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ON SOME APPROACHES TO THE STRESS-STATE DETERMINATION IN LAYERED CRACKED PLATES

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A survey of approaches to determination of the stress-state in layered plates with the cracks located symmetrically about their thickness is presented. The plate is subject to a complex biaxial loading; i.e., tension – shear or bending – torsion, while the crack of length 2a is unloaded. The plate layers can be homogeneous or non-homogeneous, rigid to shear or yielding due to shear. To study such plates the following mathematical models are employed: the plates rigid to shear are considered within the framework of classical plate theory, while those yielding due to shear are within the scope of Timoshenko's theory.
The asymptotic formulae for stress distribution near the crack tip in each layer of the plate are obtained; the limiting equilibrium of layered plate having a transverse crack is studied.
The results presented can be applied to strength evaluation of layered plates within the framework of fracture mechanics. The paper presents the results obtained by the authors recently and proposes some generalisation.

Key words: layered plates, crack, stress distribution, failure

1. Introduction

In view of their structure the construction materials can be homogeneous or non-homogeneous. There are three classes of the latter ones: micronon-homogeneous materials, layered plates and fibrous composite materials. The first group can include; e.g. dispersible hardened materials, porous materials, ceramics, concrete, powder and discretely reinforced aggregates (see: Delyavsky, 1990; Delyavsky and Korkuna, 1995; Delyavsky, 1996). While in layered composites the separate layers are to be distinguished. Within the framework of composite mechanics the two approaches (i.e, the continuous or discrete one) can be adopted when calculating the stress state and limiting equilibrium in layered plates.

The continuous approach was formulated by V. Bolotin (1980), while one of the variants was suggested by Woźniak and developed in the papers by Kaczyński and Matysiak (1989), Konieczny (1997), Matysiak and Nagórko (1989), Woźniak and Woźniak (1997).

Using a discrete approach Berezhnitsky et al. (1984b, 1985) determined the stress state in layered cracked plates subject to symmetrical and non-symmetrical bending.

The materials reinforced with fibres, i.e., fibrous composites, can be divided into two groups: short-fibre-reinforced composites and continuously reinforced ones. Opanasovich (1997a,b,c) suggested a method for determination of the stress state in short-fibre composite near the fibre tip. An approximate approach to calculation of macrostresses in a fibre-reinforced cracked composite was presented by Berezhnitsky et al. (1987).

There are three levels on which the structure of non-homogeneous materials can be studied within the framework of composite mechanics; i.e., micro-level, mezo-level and macro-level. It should be noted that when studied on macro-level such materials are considered as homogeneous with specific effective moduli. In view of their mechanical properties these materials can be considered as isotropic or anisotropic, however in special cases they can reveal orthotropic or transversely isotropic properties.

Randomly reinforced composites and micro-non-homogeneous aggregates are the isotropic materials, while orthogonally reinforced composites and materials reinforced with a fabric are the orthotropic ones. Composites reinforced in one direction and layered plates composed of isotropic or transversely isotropic layers are transversely-isotropic materials, while those reinforced with continuous fibres in various directions are anisotropic.

In view of their rigidity the materials under consideration can be rigid to shear and yielding due to shear. The rigid to shear materials can be studied within the scope of classical theory of plates, while when studying those yielding due to shear the transverse shear effects should be also included.

A variety of defects can be found in heterogeneous materials, most dangerous of them are cracks. Therefore, the stress state in such materials should be calculated within the framework of fracture mechanics. The present paper gives a brief account of these problems.

2. Formulation of the problem

We consider a thin transversely symmetrical layered plate with the orthogonal bases $\boldsymbol{e} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, such that the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are located in the plate mid-plane and $\boldsymbol{e}_3$ is directed perpendicularly to it. We assume the ideal mechanical contact on the interlayer surfaces

$$u_i^{(k-1)} = u_i^{(k)} \qquad \sigma_{i3}^{(k-1)} = \sigma_{i3}^{(k)} \qquad i = 1,2,3 \tag{2.1}$$

where $u_i^{(k)}$, $\sigma_{i3}^{(k)}$ are the components of displacement vector and transverse stress vector, respectively, in the kth layer. Let us replace the layered plate with a homogeneous one with the effective Young modulus, shear modulus and Poison ratio E, G, μ, respectively. We and introduce for such a plate the displacement field $\boldsymbol{u} = [u_1, u_2, u_3]$ and assume that the plate is subject either to the forces of intensity P_{ij} (plane stress state) or to the moments of intensity M_{ij}, $(i,j = 1,2)$ (bending stress state). Therefore, the displacement field in the plate can be represented as a sum of symmetric and antisymmetric parts $\boldsymbol{u} = \breve{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ as follows

$$u_i = \breve{u}_i + \widetilde{u}_i \qquad i = 1,2,3 \tag{2.2}$$

The first terms correspond to the plane stress state, while the second ones represent the bending stress state. In our case the load is applied to the lateral surface of the plate, while its upper and bottom surfaces are unloaded. In this case the normal stress $\sigma_{33}^{(k)}$ can be neglected, (see, e.g. Goldenveizer, 1958).

Depending on the value of transverse shear modulus G'_k of the kth layer we will consider the following two cases separately:

- Layers rigid to shear
- Layers yielding due to shear.

3. Layered cracked plate composed of the uniform isotropic rigid to shear layers

Let us assume that the transverse shear modulus of each layer $G' \to \infty$.

Then, we can neglect the stresses $\sigma_{13}^{(k)}$, $\sigma_{23}^{(k)}$ and determine the stress state in a layered plate within the framework of classical theory of plates. Hence, by virtue of Eq (2.2) we introduce the displacement vector components $u_i^{(k)}$

$$u_i^{(k)} = f_e^{(k)}(\delta)u_i^{(e)} + f_b^{(k)}(\delta)u_i^{(b)} \tag{3.1}$$

The superscript e corresponds to the plane stress state; while b indicates the bending stress state, $\delta = z/h$ is a dimensionless co-ordinate, the functions $f_e^{(k)}(\delta)$, $f_b^{(k)}(\delta)$ are represent displacement distributions over the thickness of each layer under tension and bending, respectively. The form of these functions ensures the ideal mechanical contact on the interface of layers. In the classical theory of plates (bending – according to Kirchhoff) it is assumed that

$$f_b^{(k)}(\delta) = -\delta \qquad u_i^{(b)} = u_{3,i}^{(b)} \qquad i = 1,2 \tag{3.2}$$

We assume that the displacement $u_3^{(e)}$ is equal to zero and the plate deflection $u_3^{(b)}$ remains unchanged over the thickness $u_3^{(b)} = u_3^{(b)}(x_1, x_2)$, i.e. the plate is not subjected to transverse deformations.

Let us consider the plane stress state in such a plate. In this case only the first term remains in Eq (3.1), thus we can omit the superscript e.

The components of stress tensor averaged over the plate thickness have the form

$$\sigma_{ij} = 2\sum_{k=1}^{n} \frac{1}{h} \int\limits_{h_k}^{h_{k+1}} \sigma_{ij}^{(k)}\, dz \qquad i,j = 1,2 \tag{3.3}$$

where n is the number of layers; $h_{k+1} - h_k = t_k$ is the kth layer thickness, $\sigma_{ij}^{(k)}$ denote the stress tensor components of the kth layer; h is the plate thickness.

Using the strain compatibility conditions for the layers $\varepsilon_{ij}^{(k)} = \varepsilon_{ij}$, where ε_{ij} are the plate strain tensor components, we obtain the formulae (cf Berezhnitsky et al., 1984a) for the effective moduli of layered plate

$$\mu = \frac{\sum_{k=0}^{n} \mu_k C_k}{\sum_{k=0}^{n} C_k} \qquad E = 2(1+\mu)G \tag{3.4}$$

where

$$C_k = E_k \int\limits_{h_k}^{h_{k+1}} \frac{f_k(\delta)}{1-\mu_k^2}\, dz \qquad G = 2\sum_{k=0}^{n} G_k \int\limits_{h_k}^{h_{k+1}} \frac{1}{h} f_k(\delta)\, dz \tag{3.5}$$

and for the stress distribution in each layer

$$\begin{aligned} \sigma_{ii}^{(k)} &= \frac{E_k}{E(1-\mu_k^2)} f_k(\delta)[(1-\mu\mu_k)\sigma_{ii} + (\mu_k - \mu)\sigma_{(3-i)(3-i)}] \\ \sigma_{ij}^{(k)} &= \frac{G_k f_k(\delta)\sigma_{ij}}{G} \qquad i \neq j \end{aligned} \tag{3.6}$$

where E_k, μ_k, G_k stand for the Young modulus, Poisson ratio and shear modulus of the kth layer, respectively, and $f_k(\delta) = f_e^{(k)}(\delta)$.

Let us assume that there is a crack of length $2a$ located along the plate thickness. Replacing in Eqs (3.6) σ_{ii}, σ_{ij} with the corresponding formulae for the isotropic effectively homogeneous cracked plate (Berezhnitsky et al., 1979) yields the stress distribution in each plate layer near the crack tip. After Berezhnitsky et al. (1984) we employ the polar co-ordinate system r, θ, with the origin at the crack tip

$$\begin{aligned} \sigma_r^{(k)} &= \frac{E_k f_k(\delta)}{E}\Big[\frac{K_{\rm I}}{4\sqrt{2r}}\Big(b\cos\frac{\theta}{2} - \frac{1+\mu}{1+\mu_k}\cos\frac{3\theta}{2}\Big) + \\ &+ \frac{K_{\rm II}}{4\sqrt{2r}}\Big(-b\sin\frac{\theta}{2} + 3\frac{1+\mu}{1+\mu_k}\cos\frac{3\theta}{2}\Big)\Big] \\ \sigma_\theta^{(k)} &= \frac{E_k f_k(\delta)}{E}\Big[\frac{K_{\rm I}}{4\sqrt{2r}}\Big(a\cos\frac{\theta}{2} + \frac{1+\mu}{1+\mu_k}\cos\frac{3\theta}{2}\Big) + \\ &+ \frac{K_{\rm II}}{4\sqrt{2r}}\Big(-a\sin\frac{\theta}{2} - 3\frac{1+\mu}{1+\mu_k}\sin\frac{3\theta}{2}\Big)\Big] \\ \sigma_{r\theta}^{(k)} &= \frac{G_k f_k(\delta)}{G}\Big[\frac{K_{\rm I}}{4\sqrt{2r}}\Big(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\Big) + \frac{K_{\rm II}}{4\sqrt{2r}}\Big(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\Big)\Big] \end{aligned} \tag{3.7}$$

where $K_{\rm I}$ and $K_{\rm II}$ are the stress intensity factors averaged over the plate thickness and

$$a = \frac{3(1-\mu\mu_k) + 5(\mu_k - \mu)}{1-\mu_k^2} \qquad b = \frac{5(1-\mu\mu_k) + 3(\mu_k - \mu)}{1-\mu_k^2} \tag{3.8}$$

From Eqs (3.7) it can be seen that the boundary conditions are satisfied on the crack edges in each layer of the plate only for the tangential stresses while for the normal ones, there is a requirement for the load symmetry about the crack. For the boundary conditions to be satisfied $\sigma_\theta^{(k)}(\pi) = 0$ in each layer for an arbitrary load it should be $\mu_k = \mu$. Therefore, when taking the above presented approach, in a general case the boundary conditions on the crack edges can be satisfied only for the layers, the Poisson ratios of which are equal. Otherwise, the average value should be taken in calculations. Hence, Eqs (3.7), (3.8) can be rewritten in the simplified form

$$\sigma_r^{(k)} = \frac{E_k f_k(\delta)}{E}\sigma_r \qquad \sigma_\theta^{(k)} = \frac{E_k f_k(\delta)}{E}\sigma_\theta \qquad \sigma_{r\theta}^{(k)} = \frac{G_k f_k(\delta)}{G}\sigma_{r\theta} \tag{3.9}$$

where σ_r, σ_θ, $\sigma_{r\theta}$ are the stresses near the crack tip in the effectively homogeneous isotropic plate.

Similarly, a solution to the bending problem of plate composed of the layers rigid to shear can be obtained. In this case it is necessary to use the second term appearing in Eq (3.1) and then follow the above approach.

The limiting equilibrium of cracked layered plate composed of the layers rigid to shear we evaluate using the criterion of elastic energy of a strain releasing rate. Let us assume that the crack initiates in the ith layer if the energy averaged over the plate thickness and referred to this layer reaches the value required for the crack initiation in separate layers, i.e.

$$\begin{aligned} 2\omega_i \sum_{m=0}^{n} \frac{1}{t_m} \lim_{\Delta_m\to 0} \frac{1}{\Delta_m} \int_0^{\Delta_m} \int_{h_m}^{h_{m+1}} \Big(\sigma_\beta^{(m)} u_\beta^{(m)} + \sigma_{r\beta}^{(m)} u_r^{(m)}\Big) dr\,dz = \\ = \min\Big[\frac{1}{t_i} \lim_{\Delta_i\to 0} \frac{1}{\Delta_i} \int_0^{\Delta_i} \int_{h_i}^{h_{i+1}} \Big(\sigma_\beta^{(i)} u_\beta^{(i)} + \sigma_{r\beta}^{(i)} u_r^{(i)}\Big) dr\,dz\Big]_{crit} \end{aligned} \tag{3.10}$$

where "crit" indicates the critical value of energy under tension and $\omega_i = \delta_i/h$ is the relative thickness of ith layer; Δ_m is a small crack growth in the mth layer. Let us assume that the Poisson ratios of all layers are equal. Substituting the formulae for stresses and displacements (Berezhnitsky et al., 1979) into Eq (3.10) and assuming that the crack toughness of mode I $(K_{c\mathrm{I}})$ averaged over the plate thickness is very small when compared to the crack toughness of mode II $(K_{c\mathrm{II}})$ we arrive at the equation of limiting surface in the case when a cracked laminated plate is subject to a plane stress state

$$p^{*2}\sin^2\alpha + q^{*2}\cos^2\alpha = \frac{F_i^2}{\pi a}K_{c\mathrm{I}}^2 \tag{3.11}$$

where p^* and q^* are the components of external load vector initiating a crack in the considered layer; α is the angle the direction of p component makes with the crack line, the functions F_i depend on the Young moduli of separate layers, their thicknesses and the displacement distribution over the plate thickness.

The crack toughness $K_{c\mathrm{I}}$ can be expressed in terms of the crack toughness of the mth layer $K_{c\mathrm{I}}^{(m)}$ as follows

$$K_{c\mathrm{I}} = \sqrt{2\sum_{m=0}^{n} \omega_m (K_{c\mathrm{I}}^{(m)})^2} \tag{3.12}$$

Calculations were made for the five-layer plate with given ratios of each layer thickness to its Young modulus. The Poisson ratio of each layer was equal to 0.3.

The curves determining the boundaries of regions of the admissible magnitudes of external load areas are presented in Fig.1.

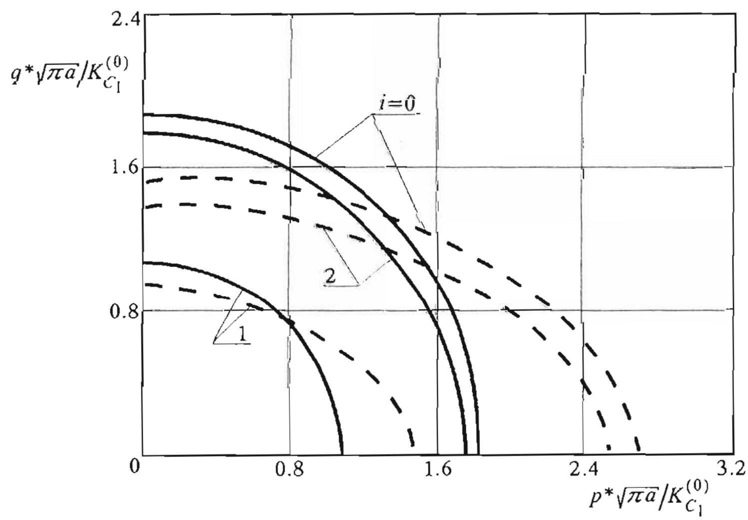

Fig. 1. Ranges of the ultimate load for the plate composed of 5 layers, the Young moduli of which are equal; thin external layers

The solid lines correspond to the external load applied at $45°$, while the dotted lines are constructed for $30°$; i represents the number of the layer, in which the limiting equilibrium is attained for the first time; $K_{c\mathrm{I}}^{(0)}$ is the crack toughness of mode I of zeroth layer. It can be seen that the area of admissible stresses for each layer of the plate depends mainly on the Young modulus and relative thickness of the layer. Namely, the area extends as the Young modulus increases and its thickness decreases.

4. Stress state in a cracked plate composed of the layers yielding due to shear

Let us consider a plate composed of isotropic homogeneous layers. In this case the transverse shear strain due to bending exerts a decisive influence on the stress state in the plate while an insignificant strain due to the tensile load can be neglected in calculation. Thus we arrive at the following calculation procedure in the case of layered plate: we determine the plane stress state in terms of the two-dimensional theory of elasticity (see Section 3) and the bending stress state within the scope of the refined plate theory.

The components of displacement vector for each layer of the plate can be written as follows (cf Berezhnitsky et al. (1984))

$$u_j^{(k)} = -h\Big[\delta\frac{\partial w}{\partial x_j} + f_k(\delta)\Big(\frac{\partial F}{\partial x_j} + (-1)^j\frac{\partial \Phi}{\partial x_{3-j}}\Big)\Big] \qquad j = 1,2 \tag{4.1}$$

where the functions $f_k(\delta)$ represent the displacement distributions over the thickness of the layered plate when subject to bending, and their form ensures the ideal mechanical contact conditions on the layer interface; w stands for the deflection of the plate middle surface; F is the function representing torsion of the each element perpendicular to the plate mid-plane before deformation; Φ is the function of transverse shear. These functions satisfy the equations

$$\Delta^2 w = 0 \qquad \Delta\Phi - k^2\Phi = 0 \qquad F = \varepsilon\Delta w \tag{4.2}$$

where the parameters k and ε depend on the elastic moduli of separate layers and their thicknesses and can be determined from the following relations

$$k^2 = \frac{4h\overline{G}}{C - B} \qquad \varepsilon = \frac{D}{2h\overline{G}} \tag{4.3}$$

were h is the plate thickness $(-1 \leqslant z/h \leqslant 1)$

$$\kappa_i = 3\int\limits_{\delta_{i-1}}^{\delta_i} \delta f_i(\delta)\, d\delta \qquad \begin{aligned}\delta_0 &= 1\\ \delta_{n+1} &= 0\end{aligned} \tag{4.4}$$

$$\overline{G} = \sum_{i=1}^{n+1} G'_i[f_i(\delta_{i-1}) - f_i(\delta_i)] \tag{4.5}$$

G'_i stands for the shear modulus of this layer in the direction perpendicular to it.

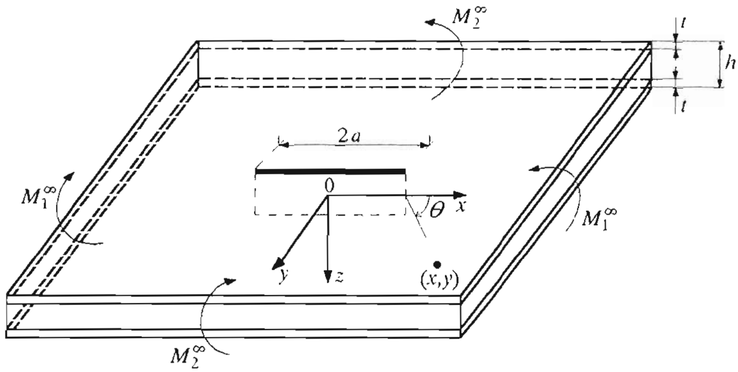

Fig. 2. Scheme of the layered cracked plate subject to bending

Let us assume that there is a crack of length $2a$ (Fig.2) along the thickness of the considered plate.

The solution to this problem can be presented as a sum of solutions to the problems of a solid layer and the cracked one, respectively, subject to the moment $M_y = -M_{22}$ applied to the crack edges. Since a solution to the first problem has no effect on the local stress state we consider only the second one.

Thus, the following boundary conditions should be satisfied along the crack

$$\begin{aligned} &Q_y = M_{xy} = 0 \qquad M_y = -M_{22}^\infty \qquad \text{for} \quad |x| < a \\ &\beta_y^i = 0 \qquad \qquad \qquad \qquad \qquad \text{for} \quad |x| > 0 \end{aligned} \tag{4.6}$$

where $\beta_y^{(i)}(x, y, \delta)$ is the angle the tangent to the deformed ith layer of plate makes with the plane $0yz$. We apply the Fourier transformations of the coordinate x to the unknown functions W and Φ, respectively. Then taking into account the conditions for stress attenuation at the infinity we can rewrite the solutions of Eqs (4.2) as follows

$$\begin{aligned} w^c &= [R_1(\alpha) + \alpha y R_2(\alpha)] \exp(-\alpha y) \\ \Phi^s &= R_3(\alpha) \exp(-y\sqrt{\alpha^2 + k^2}\ ) \end{aligned} \tag{4.7}$$

The following relations between the functions $R_1(\alpha)$, $R_2(\alpha)$ and $R_3(\alpha)$ are true

$$R_1(\alpha) = -\frac{d}{2\varepsilon\alpha^2} R_3(\alpha) \qquad R_2(\alpha) = -\frac{1}{2\varepsilon\alpha^2} R_3(\alpha) \tag{4.8}$$

where

$$d = \frac{A + D}{A - D} \tag{4.9}$$

and they satisfy the integral equation for the function $R_2(\alpha)$ (Berezhnitsky et al., 1984b)

$$\begin{aligned} &D\int_0^\infty \alpha^2 R_2(\alpha) f(\alpha)\cos(\alpha x)\, d\alpha = -\frac{\pi}{2}M_{22}^\infty && x > a \\ &\int_0^\infty \alpha R_2(\alpha)\cos(\alpha x)\, d\alpha = 0 && x > a \end{aligned} \tag{4.10}$$

where

$$f(\alpha) = 3 + \frac{A}{D} + \frac{4\alpha^2}{k^2}\Big(1 - \sqrt{1 + \frac{k^2}{\alpha^2}}\Big) \tag{4.11}$$

This equation has the solution

$$R(\alpha) = -\frac{\pi M_{22}^\infty a}{2\Big(1 + \frac{A}{D}\Big)}\Big[\Psi(1)J_1(l\alpha) - \int_0^1 J_1(a\alpha\eta)\Big(\frac{\Psi(\eta)}{\sqrt{\eta}}\Big)'\eta\, d\eta\Big] \tag{4.12}$$

where $J_n(l\alpha)$ is the cylindrical function of the first kind and nth order; the function $\Psi(\eta)$ represents the solution of the Fredholm integral equation

$$\Psi(\xi) + \int_0^1 \Psi(\eta)K(\xi,\eta)\, d\eta = \sqrt{\xi} \qquad \xi = \frac{x}{a} < 1 \tag{4.13}$$

where the kernel $K(\xi,\eta)$ reads

$$K(\xi,\eta) = \sqrt{\xi\eta}\int_0^\infty s\Big(\frac{f\left(\frac{s}{a}\right)}{1 + \frac{A}{D}} - 1\Big)J_0(\xi s)J_0(\eta s)\, ds \qquad s = a\alpha \tag{4.14}$$

Substituting into Eqs (4.11) asymptotic expansions of the functions w and Φ into the Taylor series (Berezhnitsky et al., 1984a,b) we obtain the stress distributions near the crack tip in each layer of the plate

$$\begin{bmatrix} \sigma_x^{(i)} \\ \sigma_y^{(i)} \\ \sigma_{xy}^{(i)} \end{bmatrix} = \frac{K_{\mathrm{I}}^{(i)}}{\sqrt{2r}} \begin{bmatrix} \Big[\Big(\frac{2\mu^{(i)}}{1-\mu^{(i)}} + d\Big)\delta + \varepsilon k^2 f_i\Big]\cos\frac{\theta}{2} + \frac{\delta - \varepsilon k^2 f_i}{2}\sin\theta\sin\frac{3\theta}{2} \\ \Big[\Big(\frac{2}{1-\mu^{(i)}} - d\Big)\delta - \varepsilon k^2 f_i\Big]\cos\frac{\theta}{2} - \frac{\delta - \varepsilon k^2 f_i}{2}\sin\theta\sin\frac{3\theta}{2} \\ [(d-1)\delta + \varepsilon k^2 f_i]\sin\frac{\theta}{2} - \frac{\delta - \varepsilon k^2 f_i}{2}\sin\theta\cos\frac{3\theta}{2} \end{bmatrix} \tag{4.15}$$

and

$$\sigma_{xz}^{(i)} = \sigma_{yz}^{(i)} = O(1) \tag{4.16}$$

The stress intensity factor $K_{\rm I}^{(i)}$ of the ith layer is determined in terms of the moment intensity factor $K_{\rm I} = M_{22}^{\infty}\Psi(1)\sqrt{a}$ as follows

$$K_{\rm I}^{(i)} = 3K_{\rm I}E_i\Big[2h^2(1+\mu_i)\sum_{k=1}^{n+1}\frac{E_k}{1-\mu_k}(\delta_{k-1}^3 - \delta_k^3)\Big]^{-1} \tag{4.17}$$

A similar solution was obtained for bending non-symmetrical about the crack edges (Berezhnitsky et al., 1985).

As it follows from Eqs (4.14) the boundary conditions at the crack edges are satisfied. Thus, in contrast to the classical theory of layered plates, the solution obtained within the scope of the refined theory of plates is exact. The bending M_x, M_y and twisting M_{xy} moments can be represented as functions of the angle θ in the following way

$$\left[\begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array}\right] = \frac{M_{22}^{\infty}\Psi(1)\sqrt{a}}{\sqrt{2r}}\left[\begin{array}{c} \cos\frac{\theta}{2} - \frac{1}{2}\sin\theta\sin\frac{3\theta}{2} \\ \cos\frac{\theta}{2} + \frac{1}{2}\sin\theta\sin\frac{3\theta}{2} \\ \frac{1}{2}\sin\theta\cos\frac{3\theta}{2} \end{array}\right] + O(1) \tag{4.18}$$

It should be noted that for an infinitely thin layered plate $h/a \to 0$ while for the plate treated as one rigid layer $G_i' \to \infty$

$$\Psi(1) \to \frac{D+A}{3D+A} \tag{4.19}$$

If, additionally, the Poisson ratios of all the layers are equal $\mu_i = \mu$ then

$$\Psi(1) \to \frac{1+\mu}{3+\mu} \tag{4.20}$$

This result agrees with that obtained by Knowles and Wang (1960).

The courses of the function $\Psi(1)$ versus the parameter h/a for a three-layer plate are presented in Fig.3 for a given displacement distribution over the plate thickness for different values of E_2/G_1'. From the diagrams it can be seen, that the function $\Psi(1)$ grows as one of the parameters increases with other parameters fixed, especially for small values of h/a or E_2/G_1'. For example, the function $\Psi(1)$ increases approximately by 94% as h/a varies from zero to 0.5 for $E_2/G_1' = 10$ and $E_1/E_2 = 0.2$, while it grows by 81% as

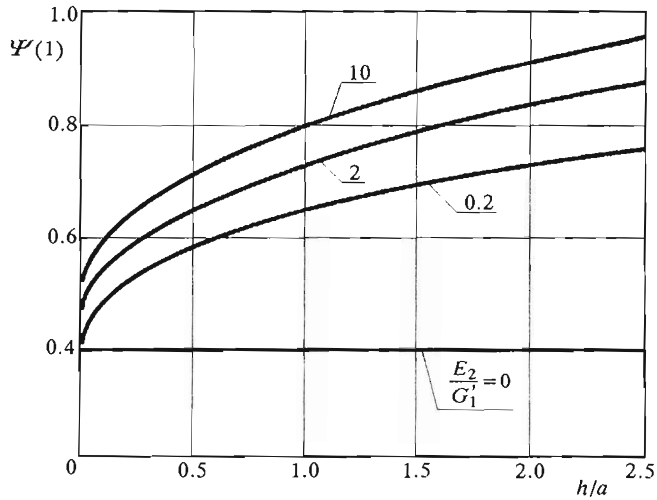

Fig. 3. Dimensionless stress intensity factor $\Psi(1)$ versus the ratio of plate thickness to the crack length

the value of parameter E_2/G_1' increases from zero to 0.5 for $h/a = 1.5$ and $E_1/E_2 = 2$. The straight line corresponds to the rigid to shear material.

As the function $\Psi(1)$ is constant over the plate thickness it can be considered as the reduced stress intensity factor determining the averaged stress field near the crack tip for the whole plate.

From the results obtained the conclusion can be drawn that lower magnitudes of the stress arise in the three-layer plate constructed in the way ensuring that the Young modulus of the middle layer is much smaller than the transverse shear modulus of the external layer for a constant value of the ratio E_1/E_2.

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Analiza stanu naprężeń w płytach warstwowych ze szczelinami

Streszczenie

W pracy przeprowadza się analizę stanu naprężeń w płytach warstwowych ze szczelinami poddanych dwuosiowemu rozciąganiu lub zginaniu stosując pewną metodę uśredniającą.

Wykazuje się, że przy rozciąganiu mozna oprzeć się na klasycznej teorii płyt warstwowych, zaś w przypadku zginania konieczne jest sterowanie teorii Timoshenki.

Wyprowadzono asymptotyczne wzory rozkładu naprężeń w pobliżu wierzchołka szczeliny w każdej warstwie płyty. Wzory te zależą od modułów efektywnych płyty warstwowej oraz od uśrednionego współczynnika intensywności naprężeń.

Przedstawiony model zawiera energetyczny test powiększania się szczeliny przy dwuosiowym rozciąganiu płyty.

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