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WHITE NOISE EXCITED VIBRATIONS OF VISCOELASTIC SHALLOW SHELLS

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The paper presents the results of the analysis of randomly excited vibrations of viscoelastic shallow shells. The parameter of interest is the dispersion of velocity normal to the element surface. The choice of parameter is motivated by estimation of the noise generated by a vibrating panel.

Key words: shallow shells, vibrations, viscoelasticity, white noise

1. Introduction

Shallow shells are commonly used as constructional elements in cages of heavy duty machines. These elements, during the machine operation, are the source of structural noise inside the cages. To estimate the total acoustic power radiated by surface elements, the value of second power of velocity normal to the middle surface and averaged over the time and space (the area of element), can be used Beranek (1988). The study of the effect of the curvature radii of shallow shells on the value of averaged second power of velocity, made by the authors for harmonic excitation has shown that this influence is strong. This analysis was carried out with no internal damping of the material taken into account.

In the present paper internal damping of the Voigt-Kelvin type is assumed and the analysis is conducted for random external force excitations. In the analysis the white noise signal is used.

2. Mathematical model of a vibrating shallow shell

Let us consider vibrations of shallow shells with internal damping. A shallow shell is after Mazurkiewicz and Nagórski (1991), a shell for which at each point of the middle surface the angle between the plane tangent to this surface and the plane of its horizontal projections is suitably small. The geometry of shallow shell is shown in Fig.1.

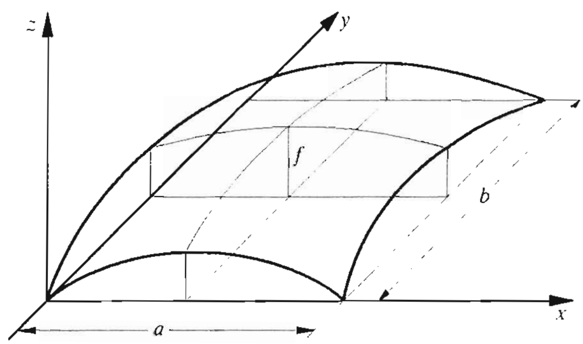

Fig. 1. Geometry of shallow shell (after Kolkunov, 1973)

The Voigt-Kelvin model is used to describe damping in the material. In this case, the total stress components consist of the two parts: elastic ones (e), and damping ones (d). The damping in the material is represented by the value of the viscous damping coefficient ε.

It was assumed in the earlier analysis, that the elastic properties of the material are described by the values of Young modulus E and Poisson ratio ν (isotropic and homogeneous material). In the present paper it is assumed, that the value of damping coefficient ε does not depend on the direction, and is given by a constant material parameter. This material is called a simple Voigt-Kelvin material

$$\sigma_x = \sigma_x^e + \sigma_x^d \qquad \sigma_y = \sigma_y^e + \sigma_y^d \qquad \tau_{xy} = \tau_{xy}^e + \tau_{xy}^d \tag{2.1}$$

In the case of shallow shell made of a simple Voigt-Kelvin material, physical equations have the form (Xia and Łukasiewicz, 1995)

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y + \dot{\varepsilon}_x + \nu\dot{\varepsilon}_y) \qquad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x + \dot{\varepsilon}_y + \nu\dot{\varepsilon}_x) \tag{2.2}$$

$$\tau_{xy} = \frac{E}{2(1+\nu^2)}(\gamma_{xy} + \nu\dot{\gamma}_{xy})$$

The set of equations of motion of the shallow shell with the geometry shown in Fig.1, written for unknown displacements u, v, w, and positive radii of curvatures R_x and R_y, has in general the form of Eqs (2.3). The analogous set of equations without damping was given e.g. by Waszczyszyn (cf Rakowski, 1982; Waszczyszyn, 1995), and was applied in our previous studies (cf Kozień, 1996; Kozień and Nizioł, 1993a,b)

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial}{\partial x}\Big(\frac{w}{R_x} + \nu\frac{w}{R_y}\Big) + \varepsilon\frac{\partial^2 \dot{u}}{\partial x^2} + \\
&+\varepsilon\frac{1-\nu}{2}\frac{\partial^2 \dot{u}}{\partial y^2} + \varepsilon\frac{1+\nu}{2}\frac{\partial^2 \dot{v}}{\partial x \partial y} + \varepsilon\frac{\partial}{\partial x}\Big(\frac{\dot{w}}{R_x} + \nu\frac{\dot{w}}{R_y}\Big) - \frac{\rho h}{B}\ddot{u} = -\frac{1}{B}X(x,y,t) \\
&\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial}{\partial y}\Big(\frac{w}{R_y} + \nu\frac{w}{R_x}\Big) + \varepsilon\frac{\partial^2 \dot{v}}{\partial y^2} + \\
&+\varepsilon\frac{1-\nu}{2}\frac{\partial^2 \dot{v}}{\partial x^2} + \varepsilon\frac{1+\nu}{2}\frac{\partial^2 \dot{u}}{\partial x \partial y} + \varepsilon\frac{\partial}{\partial y}\Big(\frac{\dot{w}}{R_y} + \nu\frac{\dot{w}}{R_x}\Big) - \frac{\rho h}{B}\ddot{v} = -\frac{1}{B}Y(x,y,t) \qquad (2.3) \\
&\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{12}{h^2}\Big(\frac{1}{R_x} + \frac{\nu}{R_y}\Big)\frac{\partial u}{\partial x} + \frac{12}{h^2}\Big(\frac{1}{R_y} + \frac{\nu}{R_x}\Big)\frac{\partial v}{\partial y} + \\
&+\frac{12}{h^2}\Big(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{2\nu}{R_x R_y}\Big)w + \varepsilon\frac{\partial^4 \dot{w}}{\partial x^4} + 2\varepsilon\frac{\partial^4 \dot{w}}{\partial x^2 \partial y^2} + \varepsilon\frac{\partial^4 \dot{w}}{\partial y^4} + \\
&+\frac{12}{h^2}\varepsilon\Big(\frac{1}{R_x} + \frac{\nu}{R_y}\Big)\frac{\partial \dot{u}}{\partial x} + \frac{12}{h^2}\varepsilon\Big(\frac{1}{R_y} + \frac{\nu}{R_x}\Big)\frac{\partial \dot{v}}{\partial y} + \\
&+\frac{12}{h^2}\varepsilon\Big(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{2\nu}{R_x R_y}\Big)\dot{w} + \frac{\rho h}{D}\ddot{w} = \frac{1}{D}Z(x,y,t)
\end{aligned}$$

where

u, v, w	–	displacements, in the directions $0x$, $0y$, $0z$, respectively
h	–	thickness
R_x, R_y	–	radii of main curvatures of the middle surface
E	–	Young modulus
ν	–	Poisson ratio
D	–	bending siffeness, $D = Eh^3/[12(1-\nu^2)]$
B	–	in-plane stiffeness, $B = Eh/(1-\nu^2)$
ε	–	viscous damping coefficient
ρ	–	material density
X, Y, Z	–	external non-inertial loads, in the directions $0x$, $0y$, $0z$, respectively.

The equations of motion of the vibrating surface elements have in general a coupled form. Often, during analysis, the inertial components which represent the motion in the tangential directions of the middle surface are neglected. In such a case the amplitudes of vibrations in the tangential directions are calculated basing on those for bending vibrations. This idea is commonly used for the dynamic analysis of shallow shells (see e.g. Kolkunov, 1973; Nath et al., 1987; Nowacki, 1972). The motivation for this approach is the following. The basic eigenfrequency of bending vibrations decreases with decreasing thickness of the element, whereas the same parameter of vibrations in the tangential directions does not depend on the thickness. So, usually the eigenfrequencies of in-plane vibrations are significantly higher then for the bending ones (Kolkunov, 1973). Therefore, it is assumed in the paper, that the velocity components in the first two equations of motion can be neglected too. Moreover, it is assumed that only the excitation in the direction normal to the middle surface ($0z$-direction) is taken into acoount. This is a function of time only (random $f(t)$) with a constant amplitude q_0. On these assumptions, the equations of motion (2.3) are simplified to the form

$$
\begin{gathered}
\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial}{\partial x}\Big(\frac{w}{R_x} + \nu\frac{w}{R_y}\Big) = 0 \\
\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial}{\partial y}\Big(\frac{w}{R_y} + \nu\frac{w}{R_x}\Big) = 0 \\
\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{12}{h^2}\underbrace{\Big(\frac{1}{R_x} + \frac{\nu}{R_y}\Big)}_{\kappa_u}\frac{\partial u}{\partial x} + \frac{12}{h^2}\underbrace{\Big(\frac{1}{R_y} + \frac{\nu}{R_x}\Big)}_{\kappa_v}\frac{\partial v}{\partial y} + \\
+\frac{12}{h^2}\underbrace{\Big(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{2\nu}{R_x R_y}\Big)}_{\kappa^2} w + \varepsilon\frac{\partial^4 \dot{w}}{\partial x^4} + 2\varepsilon\frac{\partial^4 \dot{w}}{\partial x^2 \partial y^2} + \varepsilon\frac{\partial^4 \dot{w}}{\partial y^4} + \\
+\frac{12}{h^2}\varepsilon\underbrace{\Big(\frac{1}{R_x} + \frac{\nu}{R_y}\Big)}_{\kappa_u}\frac{\partial \dot{u}}{\partial x} + \frac{12}{h^2}\varepsilon\underbrace{\Big(\frac{1}{R_y} + \frac{\nu}{R_x}\Big)}_{\kappa_v}\frac{\partial \dot{v}}{\partial y} + \\
+\frac{12}{h^2}\varepsilon\underbrace{\Big(\frac{1}{R_x^2} + \frac{1}{R_y^2} + \frac{2\nu}{R_x R_y}\Big)}_{\kappa^2}\dot{w} + \frac{\rho h}{D}\ddot{w} = \frac{1}{D}q_0 f(t)
\end{gathered}
\tag{2.4}
$$

The simply supported–free–fixed type of boundary conditions are chosen. By this we mean the simply supported case of bending vibrations

$$
w(x,y,t)\Big|_{x=0} = 0 \qquad M_x(x,y,t)\Big|_{x=0} = -D\frac{\partial^2 w(x,y,t)}{\partial x^2}\Big|_{x=0} = 0
$$

$$
\begin{aligned}
&w(x,y,t)\Big|_{x=a} = 0 && M_x(x,y,t)\Big|_{x=a} = -D\frac{\partial^2 w(x,y,t)}{\partial x^2}\Big|_{x=a} = 0 \\
&w(x,y,t)\Big|_{y=0} = 0 && M_y(x,y,t)\Big|_{y=0} = -D\frac{\partial^2 w(x,y,t)}{\partial y^2}\Big|_{y=0} = 0 \\
&w(x,y,t)\Big|_{y=b} = 0 && M_y(x,y,t)\Big|_{y=b} = -D\frac{\partial^2 w(x,y,t)}{\partial y^2}\Big|_{y=b} = 0
\end{aligned}
\tag{2.5}
$$

and the mixed free-fixed one of in-plane vibrations

$$
\begin{aligned}
&v(x,y,t)\Big|_{x=0} = 0 && N_x(x,y,t)\Big|_{x=0} = B\Big[\frac{\partial u(x,y,t)}{\partial x} + \nu\frac{\partial v(x,y,t)}{\partial y}\Big]\Big|_{x=0} = 0 \\
&v(x,y,t)\Big|_{x=a} = 0 && N_x(x,y,t)\Big|_{x=a} = B\Big[\frac{\partial u(x,y,t)}{\partial x} + \nu\frac{\partial v(x,y,t)}{\partial y}\Big]\Big|_{x=a} = 0 \\
&u(x,y,t)\Big|_{y=0} = 0 && N_y(x,y,t)\Big|_{y=0} = B\Big[\frac{\partial v(x,y,t)}{\partial y} + \nu\frac{\partial u(x,y,t)}{\partial x}\Big]\Big|_{y=0} = 0 \\
&u(x,y,t)\Big|_{y=b} = 0 && N_y(x,y,t)\Big|_{y=b} = B\Big[\frac{\partial v(x,y,t)}{\partial y} + \nu\frac{\partial u(x,y,t)}{\partial x}\Big]\Big|_{y=b} = 0
\end{aligned}
\tag{2.6}
$$

3. Method of analysis

3.1. Free vibrations

The solution to the problem of free vibrations $Z(x,y,t) = 0$, with unknown displacements $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$, can be written in a series form with unknown amplitudes $A_{mn}(t)$, $B_{mn}(t)$ and $T_{mn}(t)$, which are functions of time only, multiplied by functions of space variables (x,y) (eigenfunctions) only

$$
\begin{aligned}
u(x,y,t) &= \sum_{m,n=1}^{\infty} A_{mn}(t)\cos(\lambda_m x)\sin(\mu_n y) \\
v(x,y,t) &= \sum_{m,n=1}^{\infty} B_{mn}(t)\sin(\lambda_m x)\cos(\mu_n y) \\
w(x,y,t) &= \sum_{m,n=1}^{\infty} T_{mn}(t)\sin(\lambda_m x)\sin(\mu_n y) \\
\lambda_m &= \frac{m\pi}{a} \qquad \mu_n = \frac{n\pi}{b}
\end{aligned}
\tag{3.1}
$$

Substituting the proposed solutions (3.1) into the first two of Eqs (2.4), the functions $A_{mn}(t)$ and $B_{mn}(t)$ can be obtained

$$
\begin{aligned}
A_{mn}(t) &= \underbrace{\frac{2}{1-\nu}\lambda_m \frac{\kappa_u\Big(\mu_n^2 + \frac{1-\nu}{2}\lambda_m^2\Big) - \frac{1+\nu}{2}\kappa_v\mu_n^2}{(\lambda_m^2+\mu_n^2)^2}}_{a_{mn}} T_{mn}(t) \\
B_{mn}(t) &= \underbrace{\frac{2}{1-\nu}\mu_n \frac{\kappa_v\Big(\lambda_m^2 + \frac{1-\nu}{2}\mu_n^2\Big) - \frac{1+\nu}{2}\kappa_u\lambda_m^2}{(\lambda_m^2+\mu_n^2)^2}}_{b_{mn}} T_{mn}(t)
\end{aligned}
\tag{3.2}
$$

In the next step, the discussed functions are introduced into the third of Eqs (2.4). Thus, the differential equation of shell vibration, with unknown amplitudes T_{mn} has the follows

$$
\begin{aligned}
&\frac{\rho h}{D}\ddot{T}_{mn}(t) + \varepsilon \underbrace{\Big[(\lambda_m^2+\mu_n^2)^2 - a_{mn}\frac{12}{h^2}\kappa_u\lambda_m - b_{mn}\frac{12}{h^2}\kappa_v\mu_n + \frac{12}{h^2}\kappa\Big]}_{c_{mn}}\dot{T}_{mn}(t) + \\
&+ \underbrace{\Big[(\lambda_m^2+\mu_n^2)^2 - a_{mn}\frac{12}{h^2}\kappa_u\lambda_m - b_{mn}\frac{12}{h^2}\kappa_v\mu_n + \frac{12}{h^2}\kappa\Big]}_{c_{mn}} T_{mn}(t) = 0
\end{aligned}
\tag{3.3}
$$

Eq (3.3), after some transformations, can be rewritten in the form, which includes the parameters commonly used in dynamics; namely, eigenfrequency of undamped vibrations ω_{mn} and dimensionless damping coefficient ζ_{mn}

$$
\begin{aligned}
&\ddot{T}_{mn}(t) + 2\omega_{mn}\zeta_{mn}\dot{T}_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = 0 \\
&\omega_{mn}^2 = \frac{D}{\rho h}c_{mn} \\
&\zeta_{mn} = \frac{1}{2}\varepsilon\sqrt{\frac{D}{\rho h}c_{mn}}
\end{aligned}
\tag{3.4}
$$

The solution of Eqs (3.4), has one of the forms given in Eq (3.5), depending on whether the value of damping coefficients ζ_{mn} is smaller or greater than unity

— for $\zeta_{mn} < 1$

$$
\begin{aligned}
\overline{\omega}_{mn} &= \omega_{mn}\sqrt{1-\zeta_{mn}^2} \\
T_{mn}(t) &= L_{mn}\mathrm{e}^{-\omega_{mn}\zeta_{mn}t}\sin(\overline{\omega}_{mn}t)
\end{aligned}
\tag{3.5}
$$

— for $\zeta_{mn} > 1$

$$\begin{aligned}\widetilde{\omega}_{mn} &= \omega_{mn}\sqrt{\zeta_{mn}^2 - 1} \\ T_{mn}(t) &= L_{mn}\mathrm{e}^{-\omega_{mn}\zeta_{mn}t}\sinh(\widetilde{\omega}_{mn}t)\end{aligned} \tag{3.6}$$

3.2. Forced vibrations

A time-dependent external excitation in the form of function acting in the $0z$ direction, which does not depend on the spatial coordinates, is considered. According to the third of Eqs (2.4) it has the form $Z(x,y,t) = f(t)q_0/D$, where q_0 is a constant amplitude.

Let us propose the solution to the problem in the series form, with an unknown function $S_{mn}(t)$

$$w(x,y,t) = \sum_{m,n=1}^{+\infty} S_{mn}(t)\underbrace{\sin(\lambda_m x)\sin(\mu_n y)}_{w_{mn}(x,y)} \tag{3.7}$$

The external excitation function, can be written in the form of Fourier series (3.8), where functions $H_{mn}(t)$ are given by Eq (3.9). Due to the orthogonality of $w_{mn}(x,y)$, the form of $H_{mn}(t)$ is easily obtained

$$Z(x,y,t) = \frac{q_0}{D}f(t) = \sum_{m,n=1}^{+\infty} H_{mn}(t)w_{mn}(x,y) \tag{3.8}$$

$$H_{mn}(t) = \frac{16}{\pi^2}\frac{q_0}{D}\frac{1}{mn}f(t) \qquad m,n = 1,3,5,7,\dots \tag{3.9}$$

The differential equation for unknown functions $S_{mn}(t)$ has the form (3.10), and its solution is given by Eq (3.11), which depends on whether the value of damping coefficient ζ_{mn} is smaller or greater than unity, see Eqs (3.12) and (3.13)

$$\ddot{S}_{mn}(t) + 2\omega_{mn}\zeta_{mn}\dot{S}_{mn}(t) + \omega_{mn}^2 S_{mn}(t) = \frac{16}{\pi^2}\frac{q_0}{\rho h}\frac{1}{mn}f(t) \tag{3.10}$$

$$S_{mn}(t) = \frac{16}{\pi^2}\frac{q_0}{\rho h}\frac{1}{mn}\frac{1}{\underline{\omega}_{mn}}\int\limits_0^t h_{mn}(t-\tau)f(\tau)\,d\tau \tag{3.11}$$

and

— for $\zeta_{mn} < 1$

$$
\begin{aligned}
&\underline{\omega}_{mn} = \overline{\omega}_{mn} = \omega_{mn}\sqrt{1-\zeta_{mn}^2}\\
&h_{mn}(t-\tau) = \begin{cases} 0 & t<\tau \\ \mathrm{e}^{-\omega_{mn}\zeta_{mn}(t-\tau)}\sin[\overline{\omega}_{mn}(t-\tau)] & t \geqslant \tau \end{cases}
\end{aligned}
\tag{3.12}
$$

— for $\zeta_{mn} > 1$

$$
\begin{aligned}
&\underline{\omega}_{mn} = \widetilde{\omega}_{mn} = \omega_{mn}\sqrt{\zeta_{mn}^2-1}\\
&h_{mn}(t-\tau) = \begin{cases} 0 & t<\tau \\ \mathrm{e}^{-\omega_{mn}\zeta_{mn}(t-\tau)}\sinh[\widetilde{\omega}_{mn}(t-\tau)] & t \geqslant \tau \end{cases}
\end{aligned}
\tag{3.13}
$$

3.3. Random excitations

Assume that the excitation has the form $Z(x,y,t) = z_0 f(t)$ (see Eqs (2.4)), where $f(t)$ is a random stationary process with a known correlation function $K_{ff}(t_1,t_2) = K_{ff}(\tau)$; $\tau = t_2 - t_1$, and the mean value equal to zero $< f(t) >= 0$.

The solution to the problem is proposed in the form

$$
w(x,y,t) = \sum_{m,n=1,3,5,\dots}^{+\infty} w_{mn}(x,y)S_{mn}(t)
\tag{3.14}
$$

The correlation function of the process $w(x,y,t)$ takes the form

$$
\begin{aligned}
&K_{ww}(t_1,t_2) =\\
&= \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} w_{mn}(x,y)S_{mn}(t_1)\sum_{j,k=1}^{+\infty} w_{jk}(x,y)S_{mn}(t_2)p(z_1,z_2;t_1,t_2)\;dz_1dz_2
\end{aligned}
\tag{3.15}
$$

where $p(z_1,z_2;t_1,t_2)$ is the two-point distribution density of the random process $f(t)$.

After suitable averaging over a group of realizations one obtains the correlation function in the form

$$K_{ww}(t_1,t_2) = \frac{256}{\pi^4}\frac{z_0}{\rho^2 h^2} \sum_{m,n=1,3,5,\dots}^{+\infty} \frac{1}{mn} \sum_{j,k=1,3,5,\dots}^{+\infty} \frac{1}{jk} w_{mn}(x,y) w_{jk}(x,y) \cdot \tag{3.16}$$

$$\cdot \int\limits_0^{t_1} h_{mn}(t_1-\tau_1) \int\limits_0^{t_2} h_{jk}(t_2-\tau_2) K_{ff}(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

The dispersion of displacement $\sigma_w^2(t)$ is obtained directly from the correlation function putting $t_1 = t_2 = t$. For the known form of correlation function $K_{ff}(\tau_1,\tau_2)$ the area of integration is divided into four subareas: I, II, IIIa and IIIb depending on the relationship between t_1 and t_2 (Fig.2).

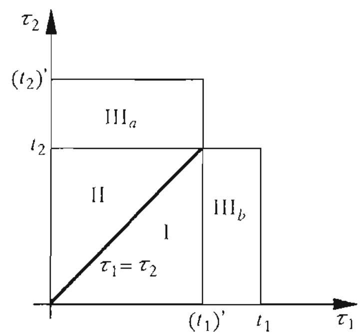

Fig. 2. Partition of the area of integration into the four subareas I, II, IIIa and IIIb

If the correlation function of the velocity normal to the surface of a shell $K_{\dot{w}\dot{w}}$ is of interest, it can be easily obtained by applying Eq (3.27). For a stationary process, Eq (3.27) takes the form (3.28). It is assumed that due to small curvatures of shallow shell, $V_z = \dot{w}$ is the most important component of the normal surface velocity vector. Therefore, the value of normal surface velocity is approximately equal to the component V_z, and the dispersion of this component only is analysed

$$K_{\dot{w}\dot{w}}(t_1,t_2) = \frac{\partial^2 K_{ww}(t_1,t_2)}{\partial t_1 \partial t_2} \tag{3.17}$$

$$K_{\dot{w}\dot{w}}(\tau) = -\frac{d^2 K_{ww}(\tau)}{d^2\tau} \tag{3.18}$$

where $\tau = t_2 - t_1$.

The dispersion of displacement or velocity can be obtained by setting $t_1 = t_2 = t$ in Eqs (3.7) or (3.18).

The dispersion of shell displacements will be obtained in the case of external force excitation by a random process of white noise type. In this case we have Eq (3.19), where $\delta(\tau_1 - \tau_2)$ is the Dirac delta function

$$\begin{aligned}\sigma_w^2(x,y,t) &= 2\frac{256}{\pi^4}\frac{z_0^2}{\rho^2 h^2}\sum_{m,n=1,3,\dots}^{+\infty}\frac{1}{mn}w_{mn}(x,y)\sum_{j,k=1,3,\dots}^{+\infty}\frac{1}{jk}w_{jk}(x,y)\cdot \\ &\cdot \int\limits_0^t h_{mn}(t-\tau_1)\,d\tau_1\int\limits_0^{\tau_1} h_{jk}(t-\tau_2)\delta(\tau_1-\tau_2)\,d\tau_2\end{aligned} \tag{3.19}$$

After integration and some arduous transformation, one obtains the formulae for dispersion of displacements in the form (3.20) and for the normal surface velocity in the form (3.21), where the functions $G_{mnjk}(t)$ have the form (3.22) for $\zeta_{mn} < 1$ or the form (3.23) for $\zeta_{mn} > 1$. In Eqs (3.20) and (3.21) underlined symbols $\underline{\omega}$, $\underline{G}$ and $\underline{K}$ denote different values of the parameters depending on the value of ζ_{mn} (see e.g. Eqs (3.5) and 93.6)) which is denoted by an overline (for $\zeta_{mn} < 1$) or a tilde (for $\zeta_{mn} > 1$)

$$\begin{aligned}\sigma_w^2(x,y,t) &= 2\frac{256}{\pi^4}\frac{z_0^2}{\rho^2 h^2}\sum_{m,n=1,3,\dots}^{+\infty}\frac{1}{mn\underline{\omega}_{mn}}w_{mn}(x,y)\cdot \\ &\cdot \sum_{j,k=1,3,\dots}^{+\infty}\frac{1}{jk\underline{\omega}_{jk}}w_{jk}(x,y)\underline{G}_{mnjk}(t)\end{aligned} \tag{3.20}$$

$$\begin{aligned}\sigma_{\dot{w}}^2(x,y,t) &= 2\frac{256}{\pi^4}\frac{z_0^2}{\rho^2 h^2}\sum_{m,n=1,3,\dots}^{+\infty}\frac{1}{mn\underline{\omega}_{mn}}w_{mn}(x,y)\cdot \\ &\cdot \sum_{j,k=1,3,\dots}^{+\infty}\frac{1}{jk\underline{\omega}_{jk}}w_{jk}(x,y)\underline{K}_{mnjk}(t)\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\overline{G}_{mnjk}(t) = \Gamma_{mnjk}\Big[\frac{1}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}+\overline{\omega}_{jk})^2} + \frac{1}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big] + \\ &-\frac{1}{2}\mathrm{e}^{-\Gamma_{mnjk}t}\Big\{\Big[\frac{\overline{\omega}_{mn}+\overline{\omega}_{jk}}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}+\overline{\omega}_{jk})^2} + \frac{\overline{\omega}_{mn}-\overline{\omega}_{jk}}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big]\cdot \\ &\cdot\sin[(\overline{\omega}_{mn}+\overline{\omega}_{jk})t] + \\ &+\Gamma_{mnjk}\Big[\frac{\Gamma_{mnjk}\cos[(\overline{\omega}_{mn}-\overline{\omega}_{jk})t]}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}-\overline{\omega}_{jk})^2} - \frac{\Gamma_{mnjk}\cos[(\overline{\omega}_{mn}+\overline{\omega}_{jk})t]}{\Gamma_{mnjk}^2 + (\overline{\omega}_{mn}+\overline{\omega}_{jk})^2}\Big] +\end{aligned} \tag{3.22}$$

$$-\Big[\frac{(\overline{\omega}_{mn}+\overline{\omega}_{jk})\sin[(\overline{\omega}_{mn}+\overline{\omega}_{jk})t]}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}+\overline{\omega}_{jk})^2}-\frac{(\overline{\omega}_{mn}-\overline{\omega}_{jk})\sin[(\overline{\omega}_{mn}-\overline{\omega}_{jk})t]}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big]\Big\}$$

$$\begin{aligned}\widetilde{G}_{mnjk}(t) &= \Gamma_{mnjk}\Big[\frac{1}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})^2}+\frac{1}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})^2}\Big]+\\ &-\frac{1}{2}e^{-\Gamma_{mnjk}t}\Big\{\Big[\frac{\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk}}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})^2}+\frac{\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk}}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})^2}\Big]\cdot\\ &\cdot\sinh[(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})t]+\\ &+\Gamma_{mnjk}\Big[\frac{\cosh[(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})t]}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})^2}-\frac{\cosh[(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})t]}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})^2}\Big]+\\ &+\Big[\frac{(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})\sinh[(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})t]}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})^2}-\frac{(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})\sinh[(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})t]}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}-\widetilde{\omega}_{jk})^2}\Big]\Big\}\end{aligned}\tag{3.23}$$

where

$$\Gamma_{mnjk}=\omega_{mn}\zeta_{mn}+\omega_{jk}\zeta_{jk}$$

Due to the fact that some of the components reach zero very fast Eqs (3.22) and (3.23) can rewritten in the approximated formulae

$$\begin{aligned}\overline{G}_{mnjk}(t)\approx\overline{G}_{mnjk} &= \Gamma_{mnjk}\Big[\frac{1}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}+\overline{\omega}_{jk})^2}+\frac{1}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big]\\ \widetilde{G}_{mnjk}(t)\approx\widetilde{G}_{mnjk} &= \Gamma_{mnjk}\Big[\frac{1}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}+\widetilde{\omega}_{jk})^2}+\frac{1}{\Gamma^2_{mnjk}-(\widetilde{\omega}_{mn}-\widetilde{\omega}^2_{jk})^2}\Big]\end{aligned}\tag{3.24}$$

The dispersion of the normal surface velocity is given by the approximate formulae

$$\begin{aligned}&\overline{K}_{mnjk}(t)\approx\overline{K}_{mnjk}=\\ &=\Gamma_{mnjk}\Big[\frac{\omega_{mn}\zeta_{mn}\overline{\omega}_{jk}+\omega_{jk}\zeta_{jk}\overline{\omega}_{mn}}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}+\overline{\omega}_{jk})^2}+\frac{\omega_{mn}\zeta_{mn}\overline{\omega}_{jk}+\omega_{jk}\zeta_{jk}\overline{\omega}_{mn}}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big]+\\ &+\Gamma_{mnjk}\Big[\frac{1}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}+\overline{\omega}_{jk})^2}+\frac{1}{\Gamma^2_{mnjk}+(\overline{\omega}_{mn}-\overline{\omega}_{jk})^2}\Big]\cdot\\ &\cdot(\Upsilon_{mnjk}+\overline{\omega}_{mn}\overline{\omega}_{jk})\end{aligned}\tag{3.25}$$

$$
\begin{aligned}
&\widetilde{K}_{mnjk}(t) \approx \widetilde{K}_{mnjk} = \\
&= \Gamma_{mnjk}\Big[\frac{\omega_{mn}\zeta_{mn}\widetilde{\omega}_{jk} + \omega_{jk}\zeta_{jk}\widetilde{\omega}_{mn}}{\Gamma^2_{mnjk} - (\widetilde{\omega}_{mn} + \widetilde{\omega}_{jk})^2} + \frac{\omega_{mn}\zeta_{mn}\widetilde{\omega}_{jk} + \omega_{jk}\zeta_{jk}\widetilde{\omega}_{mn}}{\Gamma^2_{mnjk} - (\widetilde{\omega}_{mn} - \widetilde{\omega}_{jk})^2}\Big] + \\
&+\Gamma_{mnjk}\Big[\frac{1}{\Gamma^2_{mnjk} - (\widetilde{\omega}_{mn} + \widetilde{\omega}_{jk})^2} + \frac{1}{\Gamma^2_{mnjk} - (\widetilde{\omega}_{mn} - \widetilde{\omega}_{jk})^2}\Big] \cdot \\
&\cdot(\Upsilon_{mnjk} + \widetilde{\omega}_{mn}\widetilde{\omega}_{jk})
\end{aligned}
\tag{3.26}
$$

where

$$
\Upsilon_{mnjk} = \omega_{mn}\zeta_{mn}\omega_{jk}\zeta_{jk}
$$

The dispersion of displacement and velocity are functions of the surface variables (x, y). It is useful to average them over the whole surface of the element and calculate new parameters – the surface averaged dispersions $\langle\sigma^2_w\rangle$ and $\langle\sigma^2_{\dot{w}}\rangle$, as defined by the integrals (3.27).

Due to the fact that in the definitions of the dispersions only the eigenfunctions $w_{mn}(x, y)$ are functions of surface variables, after suitable integrations, and taking into account the approximate formulae (3.24), (3.25) and (3.26), the averaged dispersions are not functions of time too, and take the final forms (3.28), where Λ_{mnjk} are defined in Eq (3.29)

$$
\begin{aligned}
\langle\sigma^2_w(t)\rangle &= \frac{1}{ab}\int\limits_0^a\int\limits_0^b \sigma^2_w(x, y, t)\; dxdy \\
\langle\sigma^2_{\dot{w}}(t)\rangle &= \frac{1}{ab}\int\limits_0^a\int\limits_0^b \sigma^2_{\dot{w}}(x, y, t)\; dxdy
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\langle\sigma^2_w\rangle &= 2\frac{256}{\pi^4}\frac{z^2_0}{\rho^2h^2}\frac{1}{4}\sum_{m,n=1,3,\dots}^{+\infty}\frac{1}{mn\underline{\omega}_{mn}}\sum_{j,k=1,3,\dots}^{+\infty}\frac{1}{jk\underline{\omega}_{jk}}\underline{G}_{mnjk}\Lambda_{mnjk} \\
\langle\sigma^2_{\dot{w}}\rangle &= 2\frac{256}{\pi^4}\frac{z^2_0}{\rho^2h^2}\frac{1}{4}\sum_{m,n=1,3,\dots}^{+\infty}\frac{1}{mn\underline{\omega}_{mn}}\sum_{j,k=1,3,\dots}^{+\infty}\frac{1}{jk\underline{\omega}_{jk}}\underline{K}_{mnjk}\Lambda_{mnjk}
\end{aligned}
\tag{3.28}
$$

$$\Lambda_{mnjk} = \int\limits_0^a \int\limits_0^b w_{mn}(x,y) w_{jk}(x,y)\,dxdy = \begin{cases} \dfrac{1}{4}ab & \text{for} \quad m = j \ \wedge \ n = k \\ 0 & \text{for} \quad m \neq j \ \vee \ n \neq k \end{cases} \tag{3.29}$$

4. Numerical example

Let us consider a shallow shell in the form of an elliptical paraboloid, whose equation of the middle surface has the form (4.1), where f is the shell deflection (see Fig.1)

$$z(x,y) = f\Big[1 - \frac{(2x-a)^2}{2a^2} - \frac{(2y-b)^2}{2b^2}\Big] \tag{4.1}$$

In this example we limit the analysis to the case of a shell whose projection on the plane xy has the form of square $a = b$. The dispersion of displacements and velocities for the shell with a deflection $f = 0.1$ m is compared with the values for the corresponding square plate $f = 0$ m. We assume the following values of parameters: $a = b = 1$ m, $h = 0.002$ m, $f = 0$ m and $f = 0.1$ m, $E = 2.1 \cdot 10^{11}$ Pa, $\rho = 7900\ \mathrm{kg/m^3}$, $\nu = 0.3$, $\varepsilon = 0.001$, $z_0 = 1\ \mathrm{N/m^2}$. For these values of parameters, the eigenfrequencies and dimensionless modal damping coefficients for the odd modes are given in Table 1.

Table 1. Values of eigenfrequencies [rad/s] and dimensionless modal damping coefficients for a plate and a shallow shell for the lowest odd modes

Mode	Plate $f = 0$		Shallow shell $f = 0.05$ m		Shallow shell $f = 0.1$ m	
(m,n)	ω_{mn}	ζ_{mn}	ω_{mn}	ζ_{mn}	ω_{mn}	ζ_{mn}
(1,1)	61.6	0.031	1033.0	0.517	2063.2	1.032
(1,3)	308.0	0.154	1076.2	0.538	2085.2	1.043
(3,3)	554.4	0.277	1170.7	0.585	2135.5	1.068
(1,5)	800.7	0.400	1305.6	0.653	2212.3	1.106
(3,5)	1047.1	0.524	1469.6	0.735	2312.9	1.157
(5,5)	1539.9	0.770	1853.2	0.927	2573.8	1.287
(1,7)	1539.9	0.770	1853.2	0.927	2573.8	1.287
(3,7)	1786.3	0.893	2062.5	1.031	2728.3	1.364
(5,7)	2279.0	1.140	2501.4	1.251	3073.6	1.537
(7,7)	3018.2	1.509	3189.4	1.595	3655.5	1.828

The values of dispersion of displacement $\langle\sigma_w^2\rangle$ and normal velocity $\langle\sigma_{\dot{w}}^2\rangle$, averaged over the surface, for the chosen values of deflection f are given in the Table 2.

Table 2. Values of averaged dispersion of displacement and normal surface velocity for some values of deflection

f [m]	0	0.01	0.02	0.05	0.2
$\langle\sigma_w^2\rangle[\mathrm{m}^2]$	$0.367\cdot10^{-6}$	$0.258\cdot10^{-8}$	$0.210\cdot10^{-9}$	$0.103\cdot10^{-10}$	$0.401\cdot10^{-13}$
$\langle\sigma_{\dot{w}}^2\rangle\left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right]$	$0.150\cdot10^{-2}$	$0.157\cdot10^{-3}$	$0.578\cdot10^{-4}$	$0.238\cdot10^{-4}$	$0.110\cdot10^{-4}$

5. Conclusions

- The analysis results show an important influence of the shallow shell deflection on dispersion of displacement and dispersion of normal surface velocity, averaged over the middle surface of the element, for external force excitation of the white noise type.
- The same conclusions have been obtained in the case of excitation by a deterministic force (for harmonic and poliharmonic excitation).
- The analysis of random vibrations of the shell excited by coloured noise should be carried out next, which is planned to be done by the authors in the near future.

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Drgania lepkosprężystych powłok małowyniosłych poddanych wymuszeniu losowemu typu białego szumu

Streszczenie

W artykule omówiono rezultaty analiz drgań powłok małowyniosłych wykonanych z materiału lepkosprężystego i poddanych wymuszeniu losowemu typu białego szumu. Jako parametr opisujący drgania przyjęto dyspersję składowej prędkości normalnej do powierzchni środkowej powłoki. Wybór tego parametru jest uzasadniony faktem, iż jego wartość może być miarą wartości mocy akustycznej promieniowanej przez drgający element.

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