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PROBLEM OF ELASTIC INTERACTION BETWEEN AN ANNULAR THICK PLATE AND ELASTIC STRATUM

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A solution to the contact pressure problem, bending moments and displacements of an axisymmetrically loaded thick annular transversely isotropic plate resting without friction on a transversely isotropic or granular half-space is presented in the paper. No singularity occurs in the contact pressure because the extensional deformation of the plate is taken into consideration. An approximate solution to the resulting integral equation is obtained using an effective numerical procedure. To assess the effects of anisotropy of the plate response, numerical results are obtained for three materials: magnesium, which is nearly isotropic; cadmium, which is moderately anisotropic; and graphite epoxy, which is highly anisotropic and the supporting half-space is modelled as granular material, like soil. The results are presented graphically.

Key word: anisotropy, contact problem, thick plate

1. Introduction

The considered linear elasticity contact problem models the actual problem of interaction between an annular flat foundation and the supporting soil. Various models used in studying soil-foundation interaction problems are discussed in the book by Selvadurai (1979). Extensive bibliography and comprehensive accounts of various contact problems can be also found in the work by Popov (1971), Poulos and Davis (1974), Hooper (1978) and Gladwell (1980).

In modelling a raft foundation, it is usually assumed that it behaves like a thin isotropic elastic plate, governed by the Kirchoff-Love plate theory. When the thickness of the plate is small, compared with the other dimensions of the plate, and the loading does not present any abrupt changes, the thin plate theory gives satisfactory results. However, when the plate is thick and anisotropic and is subjected to localized loads the influence of shearing deformations and transverse normal stresses on the plate response has to be taken into consideration. Plate theories taking into account shearing deformations and extensional deformations have been developed by Reissner (1945, 1947), Mindlin (1951), Goodier (1946) for isotropic materials and by Rogowski (1975) for orthotropic ones.

Regarding the behaviour of the supporting soil, various models have been proposed and applied within the framework of the linear elasticity. The most common of them are: (a) Winkler springs, (b) half-space continuum, and (c) layered continuum. The half-space continuum is modelled as transversely isotropic medium. Its particular case was proposed by Weiskopf (1945). The Weiskopf model takes into consideration slipping of the granules of a granular material, like soil, which causes appreciable shearing deflections. The Weiskopf model has been used previously by Misra and Sen (1975, 1976), by Ejike (1977) and by Mastrojannis (1989) in analytical studies. The transverse isotropy proposed by Weiskopf introduces two shear moduli in the plane of the isotropy and in the direction normal to the planes of the isotropy. But Young's moduli of some soil masses are also dissimilar in both directions (Dahan and Predeleanu, 1981), so the soil masses exhibit, in general, transversely isotropic mechanical behaviour.

The contact conditions also influence the soil-foundation interaction. The usual assumption is that no shearing stresses develop at the interface of the bodies in contact. This assumption is used in the present work, too. The presence of the shearing stresses, due to friction or adhesion, decreases displacements and bending moments of a thin elastic plate (Hooper, 1981; Mastrojannis et al., 1988), which probably holds also for the case of a thick plate. Accordingly, the presented solution to a frictionless contact problem gives an upper bound for the actual response and the safety aspects of the structure are quaranted. The extensional transverse deformation of the plate, which is taken into consideration, yields that no singularity occurs in the contact pressure at the end of the contact region. This is in contrast with the well-known results (Dundurs and Lee, 1971; Adams and Bogy, 1976; Gecit, 1986).

The purpose of this paper is: (a) to present a solution to the problem of a thick plate with shearing and extensional deformations taken into account and to half-space elasticity problem (with solution derived by making use of the transversely isotropic potential function method and integral transforms), to reduce contact conditions to the problem of solving the integral equation for unknown normal contact pressure, (b) to give an approximate solution to the resulting integral equation using an effective numerical procedure, and (c) to give numerical results, for some practical materials which indicate dependence of mechanical quantities on elastic constants of the plate and half-space on the plate thickness and other parameters over a certain parameter range.

2. Formulation of the problem

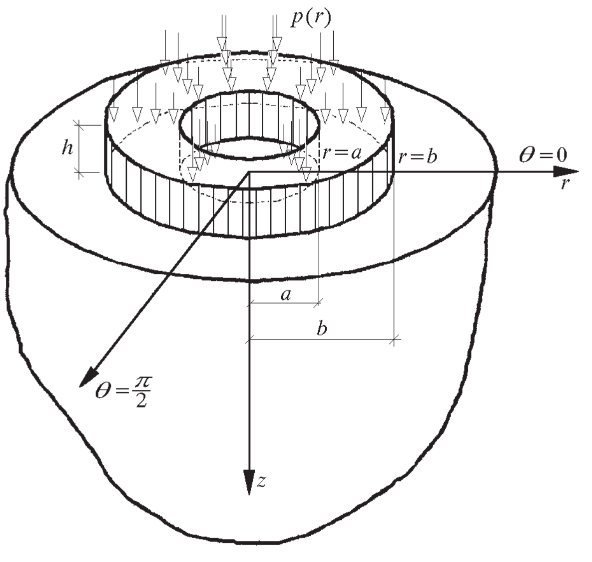

Fig. 1. Geometry of the system

An annular plate with the inner and outer radii a and b, and of the uniform thickness h is in smooth contact with the horizontal surface $z = 0$ of an elastic medium occupying the half-space $z \geqslant 0$ (Fig. 1). Due to the axisymmetrically distributed load $p(r)$ acting on the upper surface of the plate and the reactive normal pressure $q(r)$ acting on the lower surface the plate deforms and its lower surface assumes a shape described by a function $w_0(r)$ for $a \leqslant r \leqslant b$. The normal surface displacement of the half-space in the contact region is also described by the same function $w_0(r)$. Taking into consideration the theory of thick plates (Rogowski, 1975), it is required to determine the normal displacement $w(r, z)$, the contact pressure $q(r)$, the bending moments $M_r(r)$ and $M_\theta(r)$ and the shearing force $Q_r(r)$ induced by the plate. Materials of the plate and the half-space exhibit transversely isotropic behaviour. The planes of the isotropy are assumed to be parallel to the boundary $z = 0$.

The stress-strain relations for the transverse isotropy are

$$\begin{aligned}\sigma_{rr} &= c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz}\\ \sigma_{\theta\theta} &= c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz}\\ \sigma_{zz} &= c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{33}\varepsilon_{zz}\\ \sigma_{rz} &= 2c_{44}\varepsilon_{rz}\end{aligned} \tag{2.1}$$

where c_{ij} are the material constants of the transverse isotropy. The engineering constants $E_r, \nu_{r\theta}$ in the isotropic plane and E_z, ν_{rz}, G_{rz} in the principal direction of anisotropy have the following relations between the moduli of elasticity c_{ij}

$$\begin{aligned}c_{11} &= \frac{E_r(1-\nu_{rz}\nu_{zr})}{\Delta(1+\nu_{r\theta})} & c_{12} &= \frac{E_r(\nu_{r\theta}+\nu_{rz}\nu_{zr})}{\Delta(1+\nu_{r\theta})}\\ c_{13} &= \frac{E_r\nu_{rz}}{\Delta} & c_{33} &= \frac{E_z(1-\nu_{r\theta})}{\Delta}\\ c_{44} &= G_{rz} & \Delta &= 1-\nu_{r\theta}-2\nu_{rz}\nu_{zr}\end{aligned} \tag{2.2}$$

The solution to the stated axisymmetric interaction problem involves the solution to two coupled boundary value problems: one for bending and compression by the normal loads $p(r)$ and $q(r)$ with free-edge conditions, and the other for the stress and displacement fields inside the half-space $z \geqslant 0$ when the surface $z = 0$ is free to shearing traction with the normal displacement prescribed in the contact region and the normal stresses vanishing outside of the contact region. The contact surfaces separate in the neighbourhood of the point where the contact stress changes from negative to positive.

2.1. The thick plate problem

Accordingly (Rogowski, 1975), the normal displacement $w(r,z)$ of an axisymmetrically loaded thick transversely isotropic plate under bending satisfies the differential equation

$$\nabla_r^4 w(r,z) = \nabla_r^2[\nabla_r^2 w(r,z)] = [1-\beta(z)h^2\nabla_r^2]\frac{F(r)}{D} \qquad \begin{aligned}a &\leqslant r \leqslant b\\ -h &\leqslant z \leqslant 0\end{aligned} \tag{2.3}$$

where

$$D = \frac{h^3}{12}\Big(c_{11} - \frac{c_{13}^2}{c_{33}}\Big) = \frac{E_r h^3}{12(1-\nu_{r\theta}^2)}$$

$$\beta(z) = \frac{1}{4}\Big[\frac{c_{11}-c_{13}^2/c_{33}}{2c_{44}} - \frac{c_{13}}{c_{33}}\Big(1+2\frac{z(z+h)}{h^2}\Big)\Big] = \tag{2.4}$$
$$= \frac{1}{4(1-\nu_{r\theta})}\Big[\frac{G_{r\theta}}{G_{rz}} - \nu_{zr}\Big(1+2\frac{z(z+h)}{h^2}\Big)\Big]$$

denote the plate flexural rigidity and variable correction coefficient, respectively, and

$$F(r) = p(r) - q(r) \qquad\qquad \nabla_r^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{2.5}$$

The shearing force $Q_r(r)$ and bending moments $M_r(r)$ and $M_\theta(r)$ are given by expressions

$$\begin{aligned} Q_r(r) &= -D\frac{dH}{dr} \\ M_r(r) &= -D\Big(\frac{d^2}{dr^2} + \frac{\nu_{r\theta}}{r}\frac{d}{dr}\Big)[w(r) + \beta h^2 H] \\ M_\theta(r) &= -D\Big(\nu_{r\theta}\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\Big)[w(r) + \beta h^2 H] \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} H &= \nabla_r^2 w(r) + \frac{\beta h^2}{D}F(r) \\ \beta &= \frac{1}{4}\Big(\frac{c_{11}-c_{13}^2/c_{33}}{2c_{44}} - \frac{2}{3}\frac{c_{13}}{c_{33}}\Big) = \frac{1}{4(1-\nu_{r\theta})}\Big(\frac{G_{r\theta}}{G_{rz}} - \frac{2}{3}\nu_{zr}\Big) \end{aligned} \tag{2.7}$$

In equations $(2.6)_2 - (2.7)_1$

$$w(r) = \frac{1}{h}\int\limits_{-h}^{0} w(r,z)\ dz$$

is the weighted average normal displacement which satisfies a differential equation similar to (2.3), with the weighted average correction coefficient β.

For the axisymmetric and free-edge conditions of the considered annular plate the following boundary conditions apply

$$Q_r = 0 \qquad\qquad M_r = 0 \qquad\qquad \text{for} \quad r = a \ \wedge \ r = b \tag{2.8}$$

The solution to equation (2.3), satisfying boundary condition (2.8), is expressed by

$$
\begin{aligned}
w(r,z) = w_0 + \frac{r^2 - 2a^2 \ln \frac{r}{b}}{8D(b^2-a^2)} \int\limits_a^b F(\rho)\rho\Big[\eta\rho^2 + 2(b^2-a^2)\ln\frac{\rho}{b}\Big]\,d\rho \,- \\
-\frac{1}{4D}\int\limits_a^r F(\rho)\rho\Big[r^2-\rho^2+(r^2+\rho^2)\ln\frac{\rho}{b}\Big]\,d\rho + \frac{\beta(z)h^2}{D}\int\limits_a^r F(\rho)\rho\ln\frac{\rho}{b}\,d\rho
\end{aligned}
\qquad (2.9)
$$

where ρ is the variable of integration and the constant w_0 is determined by considering the contact condition, namely the equilibrium of forces in the vertical direction, i.e.

$$
\int\limits_a^b F(r)r\,dr = 0 \qquad \text{and} \qquad \eta = \frac{1-\nu_{r\theta}}{1+\nu_{r\theta}} \qquad (2.10)
$$

For a circular plate, i.e. when $a = 0$, w_0 is the central deflection of the plate.

The extensional deformation yields also transverse displacement of the plate which is expressed by equation (Rogowski, 1975)

$$
w'(r,z) = -\frac{(2z+h)h^3}{48D}\frac{c_{11}}{c_{33}}[p(r)+q(r)] \qquad (2.11)
$$

The transverse displacement is obtained as the superposition of bending deflection (2.9) with displacement (2.11). For $z = 0$ this yields

$$
\begin{aligned}
w(r,0) = w_0 + \frac{r^2 - 2a^2 \ln \frac{r}{b}}{8D(b^2-a^2)} \int\limits_a^b F(\rho)\rho\Big[\eta\rho^2 + 2(b^2-a^2)\ln\frac{\rho}{b}\Big]\,d\rho \,- \\
-\frac{1}{4D}\int\limits_a^r F(\rho)\rho\Big[r^2-\rho^2+(r^2+\rho^2)\ln\frac{\rho}{b}\Big]\,d\rho \,+ \\
+\frac{\beta_0 h^2}{4D}\int\limits_a^r F(\rho)\rho\ln\frac{\rho}{b}\,d\rho - \frac{h^4}{48D}\alpha_0[p(r)+q(r)] \qquad a \leqslant r \leqslant b
\end{aligned}
\qquad (2.12)
$$

where

$$
\begin{aligned}
\beta_0 &= 4\beta(0) = \frac{c_{11}-c_{13}^2/c_{33}}{2c_{44}} - \frac{c_{13}}{c_{33}} = \frac{1}{1-\nu_{r\theta}}\Big(\frac{G_{r\theta}}{G_{rz}} - \nu_{zr}\Big) \\
\alpha_0 &= \frac{c_{11}}{c_{33}} = \frac{E_r(1-\nu_{zr}\nu_{rz})}{E_z(1-\nu_{r\theta}^2)}
\end{aligned}
\qquad (2.13)
$$

2.2. The half-space problem

The boundary conditions for the axisymmetric indentation half-space problem are

$$
\begin{aligned}
&u_z(r,0) = w_0(r) && \text{for} \quad r \in < a, b > \\
&\sigma_{zz}(r,0) = 0 && \text{for} \quad r \in < 0, a) \cup (b, \infty) \\
&\sigma_{rz}(r,0) = 0 && \text{for} \quad r \in < 0, \infty)
\end{aligned}
\tag{2.14}
$$

with the stress and displacement vanishing at infinity. The last equation of (2.14) states that the contact is frictionless.

The normal displacement $u_z(r,0)$ on the surface $z=0$ of the half-space is determined by means of the contact pressure $\sigma_{zz}(r,0) = -q(r)$, as follows (Rogowski, 1982)

$$
u_z(r,0) = \frac{1}{G_z C} \int\limits_a^b \rho q(\rho)\, d\rho \int\limits_0^\infty J_0(\xi\rho) J_0(\xi r)\, d\xi \tag{2.15}
$$

where G_z is the shear modulus in the z-direction, and C is a material constant (Rogowski, 1982)

$$
G_z C = \frac{E_r}{(1-\nu_{r\theta}^2) s_1 s_2 (s_1 + s_2)}
$$

The improper integral in equation (2.15) has analytical representation, namely

$$
\int\limits_0^\infty J_0(\rho\xi) J_0(r\xi)\, d\xi = \frac{2}{\pi}\Big[\frac{H(r-\rho)}{r} K\Big(\frac{\rho}{r}\Big) + \frac{H(\rho - r)}{\rho} K\Big(\frac{r}{\rho}\Big)\Big] = \frac{2}{\pi}\frac{K(k)}{r+\rho} \tag{2.16}
$$

The functions $K(k)$ are complete elliptic integrals of the first kind (see equation (3.14)); the modulus $k = 2\sqrt{\rho r}/(\rho + r)$, and $H(\cdot)$ is Heaviside's function.

Then, the normal displacement is given as

$$
u_z(r,0) = \frac{2}{\pi G_z C} \int\limits_a^b \frac{q(\rho)\rho K(k)}{r+\rho}\, d\rho \tag{2.17}
$$

Substitution of equations (2.12) and (2.17) into equation (2.14) yields the following integral equation in terms of the contact pressure $q(\rho)$ for the stated contact problem

$$
\begin{aligned}
&\frac{2}{\pi G_z C}\int\limits_a^b \frac{q(\rho)\rho K(k)}{r+\rho}\,d\rho = -\frac{h^4}{48D}\alpha_0[p(r)+q(r)] + \\
&+w_0 + \frac{r^2-2a^2\ln\frac{r}{b}}{8D(b^2-a^2)}\int\limits_a^b F(\rho)\rho\Big[\eta\rho^2+2(b^2-a^2)\ln\frac{\rho}{b}\Big]\,d\rho - \\
&-\frac{1}{4D}\int\limits_a^r F(\rho)\rho\Big[r^2-\rho^2+(r^2+\rho^2)\ln\frac{\rho}{r}\Big]\,d\rho + \frac{\beta_0 h^2}{4D}\int\limits_a^r F(\rho)\rho\ln\frac{\rho}{r}\,d\rho
\end{aligned}
\tag{2.18}
$$

3. The solution to the integral equation

It is convenient to transform equation (2.18) into dimensionless form. Write

$$
\begin{array}{llll}
\rho = xb & r = yb & a = \lambda b & h = tb \\
q(xb) = P_0\overline{q}(x) & p(xb) = P_0\overline{p}(x) & K_r = \dfrac{D}{G_z C b^3} &
\end{array}
\tag{3.1}
$$

where P_0 is a constant with dimensions of pressure and K_r is the plate-to-half-space stiffness ratio. Then, the integral equation for determining the function $\overline{q}(x)$ takes the form

$$
\begin{aligned}
&\int\limits_\lambda^1 \frac{\overline{q}(x)xK(k)}{x+y}\,dx + \frac{\pi}{16K_r}\Big\{\frac{y^2-2\lambda^2\ln y}{1-\lambda^2}\int\limits_\lambda^1 \overline{q}(x)x[\eta x^2+2(1-\lambda^2)\ln x]\,dx - \\
&-2\int\limits_\lambda^y \overline{q}(x)x\Big[y^2-x^2+(y^2+x^2)\ln\frac{x}{y}\Big]\,dx + 2\beta_0 t^2\int\limits_\lambda^y \overline{q}(x)x\ln\frac{x}{y}\,dx + \\
&+\frac{t^4}{s}\alpha_0\overline{q}(y)\Big\} = w_0^* + \frac{\pi}{16K_r}T(y)
\end{aligned}
\tag{3.2}
$$

where

$$
w_0^* = \frac{\pi G_z C w_0}{2P_0 b}
$$

$$
\begin{aligned}
T(y) &= \varphi(y) + 2\beta_0 t^2 \int\limits_{\lambda}^{y} \overline{p}(x)x \ln\frac{x}{y}\,dx - \frac{t^4}{6}\alpha_0\overline{p}(y) \\
\varphi(y) &= \frac{y^2 - 2\lambda^2\ln y}{1-\lambda^2}\int\limits_{\lambda}^{1}\overline{p}(x)x[\eta x^2 + 2(1-\lambda^2)\ln x]\,dx - \\
&- 2\int\limits_{\lambda}^{y}\overline{p}(x)x\Big[y^2 - x^2 + (y^2+x^2)\ln\frac{x}{y}\Big]\,dx
\end{aligned}
\tag{3.3}
$$

We assume then, that the unknown function $\overline{q}(x)$, corresponding to the normal interfacial pressure, has the form

$$
\overline{q}(x) = w_0^* g^{(1)}(x) + g^{(2)}(x) \qquad \lambda \leqslant x \leqslant 1 \tag{3.4}
$$

where $g^{(1)}(x)$ and $g^{(2)}(x)$ are new unknown functions. Thus the problem is reduced to solving the following two integral equations for the unknown functions $g^{(1)}(x)$ and $g^{(2)}(x)$

$$
\psi^{(1)}(y) = 1 \qquad \psi^{(2)}(y) = \frac{\pi}{16K_r}T(y) \qquad \lambda \leqslant y \leqslant 1 \tag{3.5}
$$

where (for $m = 1,2$).

$$
\begin{aligned}
\psi^{(m)}(y) &= \int\limits_{\lambda}^{1}\frac{g^{(m)}(x)xK(k)}{x+y}\,dx + \\
&+ \frac{\pi}{16K_r}\Big\{\frac{y^2-2\lambda^2\ln y}{1-\lambda^2}\int\limits_{\lambda}^{1} g^{(m)}(x)x[\eta x^2 + 2(1-\lambda^2)\ln x]\,dx - \\
&-2\int\limits_{\lambda}^{y} g^{(m)}(x)x\Big[y^2 - x^2 + (y^2+x^2)\ln\frac{x}{y}\Big]\,dx + \\
&+2\beta_0 t^2\int\limits_{\lambda}^{y} g^{(m)}(x)x\ln\frac{x}{y}\,dx + \frac{t^4}{6}\alpha_0 g^{(m)}(y)\Big\}
\end{aligned}
\tag{3.6}
$$

The unknown constant w_0^* corresponding to the plate deflection w_0, is evaluated by considering the equilibrium of forces in the transverse direction, see equation $(2.10)_1$, i.e.

$$
w_0^*\int\limits_{\lambda}^{1} g^{(1)}(x)\,dx + \int\limits_{\lambda}^{1} g^{(2)}(x)\,dx = \int\limits_{\lambda}^{1}\overline{p}(x)x\,dx \tag{3.7}
$$

In general, it is not easy to obtain an analytical expression for the solution $g^{(m)}(x)$ to equations (3.5) and (3.6). An analytical solution exists for the special case of $K_r \to \infty$ and $\lambda = 0$, which corresponds to the problem of a rigid circular punch on a half-space. It is

$$\begin{aligned} & g_\infty^{(1)}(x) = \frac{4}{\pi^2\sqrt{1-x^2}} && g_\infty^{(2)}(x) \equiv 0 && w_{0\infty}^* = \frac{\pi^2}{8} \\ & \overline{q}(x) = \frac{1}{2\sqrt{1-x^2}} && w_{0\infty} = \frac{\pi P_0 b}{4G_z C} \end{aligned} \tag{3.8}$$

For an annular rigid plate $(K_r \to \infty)$ the analytical approximate solution is (Rogowski, 1982)

$$\begin{aligned} & g_\infty^{(1)}(x) \cong \frac{4}{\pi^2}\Big[\frac{1}{\sqrt{1-x^2}}\Big(1+\frac{4\lambda^3}{3\pi^2x^2}\Big) + \frac{2\lambda}{\pi\sqrt{x^2-\lambda^2}} - \frac{2}{\pi}\arcsin\frac{\lambda}{x} - \\ & \qquad -\frac{4\lambda^3}{3\pi^2}\frac{\arccos x}{x^3}\Big] \\ & g_\infty^{(2)}(x) \equiv 0 \qquad\qquad w_{0\infty}^* \cong \frac{\pi^2}{8}\Big(1-\frac{4\lambda^3}{3\pi^2}\Big)^{-1}(1-\lambda^2) \\ & \overline{q}_\infty(x) = w_{0\infty}^* g_\infty^{(1)}(x) \qquad\qquad w_{0\infty} = \frac{\pi P_0 b}{4G_z C}\Big(1-\frac{4\lambda^3}{3\pi^2}\Big)^{-1}(1-\lambda^2) \end{aligned} \tag{3.9}$$

When the plate is rigid, then the square root of the singularity exists for the contact pressure. When the bodies in contact are both elastic and the extensional deformation of the plate is taken into consideration in the governing integral equation then no singularity occurs in the normal contact pressure. This conclusion is in contrast with the well-known result (Dundurs and Lee, 1972; Adams and Bogy, 1976; Gecit, 1986), where the authors show that when the bodies in contact are elastic, the power of the singularity of the contact pressure depends upon their elastic constants.

We introduce an approximate solution using a numerical procedure. The contact length is divided into N equal segments of the length Δx. In this procedure, the pressure distribution is assumed to be piecewise constant, that is

$$g^{(m)}(x) = g_j^{(m)} \qquad x_j - \frac{\Delta x}{2} \leqslant 2 \leqslant x_j + \frac{\Delta x}{2} \tag{3.10}$$

where $x_j = \lambda + (j-1/2)\Delta$, $j = 1,2,...,N$.

On this assumption, equations (3.5) and (3.6) become

$$\psi_i^{(1)} = 1 \qquad \psi_i^{(2)} = \frac{\pi}{16K_r}T_i \qquad i = 1,2,...,N \tag{3.11}$$

where (for $m = 1, 2$)

$$\psi_i^{(m)} = \sum_{j=1}^{N} g_j^{(m)} I(y_i, x_j) + \frac{\pi}{16K_r}\Big\{\frac{y_i^2 - 2\lambda^2 \ln y_i}{1-\lambda^2} \sum_{j=1}^{N} g_j^{(m)} \Big[S(x'_j) - S(x'_{j-1})\Big] - -2\sum_{j=1}^{i} g_j^{(m)} \Big[Q(y_i, x'_j) - Q(y_i, x'_{j-1}) - \beta_0 t^2 \Big(R(y_i, x'_j) - R(y_i, x'_{j-1})\Big)\Big] + +\frac{t^4}{6}\alpha_0 g_i^{(m)}\Big\} \tag{3.12}$$

and $T_i = T(y_i)$, etc., $\lambda \leqslant y_i \leqslant 1$, $x'_j = \lambda + j\Delta x$, N_i is the number of segments in contact of the length $\lambda \leqslant x \leqslant y_i$, $y_i = \lambda + (i - 1/2)\Delta y$, $\Delta y = (1-\lambda)/N$, and $S(x)$, $Q(y_i, x)$, $R(y_i, x)$ are like influence coefficients defined by the integrals in equations (3.6), which are easily integrated

$$\begin{aligned} S(x) &= x^2\Big[\eta\frac{x^2}{4} + (1-\lambda)^2\Big(\ln x - \frac{1}{2}\Big)\Big] && j = 1, 2, ..., N \\ Q(y_i, x) &= \frac{x^2}{2}\Big[y_i^2 - \frac{5}{4}x^2 + (2y_i^2 + x^2)\ln\frac{x}{y_i}\Big] && j = 1, 2, ..., i \\ R(y_i, x) &= \frac{x^2}{2}\Big(\ln\frac{x}{y_i} - \frac{1}{2}\Big) && j = 1, 2, ..., i \end{aligned} \tag{3.13}$$

The integrals

$$I(y_i, x_j) = \int\limits_{x_j - \frac{\Delta x}{2}}^{x_j + \frac{\Delta x}{2}} \frac{x}{x + y_i} K(k_i)\, dx \qquad k_i^2 = \frac{4xy_i}{(x+y_i)^2}$$
$$K(k_i) = \int\limits_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k_i^2 \sin^2\phi}} \tag{3.14}$$

can be evaluated numerically over the N segments using Chebyshev's quadrature rule.

If the solution to two equations (3.11) is obtained, from condition (3.7) and equation (3.10) the value of w_0^* becomes

$$w_0^* = \frac{I_0}{\Delta x I_1} - \frac{I_2}{I_1} \tag{3.15}$$

where

$$I_0 = \int\limits_{\lambda}^{1} x\overline{p}(x)\, dx \qquad I_m = \sum_{j=1}^{N} g_j^{(m)} x_j \qquad m = 1, 2 \tag{3.16}$$

The discretized form of the integral equation for the stated contact problem, given by (3.11) and (3.12), yields an effective numerical procedure for evaluating $g^{(m)}(x)$ in each segment of the contact area.

The normal contact pressure takes the following form

$$\frac{q(yb)}{P_0} \equiv \overline{q}(y) = w_0^* g^{(1)}(y) + g^{(2)}(y) \qquad \lambda \leqslant y \leqslant 1 \tag{3.17}$$

Then, for instance, the transverse plate deflection and one of the moments are (function and discretized form)

$$\begin{aligned}
&\frac{w(yb,\zeta h)G_zC}{P_0 b} \equiv \overline{w}(y,\zeta) = \frac{2}{\pi}\omega_0^* + \frac{1}{8K_r}\Big\{T(y) - \frac{t^4}{6}\alpha_0(2\zeta+1)\overline{q}(y) - \\
&-\frac{t^4}{3}\alpha_0\zeta\overline{p}(y) - \frac{y^2 - 2\lambda^2\ln y}{1-\lambda^2}\int\limits_\lambda^1 x\overline{q}(x)[\eta x^2 + 2(1-\lambda^2)\ln x]\;dx + \\
&+2\int\limits_\lambda^y x\overline{q}(x)\Big[y^2 - x^2 + (y^2+x^2)\ln\frac{x}{y}\Big]\;dx - \\
&-8\beta(\zeta)t^2\int\limits_\lambda^y x\overline{q}(x)\ln\frac{x}{y}\;dx \qquad \lambda \leqslant y \leqslant 1 \qquad -1 \leqslant \zeta \leqslant 0
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\overline{w}(y_i,\zeta) = \frac{2}{\pi}\omega_0^* + \frac{1}{8K_r}\Big\{T_i - \frac{t^4}{6}\alpha_0(2\zeta+1)\overline{q}_i - \frac{t^4}{3}\alpha_0\zeta\overline{p}_i - \\
&-\frac{y_i^2 - 2\lambda^2\ln y_i}{1-\lambda^2}\sum_{j=1}^N \overline{q}_j[S(x'_j) - S(x'_{j-1})] + 2\sum_{j=1}^{N_i}\overline{q}_j\Big[Q(y_i,x'_j) - Q(y_i,x'_{j-1}) - \\
&-4\beta(\zeta)t^2[R(y_i,x'_j) - R(y_i,x'_{j-1})]\Big]\Big\} \qquad i = 1,2,...,N
\end{aligned}$$

$$\begin{aligned}
&\frac{M_r(yb)}{P_0 b^2} \equiv \overline{M}_r(y) = -\frac{1}{8}\Big(\frac{d^2\varphi}{dy^2} + \frac{\nu_{r\theta}}{y}\frac{d\varphi}{dy}\Big) + \frac{y^2(1+\nu_{r\theta}) + \lambda^2(1-\nu_{r\theta})}{4y^2(1-\lambda^2)}\cdot \\
&\cdot\int\limits_\lambda^1 x\overline{q}(x)[\eta x^2 + 2(1-\lambda^2)\ln x]\;dx + \\
&+\frac{1-\nu_{r\theta})}{4y^2}\int\limits_\lambda^y x\overline{q}(x)\Big[y^2 - x^2 + 2y^2\ln\frac{x}{y}\Big]\;dx - \int\limits_\lambda^y x\overline{q}(x)\ln\frac{x}{y}\;dx
\end{aligned} \tag{3.19}$$

$$\overline{M}_r(y_i) = -\frac{1}{8}\Big(\frac{d^2\varphi}{dy^2} + \frac{\nu_{r\theta}}{y}\frac{d\varphi}{dy}\Big) + \frac{y_i^2(1+\nu_{r\theta}) + \lambda^2(1-\nu_{r\theta})}{4y_i^2(1-\lambda^2)}\cdot$$

$$\cdot\sum_{j=1}^{N}\overline{q}_j[S(x'_j)-S(x'_{j-1})]+\frac{1-\nu_{r\theta}}{4y^2}\sum_{j=1}^{N_i}\overline{q}_j[Q^*(y_i,x'_j)-Q^*(y_i,x'_{j-1})]+$$
$$+\sum_{j=1}^{N_i}\overline{q}_j[R(y_i,x'_j)-R(y_i,x'_{j-1})] \qquad i=1,2,...,N$$

where $\zeta = z/h \in [-1,0]$.

4. Numerical results and discussion

A particular case of the transversely isotropic half-space continuum, which is used in numerical calculations, is constituted by the model proposed by Weiskopf (1945). In this model $\mathit{\Gamma} = G_r/G_z > 1$ and the material parameter C is

$$C = \mathit{\Gamma}\sqrt{\frac{2}{(1-\nu)(\mathit{\Gamma}+1-2\nu)}} \tag{4.1}$$

Write the relative stiffness parameter K_r, equation (3.1), for a granular material as

$$K_r = K_m t^3\sqrt{\frac{\mathit{\Gamma}-\nu}{2(1-\nu)}+\frac{1}{2}} \tag{4.2}$$

where

$$K_m = \frac{1}{6}\frac{E_r}{E}\frac{1-\nu^2}{1-\nu_{r\theta}^2} \tag{4.3}$$

The relative stiffness parameter K_r, the reduced relative stiffness K_m and the plate parameters $\nu_{r\theta}$, β_0 and α_0 are shown in Table 1.

In the case of a uniformly distributed load, the procedure described in the preceding sections gives the following result for the input functions from equations (3.11)

$$\begin{aligned}
\overline{p}(x) &= 1 \qquad \lambda \leqslant x \leqslant 1\\
\varphi(y) &= \frac{1}{8}\Big(y^2 - 2\frac{1+3\nu_{r\theta}}{1+\nu_{r\theta}}\Big)y^2 + \frac{\lambda^2}{8}(8y^2-5\lambda^2) -\\
&- \lambda^2\Big(y^2+\lambda^2\frac{2+\nu_{r\theta}}{1+\nu_{r\theta}} - \frac{1+3\nu_{r\theta}}{2(1+\nu_{r\theta})}\Big)\ln y + \frac{1}{2}\lambda^4(1+4\ln y)\ln\lambda = \varphi_0(y)\\
T(y) &= \varphi(y) - \frac{1}{2}\beta_0 t\Big(y^2-\lambda^2+2\lambda^2\ln\frac{\lambda}{y}\Big) - \frac{1}{6}t^4\alpha_0
\end{aligned} \tag{4.4}$$

Table 1

Parameter			Magnesium	Cadmium	Graphite/epoxy	Isotropy
$\nu_{r\theta}$			0.3711	0.1163	0.0292	0.3000
β_0			1.1312	1.6996	22.0696	1.0000
α_0			0.9144	2.3729	8.1245	1.0000
K_m			375	604	1120	220
K_r	$\Gamma = 1$	$t = 0.1$	0.375	0.604	1.12	0.22
		$t = 0.4$	24	38.7	71.7	14.1
	$\Gamma = 2.5$	$t = 0.1$	0.550	0.886	1.644	0.323
		$t = 0.4$	35.2	56.7	105.2	20.7
	$\Gamma = 5$	$t = 0.1$	0.757	1.220	2.261	0.444
		$t = 0.4$	48.4	78.1	144.7	28.4

Below is shown (Fig. 2) a numerical result of the problem of a uniformly distributed load when the evaluation is done for two different material parameters K_r, the contact area is within $a = 1 \leqslant r \leqslant b = 2$, and it is divided into $N = 10$ segments. The presented curves are approximated for the values for the middle of each segment. The continuous line corresponds to $K_r = 0.604$, the dashed one – to $K_r = 0.02$.

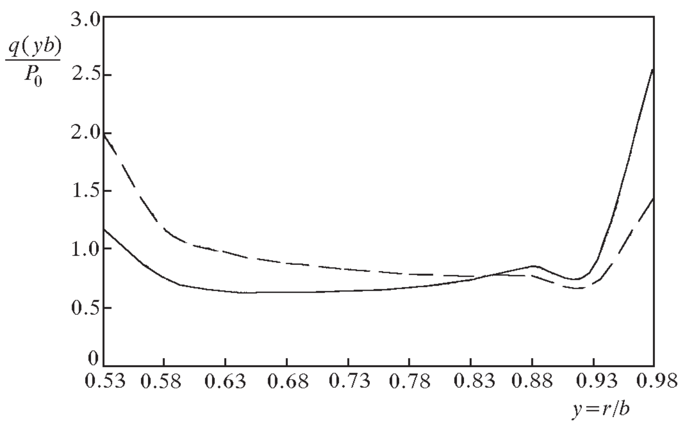

Fig. 2. Contact stresses $q(yb)/P_0$, $\lambda = a/b = 0.5$

Figures 2, 3, 4 show that the effect of the relative stiffness parameter K_r (anisotropy) on the physical quantities is strong. When this parameter is large, for instance when the plate is very rigid then the contact stresses are greater

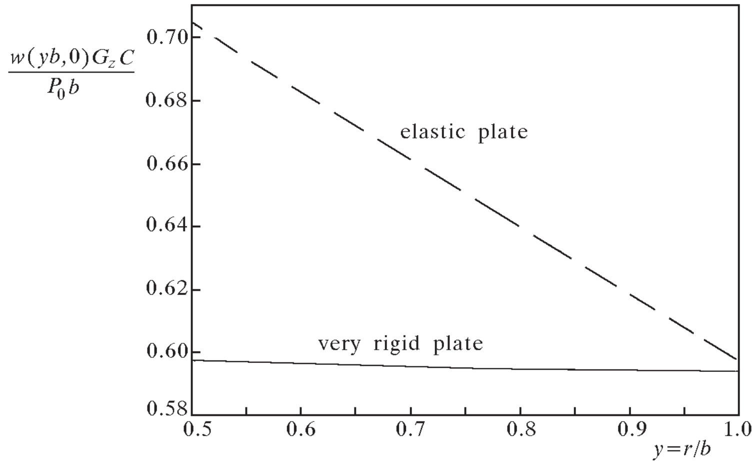

Fig. 3. Transverse deflection of the plate, $\lambda = a/b = 0.5$

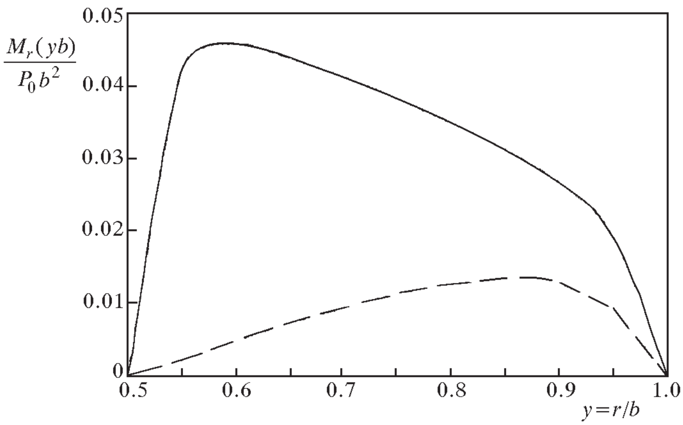

Fig. 4. Bending moment $M_r(yb)/(P_0 b^2)$, $\lambda = a/b = 0.5$

at the outer diameter than at the inner one the transverse deflection is almost constant and the moments have greater values than it is observed in plates made of much more elastic materials.

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Zagadnienie sprężystego oddziaływania między pierścieniową grubą płytą i sprężystym podłożem

Streszczenie

Otrzymano wzory dla naprężeń kontaktowych, momentów zginających i przemieszczeń osiowo symetrycznie obciążonej pierścieniowej płyty poprzecznie izotropowej, która kontaktuje się bez tarcia z poprzecznie izotropowym lub ziarnistym podłożem. Naprężenia kontaktowe nie wykazują osobliwości, dzięki temu, że uwzględniono odkształcenia wywołane ściskaniem płyty. Równanie całkowe zagadnienia rozwiązano w sposób przybliżony za pomocą efektywnego numerycznego algorytmu. Wyniki przedstawiono na wykresach.

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