. Eq. (2.22)

$$\begin{aligned} \mathrm{Cov}\,(\omega_i(x_1), \omega_j(x_2)) &= \mathrm{E}[(\omega_i(x_1) - \omega_i^0(x_1))(\omega_j(x_2) - \omega_j^0(x_2))] = \\ &= \omega_i^{;r}(x_1)\omega_j^{;s}(x_2)\,\mathrm{Cov}\,(q_r, q_s) \end{aligned} \tag{2.24}$$

It is pointed out here that the second-order accurate expectations and first-order accurate cross-covariances for eigenpairs fields are consistent with the second moment strategy, since the output second-order accurate cross-covariances would require the probabilistic moments for the random variables q_r up to the fourth order on input.

Following the same lines as for the natural frequencies, the first two probabilistic moments for the eigenvectors can be expressed as

$$\begin{aligned} &\mathrm{E}[\phi] = \phi^0 + \frac{1}{2}\phi^{(2)} \\ &\mathrm{Cov}\,(\phi_i(x_1), \phi_j(x_2)) = \phi_i^{;r}(x_1)\phi_j^{;s}(x_2)\,\mathrm{Cov}\,(q_r, q_s) \qquad i = 1, 2, \ldots \end{aligned} \tag{2.25}$$

3. First-order sensitivity of natural frequencies with respect to random variables

Putting forward the procedure introduced in Nelson (1976), a simple algorithm for the exact solution of the first partial absolute derivatives of natural frequencies with respect to random variables is discussed in this section. Recall that in contradistinction to standard nonsymmetric eigenproblems, where eigenvalues stand for independent terms consisting only in the diagonal elements of a system matrix $\Omega_{\alpha\beta}$, in our case the natural frequencies are arguments of almost all functions entering $\Omega_{\alpha\beta}$, i.e. are involved in off-diagonal elements of the system matrix as well.

Let us start with pre-multiplying Eq. (2.16) by the right-handed eigenvector ψ_α transposed to get

$$\psi_\alpha\, \Omega_{\alpha\beta}\, \phi_\beta = 0 \tag{3.1}$$

that, differentiated with respect to the random variables q_r, yields

$$\psi_\alpha^{,r}\, \Omega_{\alpha\beta}\, \phi_\beta + \psi_\alpha\, \Omega_{\alpha\beta}^{,r}\, \phi_\beta + \psi_\alpha\, \Omega_{\alpha\beta}\, \phi_\beta^{,r} = 0 \tag{3.2}$$

Since the first and last terms on the left-hand side vanish by the definition of eigenproblems, we obtain

$$\psi_\alpha\, \Omega_{\alpha\beta}^{,r}\, \phi_\beta = 0 \tag{3.3}$$

that stands for the basic formula and can be employed directly to compute the first-order sensitivity of each natural frequency $\omega = \omega_i$, with fixed $i = 1, 2, \ldots$, to a change of the random parameter q_r, $r = 1, 2, \ldots, R$. To be specific, let a crack of the length a be assumed random with the expectation a^0. Equation (3.3) reading now $(\alpha, \beta = 1, 2, \ldots, 4(N+1))$

$$\psi_\alpha\Big[\frac{\partial \Omega_{\alpha\beta}}{\partial K}\frac{dK}{da} + \Big(\frac{\partial \Omega_{\alpha\beta}}{\partial k_1}\frac{\partial k_1}{\partial \omega} + \frac{\partial \Omega_{\alpha\beta}}{\partial k_2}\frac{\partial k_2}{\partial \omega}\Big)\frac{d\omega}{da}\Big]\phi_\beta = 0 \tag{3.4}$$

implies

$$\frac{d\omega}{da} = -\Big[\psi_\alpha\Big(\frac{\partial \Omega_{\alpha\beta}}{\partial k_1}\frac{\partial k_1}{\partial \omega} + \frac{\partial \Omega_{\alpha\beta}}{\partial k_2}\frac{\partial k_2}{\partial \omega}\Big)\phi_\beta\Big]^{-1}\Big(\psi_\gamma\frac{\partial \Omega_{\gamma\eta}}{\partial K}\frac{dK}{da}\phi_\eta\Big) \tag{3.5}$$

with all functions being evaluated at $a = a^0$. We remember that the solution of type (3.5) can be treated as the exact result only in the context of the zeroth-order natural frequencies ω_i^0 obtained approximately before.

4. Illustrative examples

Two prismatic beams with the rectangular cross-section $A = b \times h$ of the length l are discussed in this section. The first one is a simple cantilever, Fig. 1 while the second is fixed at one edge and hinged at the other, Fig. 2.

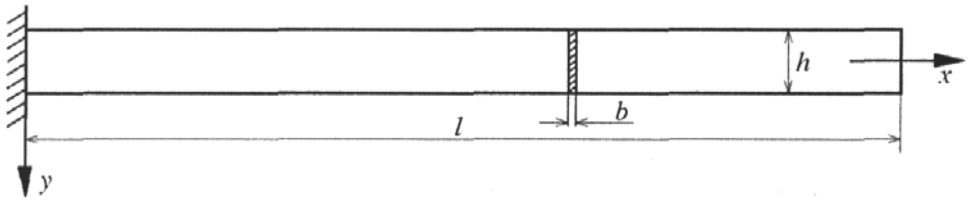

Fig. 1. Cantilever beam

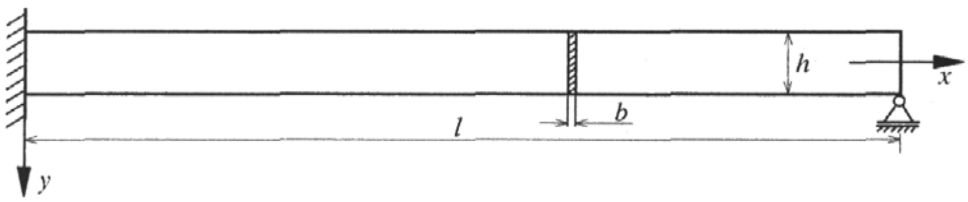

Fig. 2. Fixed-hinged beam

The SI measure units (N,mm,s) will be used throughout for input and output data. The crack length on the cross-sectional surface at x-coordinate, $a = a(x)$, is assumed to be random, defined by the first two probabilistic moments and considered for two cases:

(a) expectation $\mathrm{E}[a] = a^0 = 10$ and standard deviation $\sigma_a = 1.5$ and

(b) $\mathrm{E}[a] = a^0 = 24$ and $\sigma_a = 3.6$.

All the other input data are considered as deterministic quantities: cross-sectional dimensions $b \times h = 7 \times 40$, length $l = 600$, Young's modulus $E = 2.06 \times 10^5$, Poisson's ratio $\nu = 0.3$, mass density $\varrho = 7.83 \times 10^{-9}$. The problem is to find the first two probabilistic moments for the system natural frequencies.

The number of cracks is equal to one of the elements of the system matrix $\Omega_{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, 8$. Reducing the matrix to a (4×4)-matrix, yields:
— first two columns, $\Omega_{1\alpha}$ and $\Omega_{2\alpha}$, for both the cantilever and fixed-hinged beams

$$
\begin{aligned}
\Omega_{11} &= \sin k_2x - (k_2/k_1)\sinh k_1x \\
\Omega_{21} &= k_2^2 \sin k_2x + k_1k_2 \sinh k_1x \\
\Omega_{31} &= \Omega_{21}EJ/K + k_2(\cosh k_1x - \cos k_2x) \\
\Omega_{41} &= k_2^3 \cos k_2x + k_1^2k_2 \cosh k_1x \\
\Omega_{12} &= \cos k_2x - \cosh k_1x \\
\Omega_{22} &= k_2^2 \cos k_2x + k_1^2 \cosh k_1x \\
\Omega_{32} &= \Omega_{22}EJ/K + k_2 \sin k_2x + k_1 \sinh k_1x \\
\Omega_{42} &= k_1^3 \sinh k_1x - k_2^3 \sin k_2x
\end{aligned}
\tag{4.1}
$$

— last two columns, $\Omega_{3\alpha}$ and $\Omega_{4\alpha}$, for the cantilever beam only

$$
\begin{aligned}
\Omega_{13} &= -\sin k_2x - \gamma\cos k_2x - \vartheta\cosh k_1x \\
\Omega_{23} &= -k_2^2 \sin k_2x - \gamma k_2^2 \cos k_2x + \vartheta k_1^2 \cosh k_1x \\
\Omega_{33} &= k_2 \cos k_2x - \gamma k_2^2 \sin k_2x + \vartheta k_1 \sinh k_1x \\
\Omega_{43} &= -k_2^3 \cos k_2x + \gamma k_2^3 \sin k_2x + \vartheta k_1^3 \sinh k_1x \\
\Omega_{14} &= \eta\cos k_2x - \sinh k_1x + \zeta\cosh k_1x \\
\Omega_{24} &= \eta k_2^2 \cos k_2x + k_1^2 \sinh k_1x - \zeta k_1^2 \cosh k_1x \\
\Omega_{34} &= \eta k_2 \sin k_2x + k_1 \cosh k_1x - \zeta k_1 \sinh k_1x \\
\Omega_{44} &= -\eta k_2^3 \sin k_2x + k_1^3 \cosh k_1x - \zeta k_1^3 \sinh k_1x \\
\gamma &= \xi(k_2 \cosh k_1l \cos k_2l - k_1 \sinh k_1l \sin k_2l) \\
\zeta &= \xi(k_1 \cosh k_1l \cos k_2l + k_2 \sinh k_1l \sin k_2l) \\
\eta &= \xi k_1^3/k_2^2 \\
\vartheta &= \xi k_2^3/k_1^2 \\
\xi &= (k_1 \sinh k_1l \cos k_2l + k_2 \cosh k_1l \sin k_2l)^{-1}
\end{aligned}
\tag{4.2}
$$

— last two columns, $\Omega_{3\alpha}$ and $\Omega_{4\alpha}$, for the fixed-hinged beam only

$$
\begin{aligned}
\Omega_{13} &= -\sin k_2x + \cos k_2x \tan k_2l \\
\Omega_{23} &= k_2^2\Omega_{13} \\
\Omega_{33} &= k_2(\cos k_2x + \sin k_2x \tan k_2l) \\
\Omega_{43} &= -k_2^2\Omega_{33} \\
\Omega_{14} &= -\sinh k_1x + \cosh k_1x \tanh k_1l \\
\Omega_{24} &= -k_1^2\Omega_{14} \\
\Omega_{34} &= k_1(\cosh k_1x - \sinh k_1x \tanh k_1l) \\
\Omega_{44} &= k_1^2\Omega_{34}
\end{aligned}
\tag{4.3}
$$

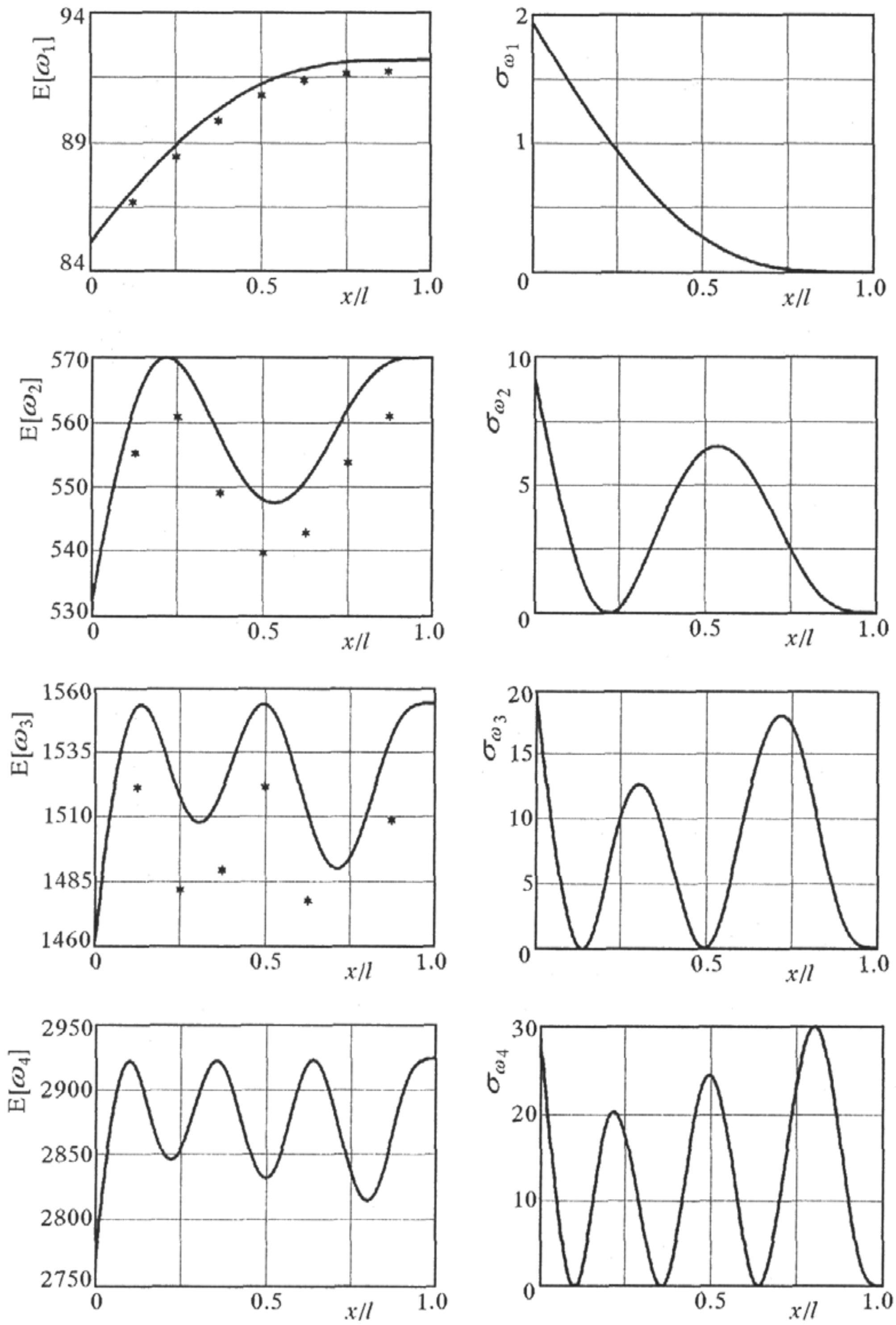

Fig. 3. Cantilever beam. First two probabilistic moments for first four natural frequencies; $a^0/h = 0.25$

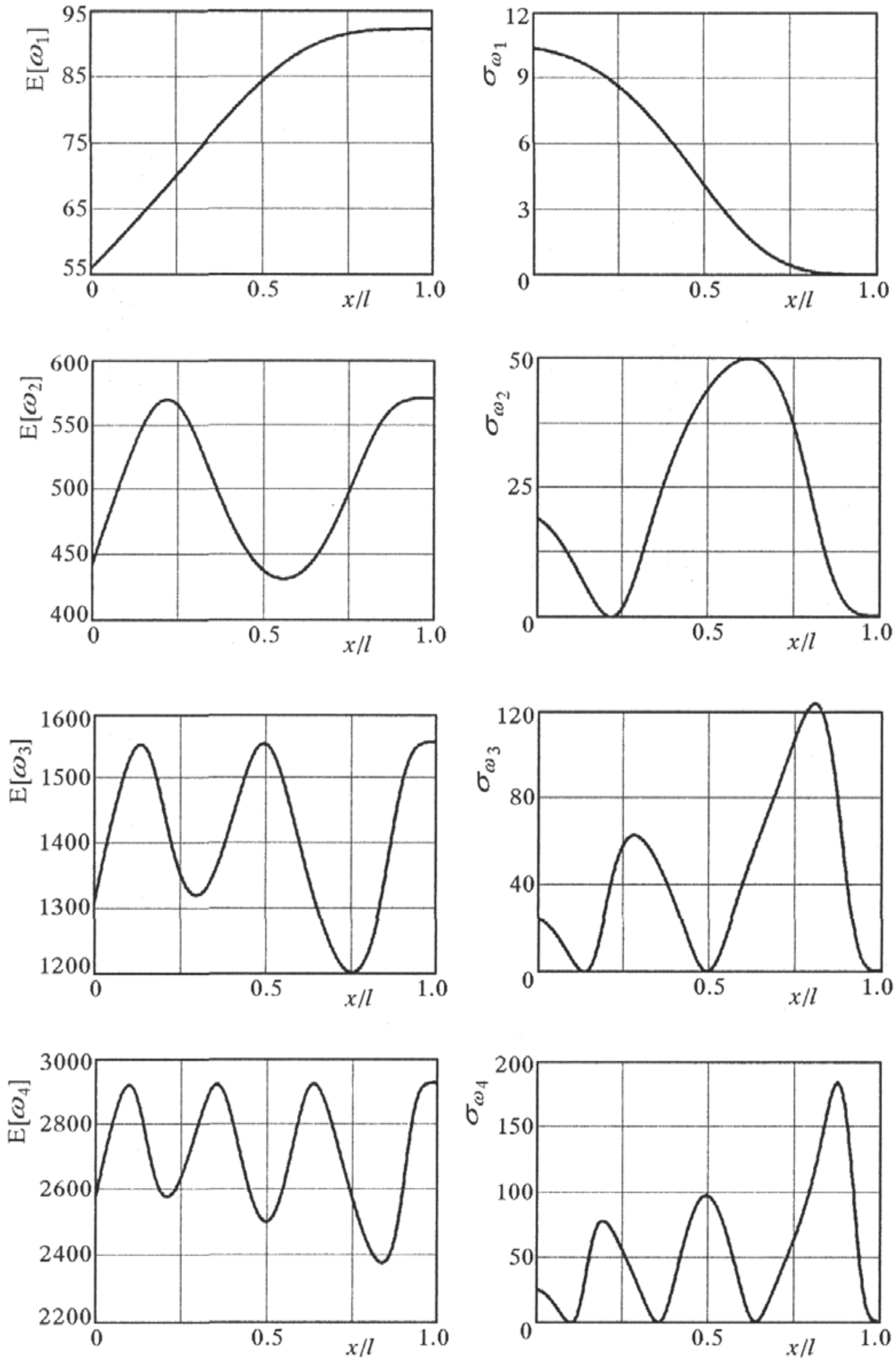

Fig. 4. Cantilever beam. First two probabilistic moments for first four natural frequencies; $a^0/h = 0.60$

Figures 3 and 4 present curves of the spatial expectations for the first four natural frequencies, $E[\omega_i]$, $i = 1, 2, 3, 4$, *versus* the dimensionless x-distance (the running position of the crack), x/l, for both the crack cases Fig. 3 and Fig. 4, respectively. The first probabilistic moments (continuous curves) are compared with the finite element-based, deterministic solutions (star symbols) obtained by using [9] with the crack length a considered as a deterministic parameter, i.e. with zero value of the second moment. The finite element model consists of 960 conventional 8-node rectangular elements and one two-dimensional crack element. It can be seen that the relative differences between the mean values and deterministic finite element-based solutions are up to about 2% for $a^0/h = 0.25$, and from 9% to 18% for $a^0/h = 0.60$; the authors' analytical deterministic solutions and finite element ones nearly coincide. The behaviour of the spatial standard deviations given for the natural frequencies shows that the values of these probabilistic second moments ω_1 decreases slowly down to zero at the free-end of the cantilever. In other words, the nearer the crack approaches the beam free end-point the less the natural frequencies are sensitive with respect to variations of the crack length a, becoming zero at the free end-point. This effect is also illustrated in Fig. 5 and Fig. 6 describing the random response of the first four frequencies, $\omega_1, \omega_2, \omega_3$ and ω_4. It is shown that with the variation α_a of the random parameter a given as 0.15 on the input its counterpart α_ω generated on the output is of the same order – a range from about 0.2 down to 0.05, decreasing with the position of the crack along the axis of the fixed-hinged beam and with the domination level of the successive natural frequencies. The latter aspect can be interpreted in the system energy context, as lower frequencies are more dominant and more sensitive to the fluctuation of the crack length rather than higher ones (the frequency terms are involved in the denominator of Eq. (3.5)).

5. Concluding remarks

The formulation discussed in the paper demonstrates that nonlinear, ill-conditioned and nonsymmetric eigenproblems for (not only beam) systems with spatially random parameters can be effectively analysed by using a non-statistical technique, basing upon the second-centered moment analysis of a function expanded through second-order power series. The strategy introduced may be considered as a general formulation for this class of homogeneous governing equations. Also, it may be put forward, with appropriately minor modifications, to cover other problems of fracture mechanics.

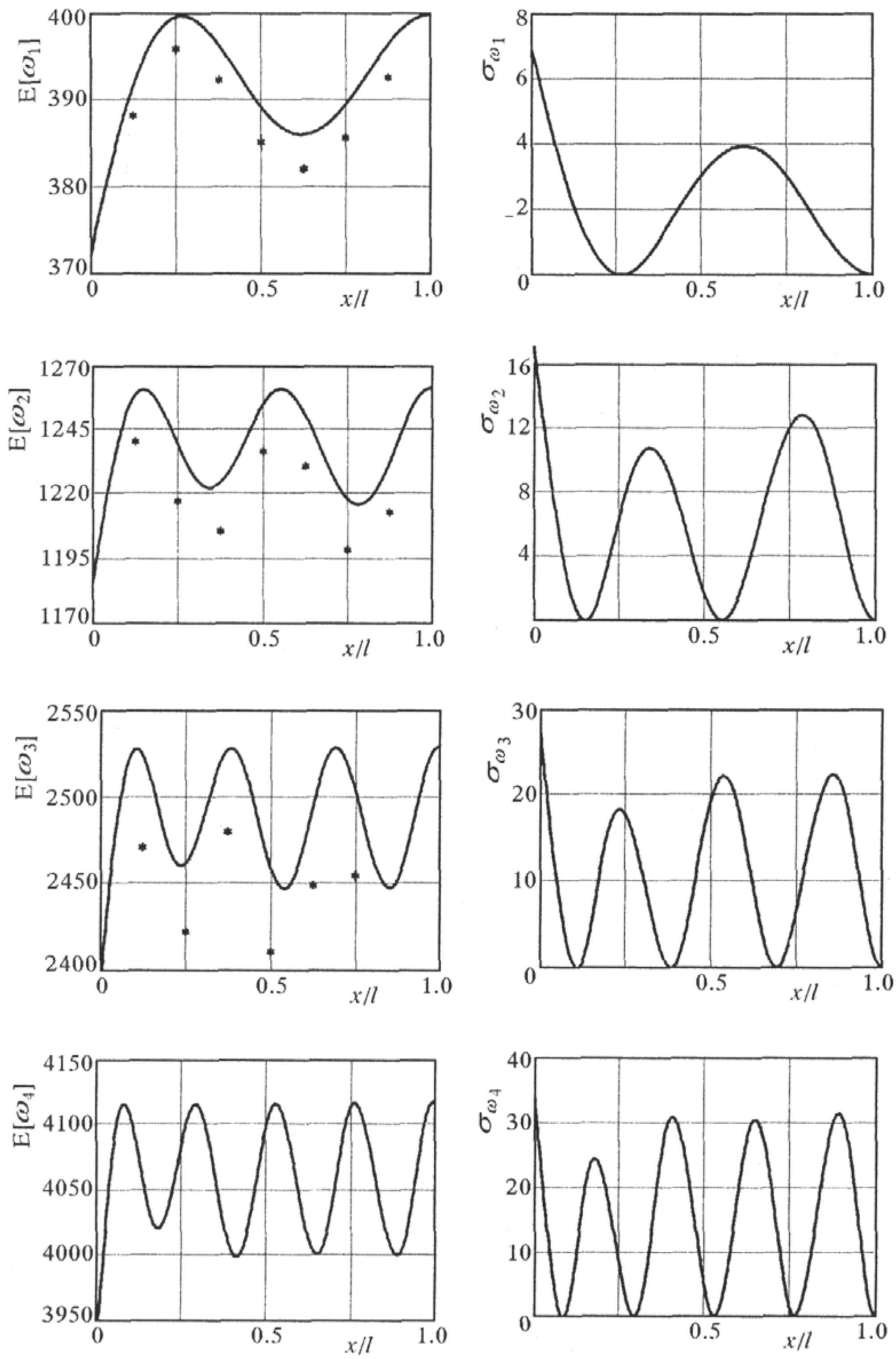

Fig. 5. Fixed-hinged beam. First two probabilistic moments for first four natural frequencies; $a^0/h = 0.25$

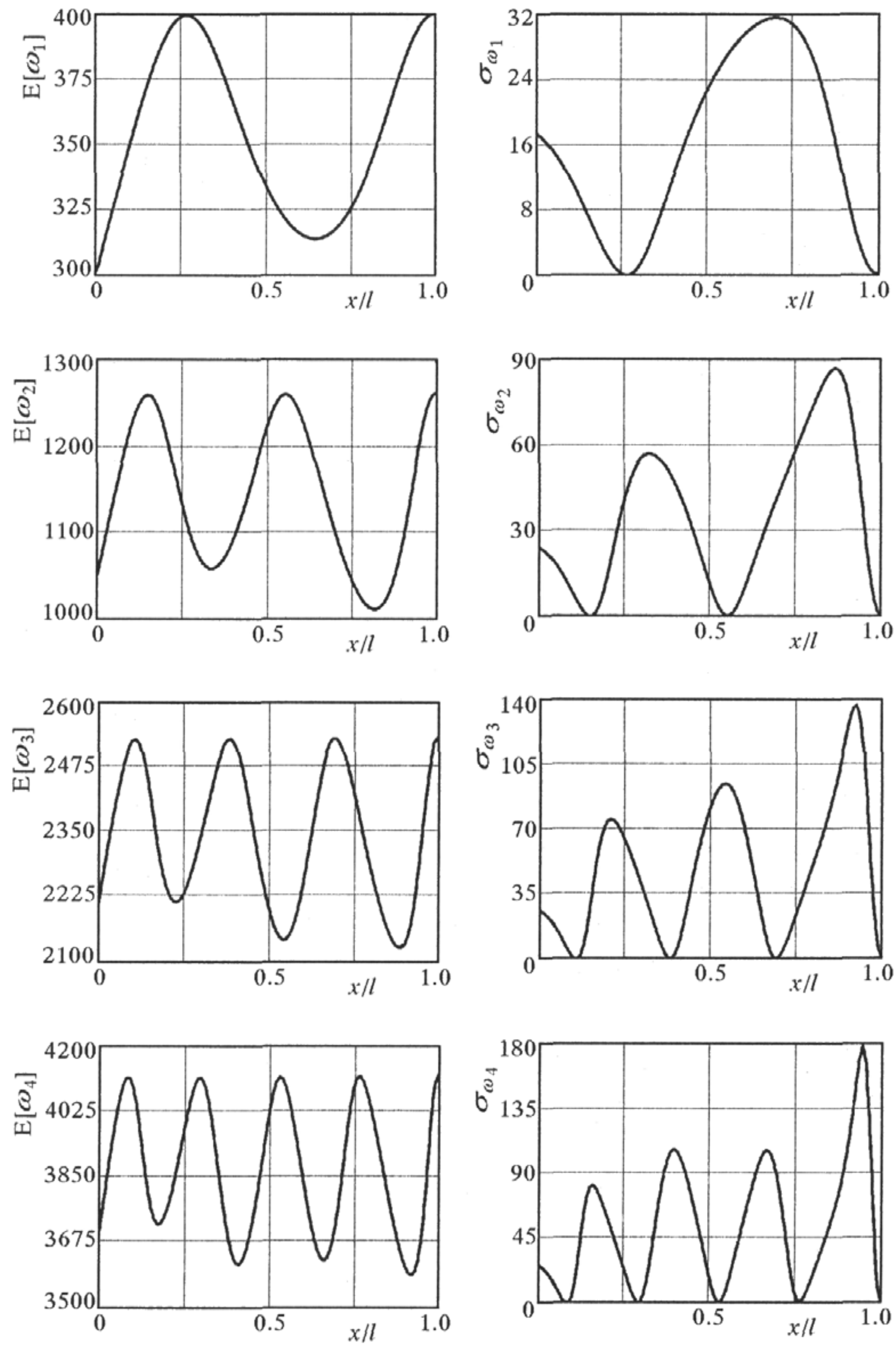

Fig. 6. Fixed-hinged beam. First two probabilistic moments for first four natural frequencies; $a^0/h = 0.60$

Despite the versatility enjoyed by the perturbation approach, we should be aware that certain subtle questions about this method remain unanswered. In particular, the power expansions are valid only if the response is analytic in ters of ϵ and the series converge. Criteria for convergence must include the magnitude of the perturbation parameter ϵ, especially for such an ill-conditioned system of eigenproblems – no such criteria have been established in the present context.

The scheme for discretization of random variables employed in the paper follows the spatial averaging method proposed in Vanmarcke and Grigoriu (1983). Alternative approaches to the approximation of random fields are:

- interpolation method (Liu and Der Kiureghian, 1986; Der Kiureghian and Jyh-Bin Ke, 1988), in which the random field is approximated through deterministic shape functions and the random nodal values of the field
- midpoint method (Vanmarcke and Grigoriu, 1983; Der Kiureghian and Jyh-Bin Ke, 1988), in which the element random variable is defined as the value of the random field at the centroid of the element, and
- series expansion method (Lawrence, 1987), in which the random field is modelled as a series of shape functions with random coefficients.

A comprehensive discussion of these methods can be found in Der Kiureghian and Liu (1988).

The analytical algorithm worked out appears to be accurate and cost-effective in particular for complex systems with correlated random parameters. There are no special difficulties in implementing stochastic options into existing source versions of conventional 'deterministic' computer codes, those like the existing packages written in the framework of the finite element methods or finite differentiation method. Adaptation of the procedure as a post-processor to commercial proprietary codes is possible but would be much more costly.

As shown above, the complete solution of the eigenproblem requires integration of $R+2$ equations. Since the number of matrix operations is proportional to $R\times(R+1)/2$ due to double summations, the computation cost would be high for large complex systems. To reduce the double summations to single ones, so that the number of the matrix operations is proportional to R, the standard normal transformation from the set of the input correlated random variables to a set of uncorrelated random variables may be applied (cf. Liu et al., 1988; Hien and Kleiber 1990; Kleiber and Hien, 1992), for instance. This aspect will be discussed in a forthcoming paper.

References

1. COLLINS J.D., THOMSON W.T., 1969, The eigenvalue problem for structural systems with statistical properties, *AIAA J.*, **7**, 4, 642-648

2. DER KIUREGHIAN A., JYH-BIN KE, 1988, The stochastic finite element method in structural reliability, *Probab. Engrg. Mech.*, **3**, 2, 83-91

3. DER KIUREGHIAN A., LIU P.-L., 1989, First- and second-order finite element reliability methods in structural reliability, in: Liu W.K., Belytschko T. (eds.), *Computational Mechanics of Probabilistic and Reliability Analysis*, Elmepress Int., 281-298

4. DREWKO J., 1999a, Elastic hinge modelling in vibration analysis of beams with cross-sections weakened by cracks, *Marine Tech. Trans.*, **10**, 93-103

5. DREWKO J., 1999b, Vibration analysis of beams with cracks, *Visnyk Lviv Univ., Ser. Mech. Math.*, **55**, 30-34

6. DREWKO J., 2000, Analysis of models of elastic-plastic hinges, *Marine Tech. Trans.*, **11**, 107-114

7. DREWKO J., SPERSKI M., 1991, Vibration of multi-chamber shell structures with discontinuously variable cross-sections, *Eng. Trans.*, **39**, 163-180

8. HIEN T.D., KLEIBER M., 1990, Finite element analysis based on stochastic Hamilton variational principle, *Comput. & Structures*, **37**, 6, 893-902

9. HIEN T.D., KLEIBER M., 1997, Stochastic finite element modelling in linear transient heat transfer, *Comput. Meth. Appl. Mech. Engrg.*, **144**, 111-124

10. HIEN T.D., KLEIBER M., 1998, On solving nonlinear heat transient heat transfer problems with random parameters, *Comput. Meth. Appl. Mech. Engrg.*, **151**, 287-299

11. HISADA T., NAKAGIRI S., 1981, Stochastic finite element method developed for structural safety and reliability, *Proc. 3rd Int. Conf. on Struct. Safety and Reliability*, 395-402

12. KLEIBER M., HIEN T.D., 1992, *The Stochastic Finite Element Method*, Wiley

13. KLEIBER M., HIEN T.D., 1997, Parameter sensitivity of inelastic buckling and post-buckling response, *Comput. Meth. Appl. Mech. Engrg.*, **145**, 239-262

14. KNOTT J.F., 1976, *Fundamentals of Fracture Mechanics*, Butterworth

15. KOBAYASHI A.S., 1971, Photoelastic studies of fracture, in: Liebowitz H. (ed.), *Fracture*, Vol. 3, Chapter 5, Acad. Press

16. LAWRENCE M.A., 1987, Basic random variables in finite element analysis, *Int. J. Num. Meth. Eng.*, **24**, 1849-1863

17. LIU W.K., BELYTSCHKO T., MANI A., 1986, Random field finite elements, *Int. J. Num. Meth. Eng.*, **23**, 1831-1845

18. LIU W.K., BELYTSCHKO T., BESTERFIELD G.H., 1986, A variational principle for probabilistic mechanics, in: T.J.R. Hughes and E. Hinton (eds.), *Finite Element Method for Plate and Shell Structures. Vol.2: Formulations and Algorithms*, Pineridge Press, 285-311

19. LIU W.K., BESTERFIELD G.H., BELYTSCHKO T., 1988, Variational approach to probabilistic finite elements, *Engrg. Mech.*, **114**, 12, 2115-2133

20. LIU P.-L., DER KIUREGHIAN A., 1986, Multivariate distribution models with prescribed marginals and covariances, *Probab. Engrg. Mech.*, **1**, 2, 105-112

21. MSC, 1994, *NASTRAN. Reference Manual, ver. 70*, MacNeal-Schwendler

22. NELSON, R.B., 1976, Simplified calculation of eigenvector derivatives, *AIAA J.*, **14**, 9, 1201-1205

23. NOWACKI W., 1972, *Dynamika budowli*, Arkady

24. TADA H., PARIS P.C., IRWIN G.R., 1973, *The Stress Analysis of Cracks. Handbook*, Hellertown

25. TOCHER K.D., 1968, *The Art of Simulation*, McGraw-Hill

26. VANMARCKE E.H., GRIGORIU M., 1983, Stochastic finite element analysis of simple beams, *J. Eng. Mech.*, ASCE, **109**, 5, 1203-1214

Stochastyczne zagadnienie własne belek ze szczelinami

Streszczenie

W pracy przedstawiono niestatystyczną metodologię w zastosowaniu do zagadnień własnych układów belkowych osłabionych szczelinami i opisanych parametrami losowymi. Problem sformułowano na podstawie kombinacji metody perturbacji drugiego rzędu i analizy drugich centralnych momentów statystycznych. Parametry losowe układu zdefiniowane są przez ich pierwsze dwa momenty statystyczne. Otrzymany hierarchiczny układ równań rozwiązano dla pierwszych dwóch momentów statystycznych częstotliwości i wektorów własnych. Ze względu na niesymetryczność macierzy układu, zaproponowano algorytm ścisłego rozwiązania dla pierwszych pochodnych częstotliwości własnych względem zmiennych losowych. Aspekty analityczne, numeryczne oraz przykłady ilustrujące sformułowanie przedyskutowano szczegółowo. Podejście ma charakter ogólny i może być stosowane do szerokiej klasy zagadnień mechaniki pękania.

Manuscript received December 18, 2001; accepted for print July 3, 2002