JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
4, **39**, 2001

MICROPOLAR PLATES SUBJECT TO A NORMAL POLYHARMONIC LOADING

Grzegorz Jemielita

Institute of Structural Mechanics, Warsaw University of Technology
e-mail: gjemiel@il.pw.edu.pl

Following the previously published considerations the present paper aims at determination of a displacement vector and infinitesimal rotation vector describing the bending of the Grioli-Toupin plate subject to a normal polyharmonic loading. The presented biharmonic representation reduces the problem of equilibrium of such a plate to a non-homogeneous biharmonic equation involving a function of plate deflection. A semi-inverse method in an explicit form has been obtained together with relationships for force and moment stresses. Formulas for determination of the functions g_i and f_i of the variable ξ and coefficients A_i in a recurrent form have been given as well.

Key words: Grioli-Toupin material, micropolar plates, polyharmonic loading

1. Introduction

Exact distributions of displacements and streses in a plate subject to loadings normal to constraining planes were found within the symmetric theory of elasticity by making use of various methods described by Jemielita (1991). The problem was reduced to the determination of a solution to a non-homogeneous biharmonic equation.

The displacement and stress distribution corresponding to the non-homogeneous equation and satisfying the equations of the symmetric theory of elasticity was found in the case of constant loadings, see Garbedian (1925), Love (1927), Sokołowski (1958) and Lekhnitskiĭ (1963), the loadings being harmonic functions, see Gutman (1940), Stevenson (1942), Hata (1953) and Negoro (1954), polyharmonic ones (Jemielita, 1993), for constant mass forces (Gutman, 1941), and biharmonic (Dougall, 1904). Numerous methods of asymptotic expansions, such as power series, Legendre's polynomials infinite differential operators, Birkhoff's method, and semi-inverse method, see Jemielita (1991, 1993), were used for that purpose.

A generalized plane stress state (GPSS) in a plate made of the Grioli-Toupin material was defined by Jemielita (1992b). The bending problem in such a plate was reduced to a single biharmonic equation involving a function of deflection. In the paper by Jemielita (1992a) non-homogeneous flexure problems were considered in a plate under its own weight and normal uniform loading acting on its faces.

Now, we investigate a plate under the normal loading $q(x^\alpha)$ acting on the faces antisymmetrical with respect to the middle plane. We assume that the function $q(x^\alpha)$ satisfies the equation

$$\nabla^{2n+2}q(x^\alpha) = 0 \tag{1.1}$$

where n is an arbitrary natural number or zero.

The summation convention is adopted. Latin indices have the range 1, 2, 3, Greek indices - take the values 1, 2 only. Comma denote partial differentiation.

We introduce also the non-dimensional variable ξ

$$\xi = \frac{2z}{h} \tag{1.2}$$

where z is the coordinate normal to the middle plane of the plate. A derivative with respect to this variable we denote as $d(\cdot)/d\xi = (\cdot)'$.

The constitutive equations of an isotropic, homogeneous and centrosymmetric medium are assumed in the following form (Nowacki, 1970; Sokołowski, 1972)

$$\begin{aligned}
\sigma_{ij} &= \mu\Big(u_{j,i} + u_{i,j} + \frac{2\nu}{1-2\nu}u^k{}_{,k}\delta_{ij}\Big) - \frac{1}{2}\epsilon_{kji}\mu^{lk}{}_{,l} \\
\mu_{ij} &= \gamma(\varphi_{i,j} + \varphi_{j,i}) + \varepsilon(\varphi_{i,j} - \varphi_{j,i}) = 4\mu l^2(\varphi_{i,j} + \eta\varphi_{j,i})
\end{aligned} \tag{1.3}$$

where

$$\varphi_i = \frac{1}{2}\epsilon_i{}^{jk}u_{k,j} \qquad l^2 = \frac{\gamma+\varepsilon}{4\mu} \qquad \eta = \frac{\gamma-\varepsilon}{\gamma+\varepsilon} \tag{1.4}$$

and

σ_{ji} – stress tensor
μ_{ij} – couple-stress tensor
ϵ_{kji} – Levi-Civitá's permutation symbol
δ_{ij} – Kronecker's delta
u_i – components of the displacement vector

φ_i	–	components of the vector of infinitesimal rotations
μ	–	Lamé's constant
ν	–	Poisson's ratio
$\gamma, \varepsilon, l, \eta$	–	material constants of the Grioli-Toupin material.

Neglecting the body forces one can write the equations of equilibrium in the following form

$$\sigma^{ji}{}_{,j} = 0 \qquad \epsilon^{ijk}\sigma_{jk} + \mu^{ji}{}_{,j} = 0 \tag{1.5}$$

while the equilibrium equations expressed in terms of displacements can be cast as follows

$$\widetilde{\nabla}^2 u_i + \frac{1}{1-2\nu} u^k{}_{,ki} + l^2 \widetilde{\nabla}^2 (u^k{}_{,ki} - \widetilde{\nabla}^2 u_i) = 0 \tag{1.6}$$

where $\widetilde{\nabla}^2$ represents the Laplace operator in R^3

$$\widetilde{\nabla}^2 = \nabla^2 + \frac{\partial^2}{\partial z^2} = \nabla^2 + \frac{4}{h^2}\frac{\partial^2}{\partial \xi^2} \tag{1.7}$$

2. Biharmonic representation of the non-homogeneous problem

Let us investigate a plate of the thickness h subject to a normal loading acting on the faces $x^3 = z = \pm h/2$ and satisfying equation (1.1).

The solution to the differential equations (1.6) with the boundary conditions on the faces

$$\sigma_{3\alpha}\Big(x^\beta, \pm\frac{h}{2}\Big) = 0 \qquad \sigma_{33}\Big(x^\beta, \pm\frac{h}{2}\Big) = \pm\frac{q}{2} \qquad \mu_{3i}\Big(x^\beta, \pm\frac{h}{2}\Big) = 0 \tag{2.1}$$

will be sought with the help of the semi-inverse method.

Let us present the components of the displacement vector u_i and the components of the infinitesimal rotation vector φ_i in the form (Jemielita, 1992b)

$$\begin{aligned} u_\alpha(x^\beta, \xi) &= -\frac{h}{2}\xi w_{,\alpha}(x^\beta) + h^3 f(\xi)\nabla^2 w_{,\alpha}(x^\beta) + \frac{1}{\mu}\sum_{i=0}^{n} h^{2i+1} f_i(\xi)\nabla^{2i} q_{,\alpha} \\ u_3(x^\beta, \xi) &= w(x^\beta) + h^2 g(\xi)\nabla^2 w(x^\beta) + \frac{1}{\mu}\sum_{i=0}^{n} h^{2i+1} g_i(\xi)\nabla^{2i} q \end{aligned} \tag{2.2}$$

$$\varphi_\alpha(x^\gamma,\xi) = \epsilon_\alpha{}^\beta\Big[w_{,\beta} + h^2 t(\xi)\nabla^2 w_{,\beta} + \frac{1}{2\mu}\sum_{i=0}^{n} h^{2i+1}\Big(g_i(\xi) - 2f_i'(\xi)\Big)\nabla^{2i} q_{,\beta}\Big] \tag{2.3}$$

$$\varphi_3 = 0$$

where $f_i(\xi)$ and $g_i(\xi)$ are the unknown functions. They satisfy the conditions

$$f_i(\xi) = -f_i(-\xi) \qquad\qquad g_i(\xi) = g_i(-\xi) \tag{2.4}$$

The functions $s(\xi)$, $f(\xi)$ and $t(\xi)$ are defined by (see Jemielita, 1992b)

$$\begin{aligned} f(\xi) &= -\frac{2-\nu}{48(1-\nu)}\xi(C_2-\xi^2) - k^2\frac{\sinh(\widehat{k}\xi)}{\sinh\widehat{k}} \\ g(\xi) &= -\frac{1}{24(1-\nu)}\Big[6\Big(1-\frac{\nu}{2}\xi^2\Big) - (2-\nu)C_2\Big] \\ t(\xi) &= \frac{1}{2}(g-2f') = \\ &= \frac{1}{24(1-\nu)}\Big[(2-\nu)C_2 - 3 - 3(1-\nu)\xi^2 + 12k(1-\nu)\frac{\cosh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big] \end{aligned} \tag{2.5}$$

where

$$k = \frac{1}{h} \qquad\qquad \widehat{k} = \frac{1}{2k}$$

The constant C_2 determines the physical meaning of $w(x^\alpha)$. In the theories of flexural plates the following functions representing the plate deflection are used:

— deflection of the plate faces $\widehat{w}(x^\alpha)$

$$\widehat{w}(x^\alpha) \stackrel{def}{=} u_3\Big(x^\alpha, \pm\frac{h}{2}\Big) \tag{2.6}$$

— deflection of the mid-plane $\overset{\diamondsuit}{w}(x^\alpha)$

$$\overset{\diamondsuit}{w}(x^\alpha) \stackrel{def}{=} u_3(x^\alpha, 0) \tag{2.7}$$

— simple average $\overset{*}{w}(x^\alpha)$

$$\overset{*}{w}(x^\alpha) \stackrel{def}{=} \frac{1}{h}\int\limits_{-h/2}^{h/2} u_3(x^\alpha, z)\, dz \tag{2.8}$$

— weighted average $\overset{\circ}{w}(x^\alpha)$

$$\overset{\circ}{w}(x^\alpha) \overset{def}{=} \frac{3}{2h} \int\limits_{-h/2}^{h/2} \Big(1 - 4\frac{z^2}{h^2}\Big) u_3(x^\alpha, z)\, dz \tag{2.9}$$

Then using the second equation of (2.2) and definitions (2.6)-(2.9) we obtain the following values of the constant C_2

— for $\widehat{w}(x^\alpha)$

$$C_2 = 3 \qquad\qquad g_i(\pm 1) = 0 \tag{2.10}$$

— for $\overset{\diamond}{w}(x^\alpha)$

$$C_2 = \frac{6}{2-\nu} \qquad\qquad g_i(0) = 0 \tag{2.11}$$

— for $\overset{*}{w}(x^\alpha)$

$$C_2 = \frac{6-\nu}{2-\nu} \qquad\qquad \int\limits_{-1}^{1} g_i(\xi)\, d\xi = 0 \tag{2.12}$$

— for $\overset{\circ}{w}(x^\alpha)$

$$C_2 = \frac{3(10-\nu)}{5(2-\nu)} \qquad\qquad \int\limits_{-1}^{1} (1-\xi^2) g_i(\xi)\, d\xi = 0 \tag{2.13}$$

It can be clearly seen that, depending on the definition of deflection, the formulae for displacements and stresses assume different forms. We assume that the function $w(x^\alpha)$ satisfies the equation

$$\mathcal{D}\nabla^4 w = \sum_{i=0}^{n} h^{2i} A_i \nabla^{2i} q \tag{2.14}$$

where $\mathcal{D}$ is the rigidity of the micropolar plate determined by

$$\mathcal{D} = D\theta \qquad\qquad \theta = 1 + 24(1-\nu)k^2 \qquad\qquad D = \frac{\mu h^3}{6(1-\nu)} \tag{2.15}$$

and A_i are some unknown constants.

Using Eqs (1.3), (2.2) and (2.3) we arrive at the following formulae for the stresses and couple stresses

$$
\begin{aligned}
\sigma_{\alpha\beta} &= -\frac{2\mu}{1-\nu}\Big\{\frac{h}{2}\xi\Big((1-\nu)w_{,\alpha\beta}+\nu\nabla^2 w\delta_{\alpha\beta}\Big)+\\
&+\frac{h^3}{48}\Big[(2-\nu)\xi(C_2-\xi^2)+48(1-\nu)k^2\frac{\sinh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big]\nabla^2 w_{,\alpha\beta}\Big\}+\\
&+\frac{2}{1-2\nu}\sum_{i=0}^{n}h^{2i}\Big\{h^2(1-2\nu)f_i\nabla^{2i}q_{,\alpha\beta}+\nu\Big[f_{i-1}+2g_i'+\frac{6(1-\nu)}{\theta}s\Big]\nabla^{2i}q\delta_{\alpha\beta}\Big\}\\
\sigma_{\alpha 3} &= -\frac{\mu h^2}{4(1-\nu)}\Big[1-\xi^2+8(1-\nu)k\frac{\cosh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big]\nabla^2 w_{,\alpha}+\\
&+\sum_{i=0}^{n}h^{2i+1}\Big\{g_i+2f_i'-k^2\Big[g_{i-1}-2f_{i-1}'+4(g_i''-2f_i''')+\frac{12(1-\nu)}{\theta}tA_i\Big]\Big\}\nabla^{2i}q_{,\alpha}\\
\sigma_{3\alpha} &= -\frac{\mu h^2}{4(1-\nu)}(1-\xi^2)\nabla^2 w_{,\alpha}+\\
&+\sum_{i=0}^{n}h^{2i+1}\Big\{g_i+2f_i'+k^2\Big[g_{i-1}-2f_{i-1}'+4(g_i''-2f_i''')+\frac{12(1-\nu)}{\theta}tA_i\Big]\Big\}\nabla^{2i}q_{,\alpha}\\
\sigma_{33} &= \frac{2}{1-2\nu}\sum_{i=0}^{n}h^{2i}\Big[2(1-\nu)g_i'+\nu\Big(f_{i-1}+\frac{6(1-\nu)}{\theta}sA_i\Big)\Big]\nabla^{2i}q\\
\mu_{\alpha\beta} &= 4\mu h^2k^2\Big[\epsilon_\beta{}^\gamma(w_{,\gamma\alpha}+h^2t\nabla^2 w_{,\gamma\alpha})+\eta\epsilon_\alpha{}^\gamma(w_{,\gamma\beta}+h^2t\nabla^2 w_{,\gamma\beta})\Big]+\\
&+2h^2k^2\Big[\epsilon_\beta{}^\gamma\sum_{i=0}^{n}h^{2i+1}(g_i-2f_i')\nabla^{2i}q_{,\alpha\gamma}+\eta\epsilon_\alpha{}^\gamma\sum_{i=0}^{n}h^{2i+1}(g_i-2f_i')\nabla^{2i}q_{,\beta\gamma}\Big]\\
\mu_{3\alpha} &= -2\mu k^2h^3\epsilon_\alpha{}^\beta\Big[\mu\Big(\xi-\frac{\sinh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big)\nabla^2 w_{,\beta}-2\sum_{i=0}^{n}h^{2i-1}(g_i'-2f_i'')\nabla^2 q_{,\beta}\Big]\\
\mu_{\alpha 3} &= \eta\mu_{3\alpha} \qquad\qquad \mu_{33}=0
\end{aligned}
\tag{2.16}
$$

3. Functions f_i, g_i and coefficients A_i

Substituting (2.2) and (2.3) into Eqs (1.6), we find nonzero solutions to this set when the function $w(x^\alpha)$ satisfies Eq. (2.14) and the functions $f_i(\xi)$

and $g_i(\xi)$ satisfy the following simple ordinary differential equations

$$
\begin{aligned}
&4k^2 f_i^{IV} - f_i'' = F(\xi)\\
&g_i'' = -\frac{1-2\nu}{8(1-\nu)}g_{i-1} - \frac{1}{4(1-\nu)}f_{i-1}' -\\
&\quad -\frac{3}{4\theta}\Big[(1-2\nu)g + 2f' - (1-2\nu)k\frac{\cosh(\widehat{k}\xi)}{\cosh\widehat{k}}\Big]A_i +\\
&\quad +\frac{1-2\nu}{8(1-\nu)}k^2\Big[g_{i-2} - 2f_{i-2}' + 4(g_{i-1}'' - 2f_{i-1}''') + \frac{12(1-\nu)}{\theta}tA_{i-1}\Big]
\end{aligned}
\tag{3.1}
$$

where $i = 0, 1, ..., n$ and

$$
\begin{aligned}
F(\xi) &= \frac{1}{2(1-2\nu)}\Big[(1-\nu)f_{i-1} + g_i' + \frac{6(1-\nu)^2}{\theta}fA_i\Big] +\\
&\quad +\frac{k^2}{2}\Big[g_{i-1}' + 4g_i''' - 2f_{i-1}'' - \frac{3(1-\nu)}{\theta}\Big(\xi - \frac{\sinh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big)A_i\Big]\\
g_i(\xi) &= f_i(\xi) = 0 \qquad \text{for} \quad i < 0
\end{aligned}
$$

Boundary conditions (2.1) can be rewritten as follows

$$
\begin{aligned}
&g_i(1) + 2f_i'(1) + k^2\Big[g_{i-1}(1) - 2f_{i-1}'(1) + 4\Big(g_i''(1) - 2f_i'''(1)\Big) +\\
&+\frac{12(1-\nu)}{\theta}t(1)A_i\Big] = 0\\
&2(1-\nu)g_i'(1) + \nu\Big[f_{i-1}(1) + \frac{12(1-\nu)}{\theta}f(1)A_i\Big] = \frac{1-2\nu}{4}\delta_{0i}\\
&g_i'(1) - 2f_{i-1}''(1) = 0 \qquad i = 0, 1, ..., n
\end{aligned}
\tag{3.2}
$$

A definite integral of the third equation of the first of the equilibrium ones (1.5) within the limits 0, 1 combined with boundary conditions given by Eqs $(2.1)_2$ and $(3.2)_3$, yields the following formula for A_i

$$
\begin{aligned}
A_i &= \delta_{0i} + \int_0^1 g_{i-1}\,d\xi + 2f_{i-1}(1) -\\
&- k^2\Big[\int_0^1 g_{i-2}\,d\xi - 2f_{i-2}(1) + \frac{12(1-\nu)}{\theta}A_{i-1}\int_0^1 t\,d\xi\Big]
\end{aligned}
\tag{3.3}
$$

The solution to system (3.1) can be written in the following form

$$
\begin{aligned}
g_i &= -\frac{1-2\nu}{8(1-\nu)}\int\limits_0^{\xi}\int\limits_0^{\xi} g_{i-1}\,d\xi d\xi - \frac{1}{4(1-\nu)}\int\limits_0^{\xi} f_{i-1}\,d\xi - \\
&- \frac{3}{4}\Big[(1-2\nu)\int\limits_0^{\xi}\int\limits_0^{\xi} g\,d\xi d\xi + 2\int\limits_0^{\xi} f\,d\xi - 4(1-2\nu)k^3\frac{\sinh(\widehat{k}\xi)}{\sinh\widehat{k}}\Big]\frac{A_i}{\theta} + \\
&+ k^2\frac{1-2\nu}{8(1-\nu)}\Big[\int\limits_0^{\xi}\int\limits_0^{\xi} g_{i-1}\,d\xi d\xi - 2\int\limits_0^{\xi} f_{i-2}\,d\xi + 4(g_{i-1}-2f'_{i-1}) + \\
&+ \frac{12(1-\nu)}{\theta}A_{i-1}\int\limits_0^{\xi}\int\limits_0^{\xi} t\,d\xi d\xi\Big] + C_i
\end{aligned}
\tag{3.4}
$$

$$
f_i = B_{1i}\widehat{k}\xi + B_{2i}\sinh(\widehat{k}\xi) - \int\limits_0^{\xi}\Big[\widehat{k}(\xi-\psi) - \sinh\Big(\widehat{k}(\xi-\psi)\Big)\Big]F(\psi)d\psi
$$

The constants B_{1i}, B_{2i} can be determined from boundary conditions $(3.2)_{1,3}$, while the coefficients C_i result from conditions (2.10)-(2.13), depending on the considered deflection. The obtained recurrent formulae (3.3) and (3.4) allow one to determine all the sought-after functions f_i, g_i and coefficients A_i.

Up till now, equations (2.2), (2-3), and (3.4) have not been reported in the literature in such a general form. These equations allow for an accurate representation of displacement vector (2.2) for a variety of deflections $w(x^\alpha)$ leading to a solution to non-homogeneous equation (2.14). The exact values of the coefficients A_i, appearing in this equation can be obtained explicitly from Eq. (3.3).

4. Plate made of the Hooke material

Stresses and displacements occurring in plate made of the Hooke material can be obtained from the limit passage as $l \to 0$. Calculating the limiting values of Eqs (2.2), (2.5), (2.16), (3.3) and (3.4) as $k \to 0$ we arrive at the formulae for displacements, stress and coefficients A_i given by Jemielita (1993).

References

1. Dougall J., 1904, An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate, *Trans. R. Soc. Edinburgh*, **41**, Part 1, 8, 129-228
2. Garabedian C.A., 1925, Solution du probléme de la plaque rectangulaire épaisse encastrée ou posée, portant une charge uniformément répartie ou concentrée en son centre, *C. R. Acad. Sci.*, **180**, 257-259, (Errata p.1191)
3. Gutman S.G., 1940, Raschet tolstykh uprugikh plit pod nepreryvno raspredelennym davleniem, *Izv. Nauch. Issl. Inst. Gidrotekh.*, **28**, 212-238
4. Gutman S.G., 1941, Raschet tolstykh uprugikh plit pod deĭstviem sobstvennogo vesa, *Izv. Nauch. Issl. Inst. Gidrotekh.*, **29**, 153-156
5. Hata K.-I., 1953, On the Thick Plate Problem I, *Mem. Fac. Eng. Hokkaido Univ.*, **9**, 3, 428-477
6. Jemielita G., 1991, Meandry teorii płyt, *Prace Naukowe Politechniki Warszawskiej. Budownictwo*, **117**, Warszawa
7. Jemielita G., 1992a, Bending of a Cosserat Plate Under its Own Weight and Normal Uniform Loading, *J. Theor. Appl. Mech.*, **30**, 2, 369-377
8. Jemielita G., 1992b, Biharmonic Representation in the Analysis of Plates Made of the Grioli-Toupin Material, *J. Theor. Appl. Mech.*, **30**, 1, 9-99
9. Jemielita G., 1993, Ścisłe równania teorii płyt i tarcz, *Prace Naukowe Politechniki Warszawskiej. Budownictwo*, **24**, Warszawa
10. Lekhnitskij S.G., 1963, Ploskoe napryazhennoe sostoyanie i izgib neodnorodnoĭ transversal'no-izotropnoĭ plity, *Izv. AN SSSR OTN 1*, 61-67
11. Love A.E.H., 1927, *A Treatise on the Mathematical Theory of Elasticity*, 4 edit., Oxford
12. Negoro S., 1954, On a Method of Solving Elastic Problems of Plates, *Proc. 4th Japan Nat. Congr. Appl. Mech.*, 153-156
13. Nowacki W., 1970, *Teoria sprężystości*, PWN Warszawa
14. Sokołowski M., 1958, The Bending of Transversally Non-Homogeneous Plates of Moderate Thickness, *Arch. Mech. Stos.*, **10**, 3, 315-328; *Bull. Acad. Pol. Sci. Sér. Sci. Tech.*, **6**, 4, 181-186
15. Sokołowski M., 1972, *O teorii naprężeń momentowych w ośrodkach ze związanymi obrotami*, PWN Warszawa
16. Stevenson A.C., 1942, On the Equilibrium of Plates, *Phil. Mag.*, Ser. 7, **33**, 224, 639-661

Poliharmoniczne obciążenia płyt mikropolarnych

Streszczenie

W niniejszej pracy (korzystając z wcześniej opublikowanych rozważań) wyznaczono przedstawienie wektora przemieszczenia i infinitezymalnego obrotu, opisujące zginanie płyty wykonanej z materiału Grioli-Toupina wywołane obciążeniem normalnym poliharmonicznym. Przedstawiona reprezentacja biharmoniczna sprowadza zagadnienie równowagi takiej płyty do rozwiązania niejednorodnego równania biharmonicznego na funkcję przedstawiającą ugięcie płyty. Metodą półodwrotną uzyskano wzory na naprężenia siłowe i momentowe w postaci jawnej. Podano też, w postaci rekurencyjnej, wzory na poszukiwane funkcje g_i, f_i zmiennej ξ oraz współczynników A_i.

Manuscript received November 22, 2000; accepted for print February 5, 2001