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NON-LINEAR ANALYSIS OF FUNCTIONALLY GRADED PLATES IN CYLINDRICAL BENDING BASED ON A NEW REFINED SHEAR DEFORMATION THEORY

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A new refined shear deformation theory for the nonlinear cylindrical bending behavior of functionally graded (FG) plates is developed in this paper. This new theory is based on the assumption that the transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The theory accounts for a quadratic variation of the transverse shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. The plates are subjected to pressure loading, and their geometric nonlinearity is introduced in the strain-displacement equations based on Von-Karman assumptions. The material properties of plate are assumed to vary according to the power law distribution of the volume fraction of the constituents. The solutions are achieved by minimizing the total potential energy and the results are compared to the classical, the first-order and other higher-order theories reported in the literature. It can be concluded that the proposed theory is accurate and simple in solving the nonlinear cylindrical bending behavior of functionally graded plates.

Key words: functional composites, plate, large deformation, energy method

1. Introduction

Composite materials have been successfully used in aircraft and other engineering applications for many years because of their excellent strength to weight and stiffness to weight ratios. Recently, advanced composite materials known as functionally graded material have attracted much attention in many engineering applications due to their advantages of being able to resist high temperature gradient while maintaining structural integrity (Koizumi, 1997). The functionally graded materials (FGMs) are microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. They are usually made from a mixture of ceramics and metals to attain the significant requirement of material properties.

Cylindrical bending analysis and also non-linear analysis of FG plates are rare in literature reviews. Qian *et al.* (2004) analysed plane strain thermoelastic deformations of a simply-supported FG plate by a meshless local Petrov-Galerkin method. Bian *et al.* (2005) extended a recently developed plate theory using the concept of shape function of the transverse coordinate parameter to investigate the cylindrical bending behavior of FG plates. Using the finite-element method, Praveen and Reddy (1998) investigated static and dynamic responses of FG plates accounting for transverse shear strains, rotary inertia, and moderately large rotations. Based on the higher-order shear deformation plate theory, Reddy (2000) developed Navier's solutions for rectangular plates and finite-element models to study the non-linear dynamic response of FG plates. An analytical solution was obtained by Woo and Meguid (2001) for large deflections of FG plates and shallow shells under transverse mechanical loads and a temperature field. Using Reddy's higher-order shear deformation plate theory, Shen (2002) studied the non-linear bending of a simply-supported FG rectangular plate subjected to mechanical and thermal loads. Yang and Shen (2003a,b) investigated large deflection and postbuckling responses of FG rectangular plates under transverse and in-plane loads, and also they investigated non-linear bending of shear deformable FG plates with various edge supports subjected to thermo-mechanical loads using a semi-analytical approach. More recently, Shen (2007) presented nonlinear thermal bending analysis for a simply-supported FG plate without or with piezoelectric actuators subjected to the combined action of thermal and electrical loads using a two step perturbation technique. We note that the majority of the above mentioned studies are concerned with analysis of functionally graded plates where material properties vary continuously through the plate thickness. However, in recent studies, Jędrysiak and Michalak (2011) and Michalak and Wirowski (2012) investigated the stability and dynamic behavior of thin plates with effective properties varying in the midplane of the plates. The formulation of mathematical models of these plates is based on a tolerance averaging technique (Woźniak *et al.*, 2008).

The purpose of this study is to develop a new shear deformation plate theory for the non-linear cylindrical bending behavior of FG plates which is simple to use. This theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The most interesting feature of this theory is that it accounts for a quadratic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. Young's modulus and mass density of the FG plate are assumed to vary according to a power law distribution of the volume fraction of the constituents whereas Poisson's ratio is constant. The solution of the nonlinear cylindrical bending problem is obtained via minimization of the total potential energy. To illustrate the accuracy of the present theory, the obtained results are compared with those of the classical, first-order and other higher-order theories reported in the literature.

2. higher order displacement theory

The displacements of a material point located at (x, y, z) in the plate under cylindrical bending may be written as

$$u = u_0(x,y) - z\frac{prtw_0}{\partial x} + \Psi(z)\theta_x \qquad\qquad w = w_0(x,y) \tag{2.1}$$

where, u and w are displacements in the x, z directions, u_0 and w_0 are midplane displacements, θ_x rotation of the yz plane due to bending. $\Psi(z)$ represents the shape function determining the distribution of transverse shear strains and stresses along the thickness. The displacement field of the classical thin plate theory (CPT) is obtained easily by setting $\Psi(z) = 0$. The displacement of the first shear deformation theory (FSDT) is obtained by setting $\Psi(z) = z$. Also, the displacement of parabolic shear deformation theory (PSDT) by Reddy (1984) is obtained by setting

$$\Psi(z) = z\Big(1 - \frac{4z^2}{3h^2}\Big) \tag{2.2}$$

The sinusoidal shear deformation theory (SSDT) of Touratier (1991) is obtained by setting

$$\Psi(z) = \frac{h}{\pi}\sin\frac{\pi z}{h} \tag{2.3}$$

3. Present new refined shear deformation theory

This theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The theory presented is variationally consistent, does not require shear correction factor, and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying the shear stress free surface conditions.

3.1. Basic assumptions

The assumptions of the present theory for the nonlinear cylindrical bending of FG plate are as follows:

(i) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of the coordinate x only

$$w(x,z) = w_b(x) + w_s(x) \tag{3.1}$$

(iii) The transverse normal stress σ_z is negligible in comparison with the in-plane stresses σ_x.

(iv) The displacement u in the x-direction consists of extension, bending, and shears components

$$u = u_0 + u_b + u_s \tag{3.2}$$

The bending component u_b is assumed to be similar to the displacement given by the classical plate theory. Therefore, the expression for u_b may be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \tag{3.3}$$

The shear component u_s gives rise, in conjunction with w_s, to the parabolic variations of the shear strain γ_{xz} and hence to the shear stress τ_{xz} through the thickness of the plate in such a way that the shear stress τ_{xz} is zero at the top and bottom faces of the plate. Consequently, the expression for u_s may be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x} \qquad f(z) = -z\Big[\frac{1}{4} - \frac{5}{3}\Big(\frac{z}{h}\Big)^2\Big] \tag{3.4}$$

3.2. Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (2.1)-(2.3) and (3.1) as

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \qquad w(x,y,z) = w_b(x,y) + w_s(x,y) \tag{3.5}$$

In the present study, we wish to investigate the effect of geometric non-linearity on the response quantities. Therefore, the Von-Karman-type of geometric non-linearity is taken into consideration in the strain-displacement relations. Substituting Eqs. (3.5) in the appropriate strain-displacement relations (Fung, 1965) results in

$$\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s \qquad \gamma_{xz} = g(z)\gamma_{xz}^s \qquad \varepsilon_z = 0 \tag{3.6}$$

where

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\Big(\frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x}\Big)^2 \qquad && k_x^b = -\frac{\partial^2 w_b}{\partial x^2} \qquad k_x^s = -\frac{\partial^2 w_s}{\partial x^2} \\ &\gamma_{xz}^s = \frac{\partial w_s}{\partial x} && g(z) = \frac{5}{4} - 5\Big(\frac{z}{h}\Big)^2 \end{aligned} \tag{3.7}$$

3.3. Constitutive relations

Consider a functionally graded plate which is made from a mixture of ceramics and metals. It is assumed that the composition properties of FGM vary through the thickness of the plate. The variation of material properties can be expressed as

$$P(z) = P_b + (P_t - P_b)V_t \tag{3.8}$$

where P denotes a generic material property like modulus, and P_t and P_b denote the corresponding properties of the top and bottom faces of the plate, respectively. Also V_t in Eq. (3.8) denotes the volume fraction of the top face constituent and follows a simple power-law as

$$V_t = \Big(\frac{z}{h} + \frac{1}{2}\Big)^n \tag{3.9}$$

where n $(0 \leqslant n \leqslant \infty)$ is a parameter that dictates the material variation profile through the thickness. Here we assume that the moduli E and G vary according to Eq. (3.8) and Poisson's ratio ν is assumed to be a constant.

The linear constitutive relations are

$$\sigma_x = Q_{11}\varepsilon_x \qquad \tau_{xz} = Q_{55}\gamma_{xz} \tag{3.10}$$

where (σ_x, τ_{yx}) and $(\varepsilon_x, \gamma_{xz})$ are the stress and strain components, respectively. Using the material properties defined in Eq. (3.8), the stiffness coefficients Q_{ij}, may be expressed as

$$Q_{11} = \frac{E(z)}{1-\nu^2} \qquad Q_{55} = \frac{E(z)}{2(1+\nu)} = g(z) \tag{3.11}$$

The stress and moment resultants of the FGM plate can be obtained by integrating Eq. (3.10) over the thickness, and are written as

$$(N_x, M_x^b, M_x^s) = \int\limits_{-h/2}^{h/2} \sigma_x(1, z, f(z))\; dz \qquad S_{xz}^s = \int\limits_{-h/2}^{h/2} \tau_{xz} g(z)\; dz \tag{3.12}$$

Substituting Eq. (3.10) into Eq. (3.12) and integrating through the thickness of the plate, the stress resultants are given as

$$\begin{aligned} &N_x = A_{11}\varepsilon_x^0 + B_{11}k_x^b + B_{11}^s k_x^s \qquad && M_x^b = A_{11}\varepsilon_x^0 + D_{11}k_x^b + D_{11}^s k_x^s \\ &M_x^s = B_{11}^s\varepsilon_x^0 + D_{11}^s k_x^b + H_{11}^s k_x^s && S_{xz}^s = A_{55}^s\gamma_{xz} \end{aligned} \tag{3.13}$$

where A_{ij}, B_{ij}, etc., are the plate stiffnesses defined by

$$\begin{aligned} &\{A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11}\} = \int\limits_{-h/2}^{h/2} \{1, z, z^2, z^3, z^4, z^6\} Q_{11}\, dz \\ &B_{11}^s = -\frac{1}{4}B_{11} + \frac{5}{3h^2}E_{11} \qquad D_{11}^s = -\frac{1}{4}D_{11} + \frac{5}{3h^2}F_{11} \\ &H_{11}^s = \frac{1}{16}D_{11} - \frac{5}{6h^2}F_{11} + \frac{25}{9h^4}H_{11} \qquad \{A_{55}, D_{55}, F_{55}\} = \int\limits_{-h/2}^{h/2} \{1, z^2, z^4\} Q_{55}\, dz \\ &A_{55}^s = \frac{25}{16}A_{55} - \frac{25}{2h^2}D_{55} + \frac{25}{h^4}F_{55} \end{aligned} \tag{3.14}$$

3.4. Solution procedure

The total potential energy Π of the FG plate is determined by the summation of strain energy and the change in potential energy of the uniform externally applied pressure, and is written as

$$\Pi = U + V \tag{3.15}$$

Here, V is the potential energy of a plate under uniform pressure, and is equal to

$$V = \int\limits_{-a}^{a} q(w_b + w_s)\, dx \tag{3.16}$$

where q is the uniformly distributed load and $2a$ is the length of the FG plate. Also, the strain energy U is defined as

$$U = \frac{1}{2}\int\limits_{-a}^{a}\int\limits_{-h/2}^{h/2} (\sigma_x\varepsilon_x + \tau_{xz}\gamma_{xz})\, dx\, dz \tag{3.17}$$

Now considering the boundary condition for the simply supported FG plate, the principal of minimum potential energy is applied assuming a first guess solution for the considered displacements (i.e. u_0, w_b and w_s) over the mid-surface of the plate as

$$\begin{aligned} &w_b = w_s = u_0 = 0 \qquad \text{at} \qquad x = -a, a \\ &M_x^b = M_x^s = 0 \qquad \text{at} \qquad x = -a, a \end{aligned} \tag{3.18}$$

The required mentioned displacement and rotation fields, which satisfy the simply supported boundary conditions, are defined as

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \begin{Bmatrix} C \sin\dfrac{\pi x}{a} \\ W_0^b \cos\dfrac{\pi x}{2a} \\ W_0^s \cos\dfrac{\pi x}{2a} \end{Bmatrix} \tag{3.19}$$

where C, W_0^b, and W_0^s are arbitrary parameters and are determined by minimizing the total potential energy as

$$\frac{\partial \Pi}{\partial(C, W_0^b, W_0^s)} = 0 \tag{3.20}$$

Equations (3.18) provides a set of three nonlinear equilibrium equations in terms of C, W_0^b, and W_0^s which should be solved. The obtained constants are then used to calculate the displacements (Eq. (3.19)) and, subsequently, the strain and stresses are found using Eqs. (3.6) and (3.10).

4. Numerical results and discussion

To examine the nonlinear bending behavior of an structure, a plate consisting of aluminum and alumina as the respective metal and ceramic substances of the FGM considered as an example. Young's modulus for aluminum is 70 GPa while for alumina it is 380 GPa. Note that Poisson's ratio is selected constant and equal to 0.3 for both constituents. For all analyses, the lower surface of the plate is assumed to be rich in metal (aluminum) and the upper surface to be rich in ceramic (alumina).

The results obtained from the analysis are presented in dimensionless parametric terms of deflections and stresses as follows:

- central deflection $W = w/h$
- loading parameter $Q = qL^4/(E_b h^4)$
- axial stress $\sigma = \sigma_x L^2/(E_b h^2)$
- shear stress $*\tau = \tau_{xz} L^2/(E_b h^2)$
- thickness coordinate $Z = z/h$

where E_b is Young's modulus of the metal (bottom surface) used in the functionally graded material, and q is uniformly distributed pressure load. The analysis is performed on a FG plate of length $L = 2a = 200\,\mathrm{mm}$ and thickness $h = 10\,\mathrm{mm}$ with the simply supported boundary condition. The shear correction factor of FSDT is fixed to be $K = 5/6$.

Figure 1 shows the inverse of the dimensionless deflection (L/w) of the homogeneous plate $(n = 0)$ as a function of the plate aspect ratio (L/h). It can be observed that the results obtained by the present new refined shear deformation theory are identical to those of the first shear deformation theory (FSDT), parabolic shear deformation theory (PSDT) and sinusoidal shear deformation theory (SSDT). It is observed that the CPT may be sufficient for thin plates.

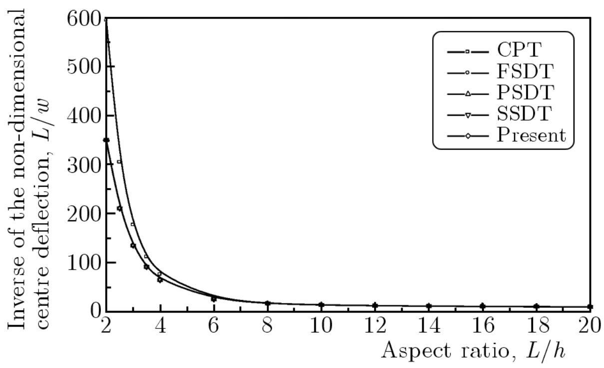

Fig. 1. Comparison of the center deflection for different theories versus aspect ratio (L, h) under constant load $q = -0.56\,\mathrm{GPa}$ and power-law index $n = 0$

Figure 2 shows the dimensionless deflection of the center of the plate with aspect ratio of $(L/h = 20)$. As it is seen, the results obtained by the present new refined shear deformation theory, FSDT, PSDT and SSDT coincide with the results of CPT. This is well explained by the large plate aspect ratio $(L/h = 20)$. For such a plate, the in-plane shear stress due to the thin thickness is negligible and, as a result, all the applied theories end up with similar results.

Figure 3 shows the dimensionless deflection of the center of the plate with aspect ratio of $(L/h = 2)$ using different plate theories. As it is seen, for small n, the plate will be rich in ceramic (alumina), which has a large Young's modulus and, as a result, its deflection will be small. Also, based on the figure, the CPT predicts lower deflection than the other theories, and the predictions by shear deformation theories are close, though the latter gives the largest deflection for the plate.

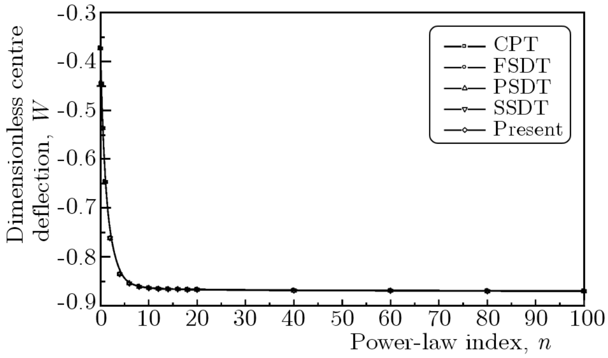

Fig. 2. Center deflection versus power-law index n under load $Q = -20$ and aspect ratio $L/h = 20$

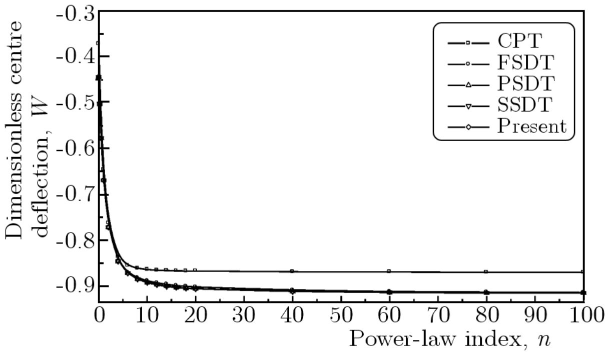

Fig. 3. Center deflection versus power-law index n under load $Q = -20$ and aspect ratio $L/h = 2$

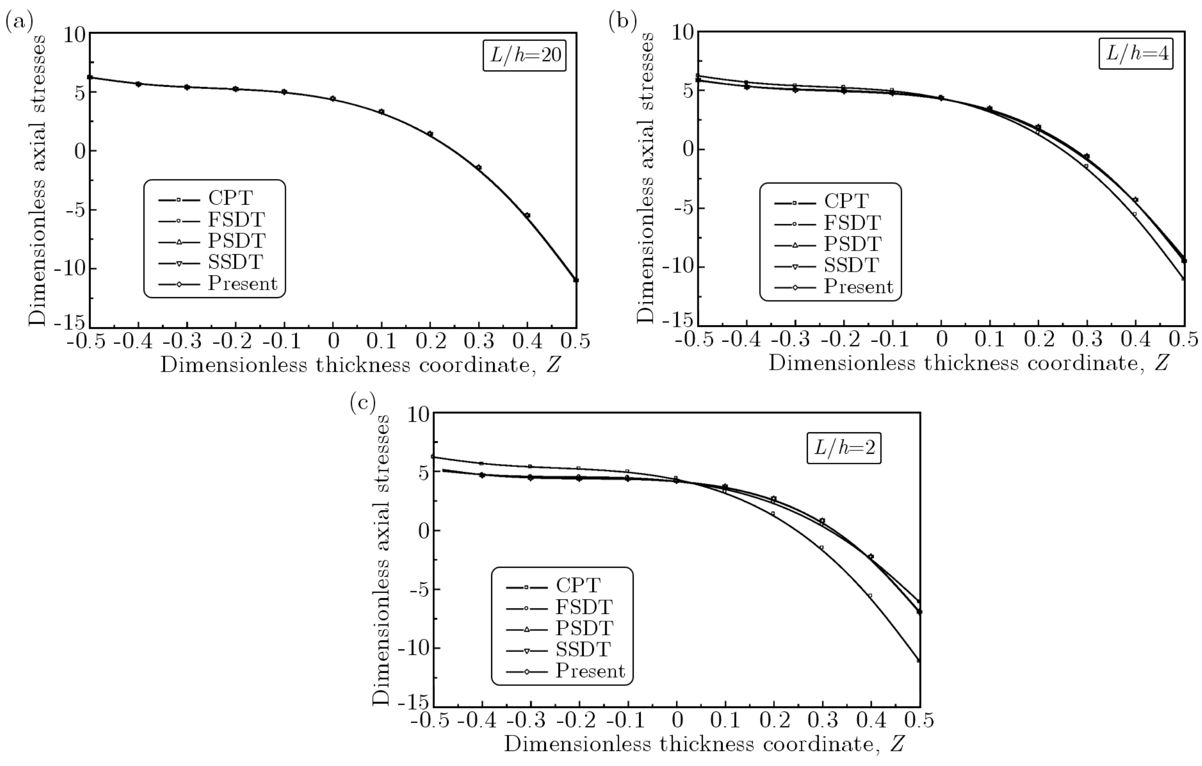

Fig. 4. Through-the-thickness axial stress σ at the center of the FG plate under load $Q = -20$ and power-law index $n = 2$: (a) $L/h = 20$, (b) $L/h = 4$, (c) $L/h = 2$

The results of the throrrasugh-the-thickness axial stress of the FG plate $(n = 2)$ are shown in Fig. 4. It is observed that the results computed using the new refined shear deformation theory are in a good agreement with those obtained by using FSDT, PSDT and SSDT. The results obtained by all models (CPT, FSDT, PSDT, SSDT and the present) are in excellent agreement even though the plate is very thin.

Figure 5 depicts through-the-thickness distributions of the shear stresses in the FG plate $(n = 2)$. It is observed from these figures that the through-the-thickness distributions of the shear stresses are not parabolic. It is to be noted that the maximum value occurs at , $Z \cong 0.2$, not at the plate center as in the homogeneous case. The results obtained by all models (PSDT, SSDT and the present) are in good agreement, particularly with PSDT.

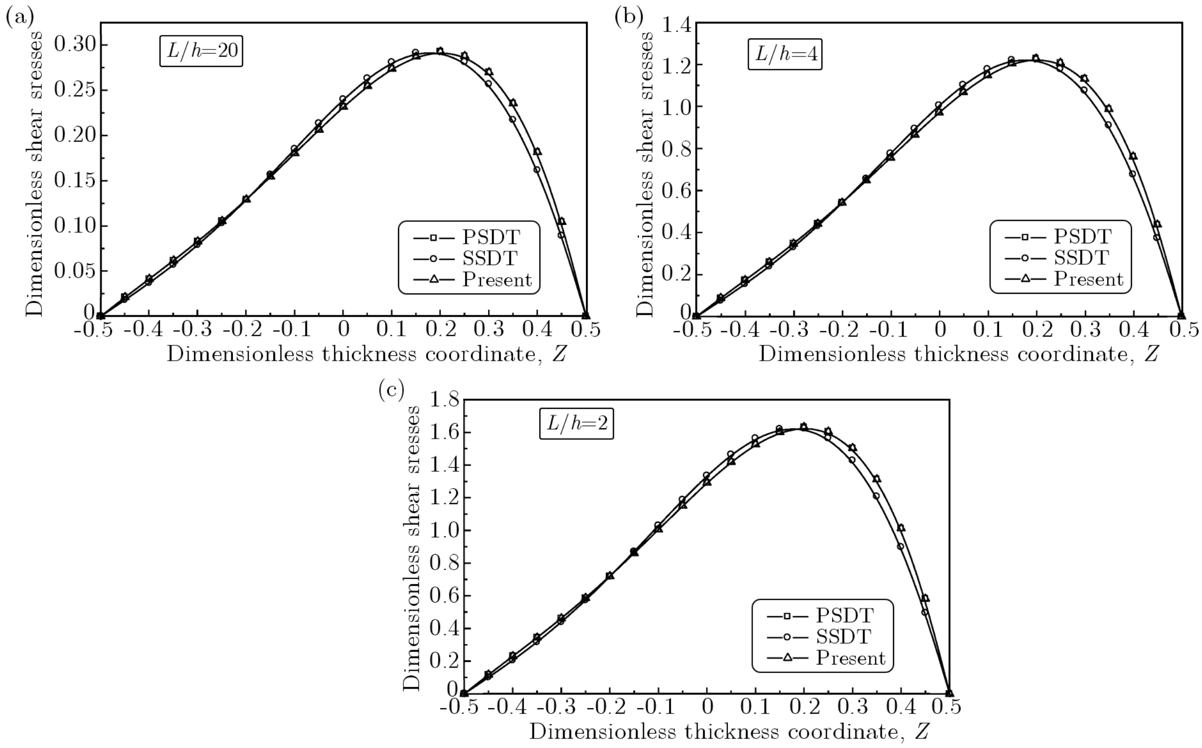

Fig. 5. Through-the-thickness shear stress τ at the center of the FG plate under load $Q = -20$ and power-law index $n = 2$: (a) $L/h = 20$, (b) $L/h = 4$, (c) $L/h = 2$

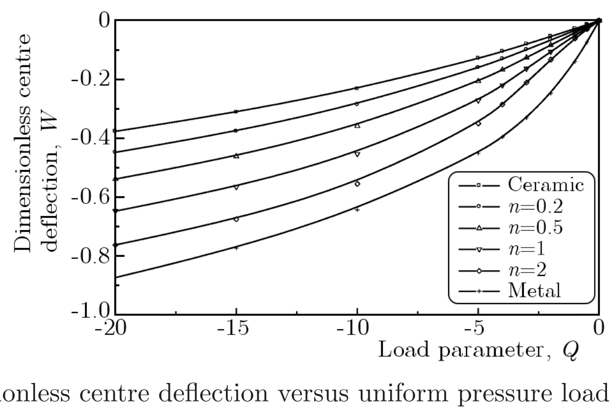

Fig. 6. Dimensionless centre deflection versus uniform pressure load with $L/h = 10$

Figure 6 illustrates variation of the non-dimensional center deflection of the FG square plate with different values of the power-law index n subjected to a uniform transverse load. The results show that a pure metal plate has the highest deflection. This is expected because the fully metallic plate is one with lower stiffness than ceramic and FG plates.

5. Conclusion

The nonlinear cylindrical bending of FG plates under pressure loading was studied using a new refined shear deformation theory. Fundamental equations for thick square FG plates have been obtained using the Von-Karman theory for large deflections. The solution of the nonlinear cylindrical bending problem is obtained via minimization of the total potential energy. The accuracy and efficiency of the present theory have been demonstrated for nonlinear bending analysis of simply supported FG plates. It can be concluded that the present theory is not only accurate but also efficient in predicting the nonlinear cylindrical bending response of FG plates compared to other shear deformation plate theories such as FSDT, PSDT and SSDT.

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Nieliniowa analiza zginania walcowego płyt wykonanych z materiału gradientowego w oparciu o nową teorię odkształcenia postaciowego

Streszczenie

W niniejszym artykule zaprezentowana została nowa, udoskonalona teoria odkształcenia postaciowego dla przypadku zginania walcowego płyt wykonanych z funkcjonalnie gradientowego (FG) materiału. Teorię tę oparto na założeniu, że przemieszczenie poprzeczne jest wynikiem efektu zginania i ścinania, ale zginanie nie wpływa na siły tnące ani ścinanie na wartość momentu gnącego. Teoria wprowadza potęgową zależność czwartego stopnia odkształceń postaciowych w funkcji grubości płyty i spełnia warunek brzegowy zerowania naprężeń ścinających na zewnętrznych powierzchniach płyty bez użycia współczynników korekcyjnych. W pracy zbadano płyty poddane ciągłym obciążeniom zewnętrznym, a uwzględniona nieliniowość miała charakter geometryczny wynikający z opisu związku von Karmana pomiędzy odkształceniem i przemieszczeniem. Gradientowe właściwości materiału ujęto potęgowym rozkładem poszczególnych składników wzdłuż grubości płyty. Rozwiązania otrzymano poprzez minimalizację całkowitej energii potencjalnej przy zginaniu, a uzyskane wyniki porównano ze znanymi w literaturze rezultatami klasycznej teorii odkształcenia pierwszego oraz wyższych rzędów. Przedstawiona analiza potwierdziła, że nowo zaproponowana teoria jest dokładna i prosta w rozwiązywaniu nieliniowego problemu zginania walcowego płyt wykonanych z materiałów gradientowych.

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