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DYNAMICS OF ELASTIC BODIES IN TERMS OF PLANE FRICTIONAL MOTION

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In the paper a general approch to dynamics of flexible systems in which displacements are resolved into displacements due to deformation and displacements due to rigid body motion, will be applied. A contact problem of bodies resting on frictional foundation and being in plane motion is stated and qualitatively discussed.

Key words: rigid body, dynamical equations, rough surface, frictional motion

1. Introduction

The majority of contact problems formulated and considered in mechanics represents two, qualitatively different approaches: the first, typical for mechanics of solids (elastic, plastic etc.) is concentrated on determination of deformations, stress distributions and interaction processes in the contact zone; the second one, typical for multibody dynamics, is looking for motion of the system described obviously as finite-dimensional. Contact is taken into account mainly by forces respresenting reactions of obstacles or interactions of contacting bodies (see e.g. Bremer and Pfeifer, 1992). One of the few explored problems in contact dynamics is a planar contact of deformable body moving on a rough surface. Some results in this topic was given by Fischer and Rammerstorfer (1991), Fischer et al. (1991), Mogilevsky and Nikitin (1997), Nikitin et al. (1996), Stupkiewicz and Mróz (1994), Stupkiewicz (1996). Bending of beams resting on frictional surface, torsion of plates pressed between two rough planes etc. are examples of this type Nikitin (1998).

In the present paper the mentioned problem of the plane motion of an elastic body resting on a rough rigid foundation will be considered. Contrary to the existing formulations, the body is treated as a highly flexible system in which the location of each particle is resolved into displacements due to deformation and displacements due to rigid body motion. Such description enables to determine the mutual interaction between rigid motion and deformation. Governing equations of dynamics in presence of two-dimensional friction and some qualitative results will be given.

The paper is organized as follows: we start with coupling of the rigid motion and deformation according to the general statment given in the paper Szefer (2000). Next we pass to the two-dimensional problem of frictional motion of an elastic body in the plane state of stress. At the end some conclusions are presented.

2. Coupling of rigid body motion and deformation

Consider a deformable body B, its motion from its reference configuration B_R into the current location B_t at instant t being measured with respect to a global inertial system $\{0x^i\}$, $i = 1,2,3$. Denoting by $\{X^K\}$, $K = 1,2,3$ the material coordinates of an arbitrary material point with its position vector $\boldsymbol{X}(X^K)$, one describes the motion $x^i = x^i(X^K,t)$ as a mapping of $\boldsymbol{X}$ onto $\boldsymbol{x}(X^K,t)$ where $\boldsymbol{x}$ means the current position vector of the point at time t. Thus the configuration B_t of the body can be treated as a result of deformation described by the displacement field $\boldsymbol{u}(X^K,t)$ followed by a rigid body motion defined by a translation vector $\boldsymbol{x}_0(t)$ and a rotation tensor $\boldsymbol{Q}(t)$ in the form (Fig. 1)

$$\boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{x}_0 + \mathbf{Q}(t)[\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X},t)] \tag{2.1}$$

Remark. We assume, that the vector $\boldsymbol{x}_0(t)$ stands for motion of the center of mass 0^*, what constituites the most convenient description.

Velocity and acceleration of each point yields (Szefer, 2000)

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_u + \boldsymbol{v}_w \qquad\qquad \ddot{\boldsymbol{x}} = \boldsymbol{a}_w + \boldsymbol{a}_u + \boldsymbol{a}_c \tag{2.2}$$

where

$$\begin{aligned}
\boldsymbol{v}_w &= \mathbf{Q}\dot{\boldsymbol{u}} & \boldsymbol{v}_u &= \dot{\boldsymbol{x}}_0 + \mathbf{Q}[\boldsymbol{\omega} \times (\boldsymbol{X} + \boldsymbol{u})]\\
\boldsymbol{a}_w &= \mathbf{Q}\ddot{\boldsymbol{u}} & \boldsymbol{a}_u &= \ddot{\boldsymbol{x}}_0 + \mathbf{Q}\{\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\boldsymbol{X} + \boldsymbol{u})] + \dot{\boldsymbol{\omega}} \times (\boldsymbol{X} + \boldsymbol{u})\}\\
\boldsymbol{a}_c &= 2\mathbf{Q}\boldsymbol{\omega} \times \dot{\boldsymbol{u}}
\end{aligned}$$

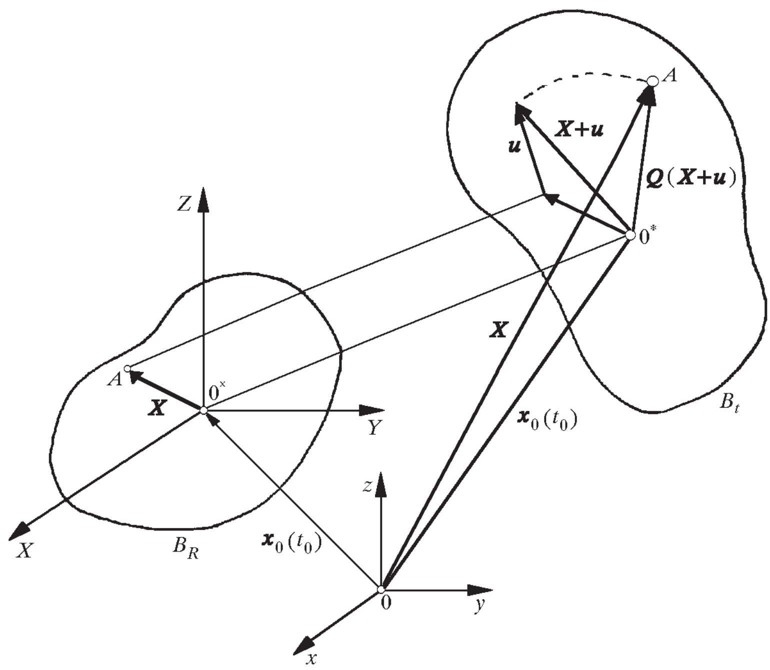

Fig. 1.

and $\omega(t)$ is the axial vector of the skew-symmetric tensor

$$\mathbf{W} = \mathbf{Q}^\top \dot{\mathbf{Q}} \Longrightarrow \mathbf{W}\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{a} \qquad \forall \boldsymbol{a}$$

The vectors $\boldsymbol{v}_w$ and $\boldsymbol{v}_u$ measured in the reference system $\{0x^i\}$ can be interpreted as the relative and transporting velocities of the particle due to deformation whereas the vectors $\boldsymbol{a}_w$, $\boldsymbol{a}_u$, $\boldsymbol{a}_c$ are the relative, transporting and Coriolis accelerations, respectively.

Thus, using the principle of momentum and the principle of the angular momentum (or the equivalent virtual power principle), one obtains the system of equations of motion of any flexible system in the form (Szefer, 2000)

$$\begin{aligned}
\boldsymbol{x}_0 : \quad & M\ddot{\boldsymbol{x}}_0 + \mathbf{Q}[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{A}(\boldsymbol{u})] + \mathbf{Q}\dot{\boldsymbol{\omega}} \times \boldsymbol{A}(\boldsymbol{u}) + 2\mathbf{Q}\boldsymbol{\omega} \times \boldsymbol{P}(\dot{\boldsymbol{u}}) + \\
& + \mathbf{Q}\boldsymbol{B}(\ddot{\boldsymbol{u}}) = \boldsymbol{F}^{ext} + \boldsymbol{F}_C \\
\boldsymbol{\omega} : \quad & \mathbf{J}^u\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}^u\boldsymbol{\omega} - 2\boldsymbol{\omega} \times \boldsymbol{K}_0(\dot{\boldsymbol{u}}) + \ddot{\boldsymbol{x}}_0 \times \mathbf{Q}\boldsymbol{A}(\boldsymbol{u}) + \boldsymbol{L}_0(\ddot{\boldsymbol{u}}) = \\
& = \boldsymbol{M}_0(\boldsymbol{u}) + \boldsymbol{M}_0^C(\boldsymbol{u}) \\
\boldsymbol{u} : \quad & \operatorname{Div} \boldsymbol{S}(\mathbf{1} + \nabla^\top \boldsymbol{u}) + \rho_R \boldsymbol{b} = \rho_R(\boldsymbol{a}_w + \boldsymbol{a}_u + \boldsymbol{a}_c)
\end{aligned} \tag{2.3}$$

Here the following notations have been used

$$
\begin{aligned}
&M = \int\limits_{V_R} \rho_R \, dV_R && \boldsymbol{A}(\boldsymbol{u}) = \int\limits_{V_R} \rho_R(\boldsymbol{X}+\boldsymbol{u}) \, dV_R \\
&\boldsymbol{P}(\dot{\boldsymbol{u}}) = \int\limits_{V_R} \rho_R \dot{\boldsymbol{u}} \, dV_R && \boldsymbol{B}(\ddot{\boldsymbol{u}}) = \int\limits_{V_R} \rho_R \ddot{\boldsymbol{u}} \, dV_R \\
&\boldsymbol{F}_C = \int\limits_{\Gamma_{C_R}} \boldsymbol{t}_R \, dS_R && \boldsymbol{F}^{ext} = \int\limits_{V_R} \rho_R \boldsymbol{b} \, dV_R + \int\limits_{S_R} \boldsymbol{p}_R \, dS_R \\
&\mathbf{J}^u = \int\limits_{V_R} \rho_R[(\boldsymbol{X}+\boldsymbol{u})(\boldsymbol{X}+\boldsymbol{u})\mathbf{1} - (\boldsymbol{X}+\boldsymbol{u}) \otimes (\boldsymbol{X}+\boldsymbol{u})] \, dV_R \\
&\boldsymbol{K}_0(\dot{\boldsymbol{u}}) = \int\limits_{V_R} \rho_R \dot{\boldsymbol{u}} \times (\boldsymbol{X}+\boldsymbol{u}) \, dV_R \\
&\boldsymbol{L}_0(\ddot{\boldsymbol{u}}) = \int\limits_{V_R} \rho_R \ddot{\boldsymbol{u}} \times (\boldsymbol{X}+\boldsymbol{u}) \, dV_R \\
&\boldsymbol{M}_0^{ext}(\boldsymbol{u}) = \int\limits_{V_R} \rho_R[\boldsymbol{b} \times \mathbf{Q}(\boldsymbol{X}+\boldsymbol{u})] \, dV_R + \int\limits_{S_R} \boldsymbol{p}_R \times \mathbf{Q}(\boldsymbol{X}+\boldsymbol{u}) \, dS_R \\
&\boldsymbol{M}_0^C = \int\limits_{\Gamma_{C_R}} \boldsymbol{t}_R \times \mathbf{Q}(\boldsymbol{X}+\boldsymbol{u}) \, dS_R
\end{aligned}
\tag{2.4}
$$

where

ρ_R	–	mass density in B_R
V_R	–	volume domain in B_R
S_R	–	boundary surface loaded by prescribed external tractions $\boldsymbol{p}_R$
Γ_{C_R}	–	contact zone with contact tractions $\boldsymbol{t}_R$
$\boldsymbol{b}$	–	body forces
∇	–	stands for the gradient operator with respect to B_R
$\mathbf{1}$	–	identity tensor
$\mathbf{S}$	–	second Piola-Kirchhoff stress tensor.

Vectors $\boldsymbol{F}_C$ and $\boldsymbol{M}_0^C$ expresses the presence of contact forces or reactions of constraints.

Remark. Denoting the sum of the second to the fifth terms on the left-hand side of equation $(2.3)_1$ by

$$
\boldsymbol{F}_u = \mathbf{Q}[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{A}(\boldsymbol{u})] + \mathbf{Q}\dot{\boldsymbol{\omega}} \times \boldsymbol{A}(\boldsymbol{u}) + 2\mathbf{Q}\boldsymbol{\omega} \times \boldsymbol{P}(\dot{\boldsymbol{u}}) + \mathbf{Q}\boldsymbol{B}(\ddot{\boldsymbol{u}}) \tag{2.5}
$$

and similarly the coresponding sum in $(2.3)_2$ by

$$
\boldsymbol{M}_0^u = -2\boldsymbol{\omega} \times \boldsymbol{K}_0(\dot{\boldsymbol{u}}) + \ddot{x}_0 \times \mathbf{Q}\boldsymbol{A}(\boldsymbol{u}) + \boldsymbol{L}_0(\ddot{\boldsymbol{u}}) \tag{2.6}
$$

one can write the mentioned equations in the form

$$
\begin{aligned}
M\ddot{\boldsymbol{x}}_0 &= \boldsymbol{F}^{ext} + \boldsymbol{F}_C - \boldsymbol{F}_u \\
\mathsf{J}^u\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathsf{J}^u\boldsymbol{\omega} &= \boldsymbol{M}_0^{ext}(\boldsymbol{u}) + \boldsymbol{M}_0^C(\boldsymbol{u}) - \boldsymbol{M}_0^u
\end{aligned}
\tag{2.7}
$$

This form coincides with the known system of equations of rigid body dynamics but with the inertial tensor J^u (see Eqs (2.4)). System (2.7) together with equation $(2.3)_3$ possess a clear structure and provide a simple interpretation for the coupled rigid motion and deformation: translation and rotation influences the motion of a continuum by additional transportation and the Coriolis acceleration, whereas deformation influences the rigid body motion by configuration-dependent force (2.5), moment (2.6) and inertial tensor J^u. System (2.3) must be completed by the constitutive equations

$$
\mathsf{S} = \mathcal{F}(\boldsymbol{X}, \mathsf{E}) \tag{2.8}
$$

the kinematical equations for Green's strain tensor

$$
\mathsf{E} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^\top\boldsymbol{u} + \nabla\boldsymbol{u}^\top\nabla\boldsymbol{u}) \tag{2.9}
$$

and by the boundary and initial conditions

$$
\mathsf{S}(\mathbf{1} + \nabla^\top\boldsymbol{u})\boldsymbol{N} = \begin{cases} \boldsymbol{p}_R & \text{on} \quad S_R \\ \boldsymbol{t}_R & \text{on} \quad \mathit{\Gamma}_{C_R} \end{cases} \tag{2.10}
$$

$$
\begin{array}{lll}
\boldsymbol{x}_0(t_0) = \boldsymbol{r}_0 & \dot{\boldsymbol{x}}_0(t_0) = \boldsymbol{v}_0 & \boldsymbol{\omega}(t_0) = \boldsymbol{\omega}_0 \\
\boldsymbol{u}(\boldsymbol{X}, t_0) = \boldsymbol{u}_0(\boldsymbol{X}) & \dot{\boldsymbol{u}}(\boldsymbol{X}, t_0) = \overline{\boldsymbol{v}}_0(\boldsymbol{X}) & \boldsymbol{X} \in B_R
\end{array}
\tag{2.11}
$$

Here $\boldsymbol{N}$ means the unit outward vector normal to S_R and $\mathit{\Gamma}_{C_R}$ is the mapping of the contact zone $\mathit{\Gamma}_C$ onto the reference configuration. Equations (2.3), (2.8), (2.9) constitute a coupled system with unknown functions $\boldsymbol{x}_0(t)$, $\boldsymbol{\omega}(t)$ and $\boldsymbol{u}(X^K, t)$, $K = 1, 2, 3$, which describe the complex motion of any flexible body with displacements explicitely decomposed into rigid motion and pure deformation. Such statement of any dynamical problem represents a third, and in fact, the most general approach to dynamics of deformable bodies. The system (2.3) which consist of two ordinary and one partial differential equations shows evidently the mutual dependence of translation, rigid rotation and deformation. Simultaneously, the displacements due to deformations depend strongly on rigid motion what can be seen from $(2.3)_3$, where dynamic body forces are supplemented by transportation and Coriolis members. The presence of deformation shows additionally that contrary to pure rigid motion, there is a coupling between translation and rotation.

It is worth to observed that equations (2.3) (or in the form (2.7)) are valid for elastic bodies the external constraints of which may be nonholonomic, rheonomic, unilateral and rough. The material system may posses large displacements and rotations, too. Thus the impact, friction, rolling with and without sliding etc. can be taken into account.

3. Elastic plate undergoing frictional motion

Consider a thin elastic plate resting on a rough rigid foundation loaded by prescribed tangential boundary tractions $\boldsymbol{p}_t$ and compressed by normal forces with density p_n (Fig. 2).

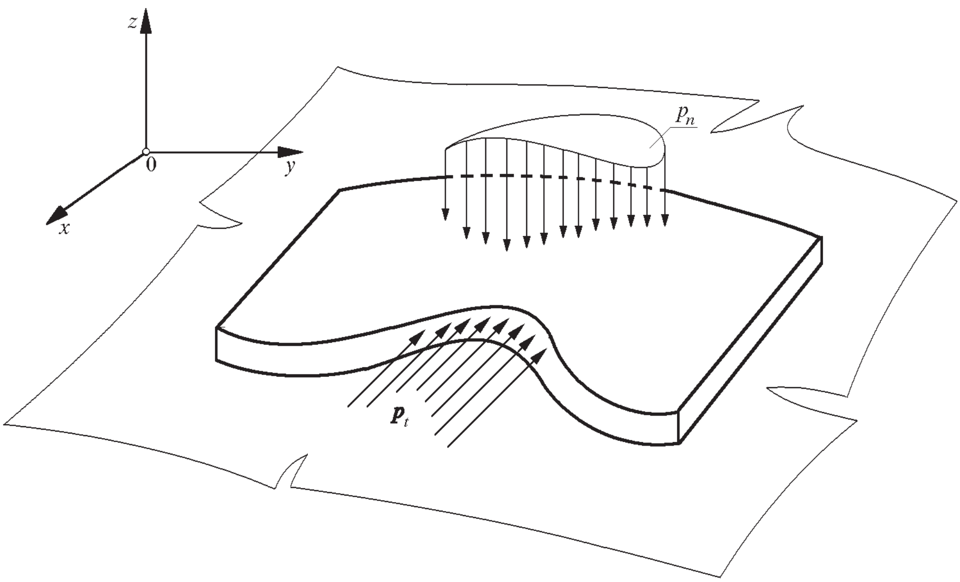

Fig. 2.

When the body starts to move due to the external boundary load $\boldsymbol{p}_t$, friction occurs. Thus the body forces

$$\boldsymbol{b} = -\mu p_n(\boldsymbol{X}, t) \frac{\boldsymbol{v}_T}{|\boldsymbol{v}_T|} \tag{3.1}$$

where $\boldsymbol{v}_T$ means the sliding velocity, arise at all points of the plate area where $p_n(\boldsymbol{X},t) \neq 0$. The intensity of the friction forces $|\boldsymbol{b}| = \mu p_n$ is known whereas their direction results from the Coulomb law (3.1). They have the body force character since they act on the internal points of the plane body. For sticking it will be $\boldsymbol{v}_T = \mathbf{0}$; otherwise the body is sliding. Let the material reference frame be a cartesian coordinate system $(0^*, X, Y, Z)$ with its origin in the center of mass and let the global inertial system will be denoted by $(0, x, y, z)$. Thus the kinematics of the body yields

$$\boldsymbol{\omega} = [0, 0, \omega = \dot{\alpha}] \qquad \boldsymbol{u} = [u_x, u_y] \qquad \boldsymbol{x}_0 = [x_0, y_0]$$

$$\mathbf{Q} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dot{\mathbf{Q}} = -\dot{\alpha} \begin{bmatrix} \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{x}^\top = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + \mathcal{A}\cos\alpha - \mathcal{B}\sin\alpha \\ y_0 + \mathcal{A}\sin\alpha + \mathcal{B}\cos\alpha \end{bmatrix}$$

$$\boldsymbol{v}^\top = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{x}_0 - \omega[\mathcal{A}\sin\alpha + \mathcal{B}\cos\alpha] + \dot{u}_x\cos\alpha - \dot{u}_y\sin\alpha \\ \dot{y}_0 - \omega[-\mathcal{A}\cos\alpha + \mathcal{B}\sin\alpha] + \dot{u}_x\sin\alpha + \dot{u}_y\cos\alpha \end{bmatrix} \tag{3.2}$$

$$\boldsymbol{a}_w^\top = \begin{bmatrix} a_{wx} \\ a_{wy} \end{bmatrix} = \begin{bmatrix} \ddot{u}_x\cos\alpha - \ddot{u}_y\sin\alpha \\ \ddot{u}_x\sin\alpha + \ddot{u}_y\cos\alpha \end{bmatrix}$$

$$\boldsymbol{a}_u^\top = \begin{bmatrix} a_{ux} \\ a_{uy} \end{bmatrix} = \begin{bmatrix} \ddot{x}_0 - \omega^2[\mathcal{A}\cos\alpha + \mathcal{B}\sin\alpha] - \dot{\omega}[\mathcal{A}\sin\alpha + \mathcal{B}\cos\alpha] \\ \ddot{y}_0 - \omega^2[\mathcal{A}\sin\alpha - \mathcal{B}\cos\alpha] - \dot{\omega}[-\mathcal{A}\cos\alpha + \mathcal{B}\sin\alpha] \end{bmatrix}$$

$$\boldsymbol{a}_c^\top = \begin{bmatrix} a_{cx} \\ a_{cy} \end{bmatrix} = \begin{bmatrix} -2\omega(\dot{u}_x\sin\alpha + \dot{u}_y\cos\alpha) \\ -2\omega(-\dot{u}_x\cos\alpha + \dot{u}_y\sin\alpha) \end{bmatrix}$$

where $\mathcal{A} = X + u_x$, $\mathcal{B} = Y + u_y$.

Taking into account the fact that in a plane motion it is $J\boldsymbol{\omega}\|\boldsymbol{\omega}$, $\boldsymbol{K}_0\|\boldsymbol{\omega}$, calculating next all the integrals (2.4)

$$\begin{aligned} A_x(t) &= \int\limits_{V_R} \rho_R u_x\, dV_R & A_y(t) &= \int\limits_{V_R} \rho_R u_y\, dV_R \\ P_x(t) &= \int\limits_{V_R} \rho_R \dot{u}_x\, dV_R & P_y(t) &= \int\limits_{V_R} \rho_R \dot{u}_y\, dV_R \\ B_x(t) &= \int\limits_{V_R} \rho_R \ddot{u}_x\, dV_R & B_y(t) &= \int\limits_{V_R} \rho_R \ddot{u}_y\, dV_R \end{aligned} \tag{3.3}$$

$$L_{0z}(t) = \int\limits_{V_R} \rho_R(\mathcal{B}\ddot{u}_x - \mathcal{A}\ddot{u}_y)\,dV_R \qquad J_{zz}(t) = \int\limits_{V_R} \rho_R(\mathcal{A}^2 + \mathcal{B}^2)\,dV_R$$

and introducing for clarity the unknown matrix $\boldsymbol{q}^\top = [x_0, y_0, \alpha]$, one obtains the system of equations (2.3) for the two-dimensional case as follows

$$\begin{bmatrix} M & 0 & -A_x\sin\alpha + A_y\cos\alpha \\ 0 & M & A_x\cos\alpha - A_y\sin\alpha \\ A_x\sin\alpha + A_y\cos\alpha & -A_x\cos\alpha + A_y\sin\alpha & J_{zz} \end{bmatrix}\ddot{\boldsymbol{q}} -$$

$$-2\begin{bmatrix} 0 & 0 & P_x\sin\alpha + P_y\cos\alpha \\ 0 & 0 & P_x\cos\alpha - P_y\sin\alpha \\ 0 & 0 & 0 \end{bmatrix}\dot{\boldsymbol{q}} + \begin{bmatrix} 0 & 0 & -A_x\cos\alpha - A_y\sin\alpha \\ 0 & 0 & -A_x\sin\alpha + A_y\cos\alpha \\ 0 & 0 & 0 \end{bmatrix}\dot{\boldsymbol{q}}^2 +$$

$$+\begin{bmatrix} B_x\cos\alpha - B_y\sin\alpha \\ B_x\sin\alpha + B_y\cos\alpha \\ L_{0z} \end{bmatrix} = \begin{bmatrix} F_x^{ext} + F_{Cx} \\ F_y^{ext} + F_{Cy} \\ M_{0z}^{ext} + M_{0z}^C \end{bmatrix} \tag{3.4}$$

$$\mathrm{Div}\begin{bmatrix} S_{xx}(1+u_{x,x}) + S_{xy}u_{y,y} & S_{yx}(1+u_{x,x}) + S_{yy}u_{y,y} \\ S_{xx}u_{y,x} + S_{xy}(1+u_{y,y}) & S_{yx}u_{y,x} + S_{yy}(1+u_{y,y}) \end{bmatrix} -$$

$$-\mu p_n\begin{bmatrix} \dfrac{v_x}{\sqrt{v_x^2+v_y^2}} \\ \dfrac{v_y}{\sqrt{v_x^2+v_y^2}} \end{bmatrix} = \rho_R\begin{bmatrix} a_{wx} + a_{ux} + a_{cx} \\ a_{wy} + a_{uy} + a_{cy} \end{bmatrix}$$

The quantity $\dot{\boldsymbol{q}}^2$ means multiplication of matrices $\dot{\boldsymbol{q}}^\top\dot{\boldsymbol{q}}$.

The external resultant force $\boldsymbol{F}^{ext}$ and moment $\boldsymbol{M}_{0*}^{ext}$ have the components

$$F_x^{ext} = -\mu\int\limits_{V_R}\frac{p_n v_x}{\sqrt{v_x^2+v_y^2}}\,dV + \int\limits_{S_R} p_{tx}\,dS$$

$$F_y^{ext} = -\mu\int\limits_{V_R}\frac{p_n v_y}{\sqrt{v_x^2+v_y^2}}\,dV + \int\limits_{S_R} p_{ty}\,dS \tag{3.5}$$

$$M_{0z}^{ext} = \int\limits_{S_R}(p_{tx}r_y - p_{ty}r_x)\,dS - \mu\int\limits_{V_R} p_n\frac{v_x r_y - v_y r_x}{\sqrt{v_x^2+v_y^2}}\,dV$$

$$r_x = \mathcal{A}\cos\alpha - \mathcal{B}\sin\alpha \qquad r_y = \mathcal{A}\sin\alpha - \mathcal{B}\cos\alpha$$

The system (3.4) is strongly nonlinear and can be solved numerically only.

It is seen from (3.4) that the rigid part of plane motion depends on deformation through functions (3.3) and (3.5) only. This property makes it possible to solve the system $(3.4)_1$ (with suitable initial conditions) formally independently on $(3.4)_2$ (e.g. by means of the Runge-Kutta method).

On the other hand, the nonlinearity of $(3.4)_2$ causes that the incremental approach is obviously used. The incremental form of $(3.4)_2$ is then as follows

$$\mathrm{Div}\,[\mathbf{S}\Delta\mathbf{H} + \Delta\mathbf{S}(\mathbf{1}+\mathbf{H})] + \Delta\boldsymbol{b} = \rho_R(\Delta\boldsymbol{a}_w + \Delta\boldsymbol{a}_u + \Delta\boldsymbol{a}_c) \tag{3.6}$$

where $\mathbf{H} = \nabla\boldsymbol{u}$.

Thus the system $(3.4)_1$ must be solved iteratively for any increment $\Delta\boldsymbol{u}$.

Leaving the numerical details and analysis for separate discussion, one can however, in particular cases, come to some general qualitative conclusions

A. Constant body force

If the density of the external body force is constant, then

$$\boldsymbol{F}^{ext} = \int\limits_{V_R} \rho_R\boldsymbol{b}\,dV = M\boldsymbol{b} \tag{3.7}$$

and we obtain from $(2.7)_1$

$$\ddot{\boldsymbol{x}}_0 = \boldsymbol{b} + \frac{1}{M}(\boldsymbol{F}_C - \boldsymbol{F}_u) \tag{3.8}$$

Substituting this expression into $(2.3)_3$ we obtain

$$\mathrm{Div}\,\mathbf{S}(\mathbf{1} + \Delta\boldsymbol{u}^\top) + \rho_R\boldsymbol{b} = \rho_R\mathbf{Q}\ddot{\boldsymbol{u}} + \rho_R\Big[\boldsymbol{b} + \frac{1}{M}(\boldsymbol{F}_C - \boldsymbol{F}_u) + \boldsymbol{a}_u^\omega + \boldsymbol{a}_c\Big] \tag{3.9}$$

where $\boldsymbol{a}_u^\omega$ means this part of $\boldsymbol{a}_u$ which results from rotation (see (2.2)). One can see from the above equation that the term $\rho_R\boldsymbol{b}$ vanishes and it reads finally

$$\mathrm{Div}\,\mathbf{S}(\mathbf{1} + \Delta\boldsymbol{u}^\top) = \rho_R(\mathbf{Q}\ddot{\boldsymbol{u}} + \boldsymbol{a}_u^\omega + \boldsymbol{a}_c) + \frac{\rho_R}{M}(\boldsymbol{F}_C - \boldsymbol{F}_u) \tag{3.10}$$

This result means that, in the case of constant body force, pure deformation does not depend on $\boldsymbol{b}$; the constant body force density influences translation only. This fact is invisible if displacements are not presented in the form (2.1).

B. Sliding without rotation under symmetric monotonic load and uniform pressure

The result obtained above can be applied to a plate being in translatory sliding motion (Fig. 3a). Thus $\boldsymbol{\omega} = \mathbf{0}$. Assume that the lateral velocity v_y is small (e.g. if the plate have dimensions of a rod). Then $\boldsymbol{b} = -\mu p_n[1, 0]$ (since the direction of velocity $\boldsymbol{v} = \dot{\boldsymbol{u}} + \dot{\boldsymbol{x}}_0$ for all points of the body is the same and known) and the property of case A holds true.

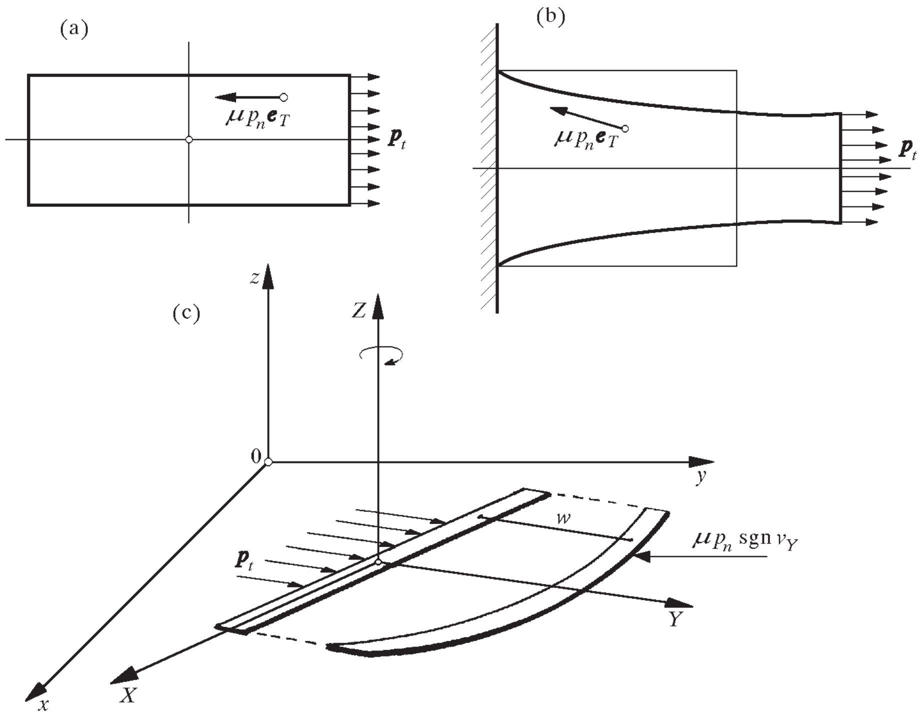

Fig. 3.

So, friction disappears in equations of motion $(3.4)_2$. If stick-slip process occurs (and this takes place when $\boldsymbol{v}$ have to vary under nonmonotonic or nonsymmetrical loads $\boldsymbol{p}_t$) then friction influences the deformations.

On the other hand, when the lateral velocity v_y cannot be neglected and the plate will be clamped on one side (Fig. 3b), rigid rotation vanishes $\boldsymbol{\omega} = \mathbf{0}$, $\mathbf{Q} = \mathbf{1}$ and pure deformation results now from the equation

$$\operatorname{Div} \mathbf{S}(\mathbf{1} + \Delta \boldsymbol{u}^{\top}) - \mu p_n \boldsymbol{e}_T = \rho_R(\ddot{\boldsymbol{u}} + \ddot{\boldsymbol{x}}_0) \tag{3.11}$$

where

$$\boldsymbol{e}_T = \left[\frac{v_x}{\sqrt{v_x^2+v_y^2}}, \frac{v_y}{\sqrt{v_x^2+v_y^2}}\right] \qquad v_x = \dot{u}_x + \dot{x}_0 \qquad v_y = \dot{u}_y$$

whereas the system (2.7) (and hence ; $(3.4)_1$) yields

$$M\ddot{x}_0 = F_x^{ext} + F_{Cx} - B_x \tag{3.12}$$

Remark. In case of symmetry all the terms in $(2.7)_2$ vanishes. Simultaneously it is $y_0 = 0$. Hence the above result.

C. Dynamic bending of a beam

Consider an elastic uniform slender beam using the standard Bernoulli-Euler model of small deformation but with large rigid rotations. Let the beam of length L, cross-sectional area A, inertia moment J and Young modulus E rest on the plane $\{0xy\}$ (Fig. 3c). The material coordinate system $\{0^*XYZ\}$ rotates with the beam. The centroidal axis is assumed to be inextensible. Under the action of prescribed load $\boldsymbol{p}_t(X,t)$, the beam moves and bends laterally with the deflection $w(X,t)$.

Thus the functions (3.3) take the values

$$\begin{aligned}
&u_x \equiv 0 \implies A_x = P_x = B_x = 0 \\
&A_y = \rho A \int_{-L/2}^{L/2} w(X,t)\,dX \qquad P_y = \rho A \int_{-L/2}^{L/2} \dot{w}(X,t)\,dX \\
&B_y = \rho A \int_{-L/2}^{L/2} \ddot{w}(X,t)\,dX \qquad L_{0z} = -\rho A \int_{-L/2}^{L/2} \ddot{w}(X,t)X\,dX
\end{aligned} \tag{3.13}$$

For the loading terms one obtains the components

$$\begin{aligned}
F_x &= \int_{-L/2}^{L/2} p_{tx}\,dX - \mu b \int_{-L/2}^{L/2} p_n \frac{v_x}{\sqrt{v_x^2+v_y^2}}\,dX \\
F_y &= \int_{-L/2}^{L/2} p_{ty}\,dX - \mu b \int_{-L/2}^{L/2} p_n \frac{v_y}{\sqrt{v_x^2+v_y^2}}\,dX \\
M_{0z} &= \int_{-L/2}^{L/2} \boldsymbol{p}_t(X,t)\boldsymbol{X}\,dX - \mu b \int_{-L/2}^{L/2} p_n \frac{v_x X \sin\alpha - v_y X \cos\alpha}{\sqrt{v_x^2+v_y^2}}\,dX
\end{aligned} \tag{3.14}$$

where b is the width of the beam.

From (3.4) result the equations of plane rigid motion of the beam

$$
\begin{aligned}
&M\ddot{x}_0 + A_y(t)(\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha) - 2P_y(t)\dot{\alpha}\cos\alpha - B_y(t)\sin\alpha = F_x(t)\\
&M\ddot{y}_0 + A_y(t)(\ddot{\alpha}\sin\alpha - \dot{\alpha}^2\cos\alpha) + 2P_y(t)\dot{\alpha}\sin\alpha + B_y(t)\cos\alpha = F_y(t)\\
&\rho\frac{bL^3}{12}\ddot{\alpha} + A_y(t)(\ddot{x}_0\cos\alpha + \ddot{y}_0\sin\alpha) + L_{0z} = M_{0z}(t)
\end{aligned}
\tag{3.15}
$$

To obtain the most convenient form of bending, the local coordinate system $\{0^*XY\}$ which is moving together with the beam will be used (see Fig. 3c). We then get

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{x}_0 + (\boldsymbol{X} + \boldsymbol{w}) = [x_0, y_0] + [X, w]\\
\boldsymbol{v} &= [\dot{x}_0 - \omega w, \dot{y}_0 + \dot{w} + \omega X]\\
\boldsymbol{a} &= [\ddot{x}_0 - \dot{\omega} w - \omega^2 X - 2\omega\dot{w}, \ddot{y}_0 + \ddot{w} + \dot{\omega}X - \omega^2 w]
\end{aligned}
\tag{3.16}
$$

Using the lateral components of velocity and acceleration, one obtains the dynamical equation of the beam

$$
EJ\frac{\partial^4 w}{\partial X^4} = p_t(X,t) - \mu p_n(X,t)\,\mathrm{sgn}\,(\dot{y}_0 + \dot{w} + \omega X) - \rho A(\ddot{y}_0 + \ddot{w} + \dot{\omega}X - \omega^2 w)
\tag{3.17}
$$

This equation generalizes the static case discovered by Nikitin (1992) and Stupkiewicz (1996). If the beam move translational one get

$$
EJ\frac{\partial^4 w}{\partial X^4} = p_t - \mu p_n\,\mathrm{sgn}\,(\dot{y}_0 + \dot{w}) - \rho A(\ddot{y}_0 + \ddot{w})
\tag{3.18}
$$

Finally, if only pure deformation (bending) is taken into account, one obtains the standard dynamic equation in terms of frictional contact

$$
EJ\frac{\partial^4 w}{\partial X^4} = p_t - \mu p_n\,\mathrm{sgn}\,\dot{w} - \rho A\ddot{w}
\tag{3.19}
$$

4. Concluding remarks

The presented approach to dynamics based on formula (2.1) differs from the standard procedure where elastic strains and stresses result from the prescribed rigid motion (obviously used in multibody dynamics of elastic systems). No restrictions on displacements, velocities and deformation gradients were introduced. Thus the systems with high flexibility and large rigid motion can be analyzed. Plane friction constitutes still a challenge in contact dynamics. Few numerical results of plane sliding motion are known up to now (some of them were mentioned in the Introduction). The equations derived in the paper give the possibility to analyze the mutual interaction between rigid motion and deformation which is of great interest today.

Some simple qualitative examples of sliding were presented only.

More complex cases of coupling of rigid motion and large deformation in terms of contact will be discussed separately.

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Dynamika ciał sprężystych w warunkach płaskiego ruchu szorstkiego

Streszczenie

W pracy zastosowano ogólny opis dynamiki układów odkształcalnych, w których przemieszczenia są dekompozycją części wynikającej z deformacji oraz części wywołanej ruchem sztywnym. Sformułowano i przedyskutowano jakościowo problem kontaktu ciała leżącego na chropowatym podłożu i będącego w ruchu płaskim.

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