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A NON-SINGULAR DESCRIPTION OF PARAMETRICAL RESONANCE

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Employing the Mathieu equations we present a method for construction of Ince-Strutt diagrams, in which no cut-off of the infinite chain of equations for Fourier coefficients is necessary.

Key words: stability, infinite chain of equations, truncation, Gibbs phenomenon

1. Introduction

Many problems arising in different fields of technology and physics can be described by means of linear equations with periodic coefficients. Equation of this type is satisfied by state vectors and can be used in description of a wide range of phenomena such as motion of electrons in periodic environments (crystals), strain of construction an elements or electrical non-stationary circuits (parametrical circuits). Linear equations with periodical coefficients are also used when investigating the solution stability of non-linear equations with periodical solutions (the first variational equations).

The main purpose of the paper is to investigate the stability of the trivial solution to the Mathieu equation (ME) (Bolotin, 1956; Gutowski, 1971), with the help of Fourier series. We do this without a common cut-off of the infinite chain of equations for the corresponding Fourier coefficients (variables). In this way, the problem of searching for untrivial solutions to those equations (banal in the case of Eqs (4.1)) is separated from the problem of looking for the boundary parameters dividing the space F of parameters (δ, ε) into the regions of stable and unstable solutions to the MEs (Ince-Strutt diagrams). In other words, we replace the obscure, in our opinion, formalism of zero determinants with a new formalism of zero discriminants, heuristically justified by the qualitative properties of Fourier coefficients discussed in the paper. This also has the advantage that it associates our approach in a natural way with the biffurcation theory (Kurnik, 1997).

In the paper we adopt a methodological procedure similar in some way to the adiabatic principle of the Quantum Field Theory (Bogolubov and Shirkov, 1976), according to which there exists a limit of the weak coupling. In our case this principle simply means that the end points of unstable regions of the Ince-Strutt diagrams are situated exactly at these points of the parametrical space F where they should be according to the monodromy matrix theory for strongly stable systems (Arnold, 1974). We cannot say this, however, about other parts of the corresponding regions and in fact more extensive instability regions were obtained than those usually presented in the literature. However, the practical meaning of these differences may be important only for low values of the parameter δ (Arnold, 1974, 1975).

2. Mathieu-Hill equations – stability

One of the simple representatives of differential, linear equations with periodic coefficients is Mathieu-Hill equation (MHE)

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon\psi)x = 0 \tag{2.1}$$

with two arbitrary parameters δ and ε forming the 2D space F and the periodic function ψ with period T. This equation is related to the very important issue of determining the regions of stability and instability in space F of the trivial solution

$$x \equiv 0 \tag{2.2}$$

It is true that the boundaries of stability regions in the space F (Ince-Strutt diagrams) are determined by the values of parameters δ and ε for which periodic solutions with minimal periods T and $2T$ exist (Bolotin, 1956; Mierkin, 1987; Arnold, 1974, 1975; Gutowski, 1971). See also Rouche et al. (1977), Bogusz (1972), Kurnik (1997).

In the simplest case, the MHEs take the shape of Mathieu's equations (MEs) (Landau and Lifshic, 1966; Arnold, 1974, 1975), which, after the corresponding change of the variables, can be described in the standard form

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t)x = 0 \tag{2.3}$$

where constants δ, ε are related to physical parameters.

3. Fourier series method

The traditional method for looking for the boundaries of stability regions in the parametrical space F for MHEs (2.1) consists in a representation of periodic solutions with period T and $2T$ with the help of the Fourier series. In the case of ME, where the perturbative function $\psi = \cos t$, we consider the solutions with periods 2π and 4π presented as follows

$$\begin{gathered} x = \frac{\eta_0}{2} + \sum_j^{\infty}(n_j \cos jt + \lambda_j \sin jt) \\ x = \frac{\alpha_0}{2} + \sum_j^{\infty}\Big(\mu_j \cos\frac{jt}{2} + \nu_j \sin\frac{jt}{2}\Big) \end{gathered} \tag{3.1}$$

These series substituted into Eq (2.3) lead to the appropriate equations for the expansion coefficients

$$\{\eta_j\}, \ \{\lambda_j\}, \ \{\mu_j\}, \ \{\nu_j\} \tag{3.2}$$

In this way the four independent, infinite chains of equations similar to three point equations (Samarskij and Nikolajev, 1988), are obtained

$$-a_j y_{j-1} + c_j y_j - b_j y_{j+1} = 0 \tag{3.3}$$

with coefficients a_j, b_j and c_j specified properly, where the index $j = 0, 1, 2, ...$ up to infinity and y_j represent one of the sets of Fourier coefficients, Eq (3.2). The last equation contains the variable y with a negative index j, which is equalled to zero. In other words, the infinite chains of equations for the Fourier coefficients considered here and represented by y variables are truncated only from below which is a consequence of the properties of Fourier coefficients. The fourfold versions of these equations resalt from the fact that for every period there exist odd and even solutions due to replacing the variable t with $-t$.

For example, for the coefficients η, which correspond to the even solution with the period 2π, we obtain

$$a_j = b_j = -\frac{\varepsilon}{2} \qquad c_j = \delta - j^2 \tag{3.4}$$

for j = $1, 2, ...$, whereas, for $j = 0$, $a_0 = 0$, $c_0 = \delta$, $b_0 = -\varepsilon$. The coeficients in the remaining versions of the infinite chains of equations are similar, see Section 5.

At first glance, the problem of finding the parameters δ and ε which correspond to the periodic solutions to the ME with the periods 2π and 4π is now reduced to the problem of finding those values of the parameters for which solutions to Eqs (3.3) exist. The usual reasoning is the following: after making a cut-off at some point of the infinite chain of equations considered, Eqs (3.3), for the sake of obtaining a nontrivial solution we require, that the determinant of the resulting homogenous, finite system of equations for the Fourier coefficients be equal to zero. In this way the limitations for the parameters of ME are usually obtained. However, in the case of the infinite chains of equations, the homogeneity of the equations is not an obstacle to obtaining untrivial solutions. This property of infinite chains of equations and in particular of Eqs (3.3) is due to their inherent incompleteness, i.e., in each system of equations obtained from (3.3), for a finite j, the number of variables exceeds the number of equations. This additional variable(s) means that a linear infinite homogeneous system may have other than trivial solutions. It can be clearly seen from Eqs (3.3) when we do not use a cut-off at any point on the infinite chain of equations but, we treat this system of equations as a reccurent formula allowing us to calculate subsequent Fourier coefficients from the first one (y with $j = 0$). In fact, this methodology is adopted in paper. As a result the problem of finding non-trivial solutons to Eqs (3.3) (in fact a trivial one) is separated from the problem of finding the boundary values of parameters δ and ε in the space F which separate the stable from unstable trivial solutions to ME.

In adopting this methodological principle however, we face the problem of how to find the parameters of MEs related to periodic solutions with the period 2π and 4π. As we well know, general classes of functions, by no means periodic, can be expanded in the Fourier series. For it to be impossible to expand a function $f(t)$ in a trigonometric series, $f(t)$ would have to have, in its domain (or over one period if $f(t)$ is periodic), an infinite number of discontinuities (or jumps) or an infinite number of maxima and minima (Zeldovich and Yaglom, 1987). There is, however, a certain property of Fourier coefficients, which distinguishes the expansions of periodic functions from aperiodic. This property is expressed by means of the Gibbs phenomenon (Bracewell, 1968; Edwards, 1979) which consists in a worse convergence of the Fourier series in the neighbourhoods of points at which the functions analised have jumps or rapidly change directions. Since in a finite interval every aperiodic function can be treated as a periodic function with jumps defined upon the infinite interval, we should expect the Gibbs phenomenon when aperiodic solutions are expanded with the help of series (3.1). So, based on the Gibbs phenomenon, we can claim that a convergent Fourier series with the same periods, without the Gibbs phenomenon upon the intervals $< 0, T >$ or $< 0, 2T >$ correspond to periodic solutions to the MHEs, with the periods T and $2T$. So our search for the boundary values of parameters δ and ε can be reduced to a search for the Fourier series with the periods T and $2T$ without the Gibbs phenomena.

In the case of MEs the above reduction refers to solutions and series (3.1) with the periods 2π and 4π.

The case in which the aperiodic solution incidently has the same value at $t = T$, for parameters δ, ε not lying on a border, is also eliminated by the absence of Gibbs phenomenon since then the first derivative of the considered aperiodic solution should have a jump. Otherwise, from the uniqueness theorem, it would be a periodic solution.

Taking into account the above heuristic reasoning we accept the following hypothesis: a good, or even an exact, approximation to a periodic solution to *the ME with period 2π or 4π can be obtained by means of the appropriate*, uniformly convergent Fourier series (3.1).

In fact, we make an additional assumption of the multiplicative structure of Fourier coefficients y, see Eq (4.5). We assume that the condition of Eq (4.6) type, after separating the leading terms in j from α coefficient (conditions for γ), depend on parameters δ, ε in a "uniform" way. It turns out that this assumption leads to consistent results, Section 5. Moreover, in this way, the boundary values of parameters δ, ε resulting from dynamical analysis are related to the border values of these parameters related to the computational stability in the case of "final" problems to three point equations, Section 4. These two sets of parameters may be identical.

4. General remarks concerning three point equations

Let us consider inhomogeneous Eqs (3.3)

$$-a_j y_{j-1} + c_j y_j - b_j y_{j+1} = w_j \tag{4.1}$$

also known as the three-layer equations because of their relations to a discrete version of the 1D differential equation

$$f\frac{d^2y}{dx^2} + g\frac{dy}{dx} + hy = w \tag{4.2}$$

see Potter (1973). Depending on the range of the discrete variable j, Eq (4.1) corresponds to an initial or boundary problem. In a typical boundary problem $0 \leqslant j \leqslant N$. We consider "the initial" problem in which $j = 0,1,2...$, up to infinity, see also (3.4). In this case

$$y_{j+1} = \frac{1}{\alpha_{j+1}}(y_j - \beta_{j+1}) \tag{4.3}$$

for $j = 0,1,2...$, where

$$\alpha_{j+1} = \frac{b_j}{c_j - a_j\alpha_j} \qquad\qquad \beta_{j+1} = \frac{w_j + a_j\beta_j}{c_j - a_j\alpha_j} \tag{4.4}$$

and $\alpha_1 = b_0/c_0$, $\beta_1 = w_0/c_0$. It can shown that Eqs (4.3) and (4.4) are equivalent to Eqs (4.1).

It can be seen that for homogeneous Eqs (4.1) coefficients $\beta_j \equiv 0$ and in this case the general solution to Eqs (4.1) can be obtained using the recurrence formula

$$y_{j+1} = \frac{y_j}{\alpha_{j+1}} \tag{4.5}$$

in which α_j are expressed by Eq (4.4). In other words, the jth Fourier coefficient of series (3.1) is the product of inverse powers of the consecutive α coefficients and the zero order Fourier coefficient y $(j = 0)$.

This structure of the Fourier coefficients and the requirement for the hypothesis postulated in Section 3 to be satisfied suggest the following limitations

$$|\alpha_j| > 1 \tag{4.6}$$

for a suitably large $j > J$ or equivalently

$$|y_j| > |y_{j+1}| \tag{4.7}$$

The above inequalities are indeed expected from the consecutive Fourier coefficients constructed by means of more and more intensively oscillating sinusoidal functions. From Eqs (4.5) and (4.6) we obtain

$$|y_j| \to 0 \qquad \text{for } j \to \infty \tag{4.8}$$

which is again expected from the Fourier coefficients resulting from the MEs solutions.

Of course, it is not true that every convergent series must satisfy conditions (4.6) or (4.7). If, however, these conditions are satisfied and, additionally, the values of Fourier coefficients have the same sign, for almost all j, then the corresponding Fourier series are uniformly convergat (Edwards, 1979), i.e., the hypothesis of Section 3 can be realized by conditions (4.6) or (4.7).

Below, we show that these and other, also plausible conditions; namely, that

$$|\alpha_j| \geqslant M > 0 \tag{4.9}$$

for any finite j, lead to concise expressions for Fourier coefficients with large indices j. We could call the last ones the anticlosure conditions because they are not satisfied if one of the consecutive y is made equal to zero. In this case the correspond α should tend to infinity and then, from (4.4), the next α is equal to zero, and conditions (4.9) are not satisfied.

It turns out that the above limitations of the Fourier coefficients are not satisfied by the formulas obtained from Eq (4.4) in terms of the perturbation theory, see Eqs (5.1) and (5.2). But even in the cases leading to divergent Fourier series we can gain certain information about the Ince-Strutt diagrams.

At this point Eqs (4.1) should be noted in the context of a "final" problem. Given y, for $j = N$, and y with the lower subscript j are calculated. In this case, conditions

$$|\alpha_j| < 1 \tag{4.10}$$

correspond to the computational stability for Eqs (4.1). They are satisfied when the coefficients of Eqs (4.1) satisfy the following inequalities

$$|c_j| \geqslant |a_j| + |b_j| \tag{4.11}$$

see Samarskij and Nikolajev (1988), Potter (1973). In other words, unstable computation algorithm for the "final" problem of Eqs (4.1) correspond to conditions (4.6).

5. Ince-Strutt diagrams

The analysis presented aims at finding, in the space F, the Ince-Strutt diagrams for the MHEs. The canonical method of putting the determinant(s) equal to zero is replaced here with inequalities like (4.6) and (4.7). To see whether it is easy to satisfy these inequalities let us look more closely at the recurrent formulas (4.4), in the case of (3.4)

$$\alpha_{j+1} = \frac{\varepsilon/2}{j^2 - \delta - \alpha_j \varepsilon/2} \tag{5.1}$$

for $j = 1, 2...$, where $\alpha_1 = b_0/c_0 = -\varepsilon/\delta$. First, let us consider the case of small values of $|\varepsilon|$ and the assumption of regularity of α. We should expect accuracy α approximatons in terms of the simplified recurrent formulas

$$\alpha_j \cong \frac{\varepsilon/2}{j^2 - \delta} \tag{5.2}$$

for $j = 2, 3....$ At arbitrarily fixed ε and δ, α coefficients tend to zero as j tends to infinity. So, the conditions, Eq (4.6), are not satisfied. Moreover, the Fourier series (3.1) with the coefficients resulting from Eqs (5.2) and (4.5) are not even convergent. In other words, from the point of view of Fourier series, canonical perturbation theory is completely useless even in the case when the parameter ε does not appear in equations in front of the coefficients with subscript $j+1$, see Eq (5.1). It is astonishing that in spite of that, some exact information about the Ince-Strutt diagrams can be obtained from Eq (5.2) when

$$\delta = m^2 \tag{5.3}$$

where m – integer. In such a case, the coefficient α for $j = m$, tends to infinity and in spite of the fact that the remaining coefficient α do not satisfy Eq (4.6), a "very lage" value of one of the coefficients α compensates for that. As a result Eq (5.3) represents the system on the stability border. The above instability points, obtained by means of the Fourier series method, are in agreement with the instability points obtained within the monodromy matrix theory (Arnold, 1974, 1975).

Now, we find approximations of α coefficients in different than Eq (5.2). Looking once more at the recurrent formula (5.1) we see that the new γ coeeficients defined by

$$\alpha_j = \frac{2(j^2 - \delta)}{\varepsilon} + \gamma_j \tag{5.4}$$

by virtue of Eq (5.1) yield

$$\alpha_{j+1} = -\frac{1}{\gamma_j} \tag{5.5}$$

Now, the conditions unbounded from above, see Eq (4.6), are replaced with the bounded conditions for the coefficient γ. In the new variables, the recurrent formula (5.1) can be rewritten as

$$-\frac{1}{\gamma_j}=\frac{\varepsilon/2}{j^2-\delta+\frac{\varepsilon}{2\gamma_{j-1}}}=\frac{\varepsilon\gamma_{j-1}}{2\gamma_{j-1}(j^2-\delta)+\varepsilon}$$

or, equivalently, as

$$\varepsilon\gamma_j\gamma_{j-1}+2\gamma_{j-1}(j^2-\delta)+\varepsilon=0 \tag{5.6}$$

By introducing the differences $\Delta\gamma$

$$\gamma_j=\gamma_{j-1}+\Delta\gamma_j$$

one can describe these equations as

$$\varepsilon\gamma_{j-1}^2+\varepsilon\gamma_{j-1}\Delta\gamma_j+2\gamma_{j-1}(j^2-\delta)+\varepsilon=0 \tag{5.7}$$

By treating the above differences as given quantities we can write the formula for γ_{j-1}, for $j=2,3,...$, as follows

$$\gamma_{j-1}=\frac{1}{\varepsilon}\Big[\delta-j^2-\frac{\varepsilon}{2}\Delta\gamma_j\pm\sqrt{\Big(\delta-j^2-\frac{\varepsilon}{2}\Delta\gamma_j\Big)^2-\varepsilon^2}\Big] \tag{5.8}$$

Hence, and in view of the conditions for $\Delta\gamma$ (see Eqs (5.5) and (4.6)) we get, for $j>J(\delta,\varepsilon)$

$$\gamma_{j-1}\cong\frac{1}{\varepsilon}\Big(\delta-j^2\pm|\delta-j^2|\sqrt{1-\frac{\varepsilon}{|\delta-j^2|^2}}\Big) \tag{5.9}$$

Choosing the sign $+$ and taking into account Eq (5.4) we have, for $|\delta-j^2|$ large enough

$$\alpha_j=\frac{2(j^2-\delta)}{\varepsilon}-\frac{\varepsilon}{2[(j+1)^2-\delta]} \tag{5.10}$$

The above formula probably properly reflects the symmetry of coefficients of a_j and b_j in equations of (3.3), (4.1) type, see (3.4). In contrast to Eq (5.2) this formula satisfies conditions (4.6) and α with sufficiently large j are positive. Therefore, the postulates (4.6) are consistent with the equations considered for the coefficients.

Moreover, Eq (5.9) and the real values of coefficients γ bound the parameters ε and δ in the following way

$$(\delta-j^2)^2\geqslant\varepsilon^2 \tag{5.11}$$

for $j > J(\delta, \varepsilon)$. Since the existence requirement of solutions to the MEs or Eqs (4.1) is not imposed on the parameters ε and δ the above inequalities are the necessary conditions for the parameters ε and δ for our hypothesis that they are situated on the border of instability regions to be true.

We show that conditions (5.11) are satisfied by the following relations between ε and δ

$$\delta = m^2 + \varepsilon \tag{5.12}$$

where m is an arbitrary natural number and ε is positive. This is obviously true for $j = 1, 2, ..., m$. For $j > m$, it means for $j = m + k$, where $k = 1, 2, ...$ we get from Eq (5.11)

$$(\varepsilon - 2km - k^2)^2 \geqslant \varepsilon^2$$

This leads to the following conditions to ε

$$k(2m + k) \geqslant 2\varepsilon \tag{5.13}$$

For

$$\delta = m^2 - \varepsilon \tag{5.14}$$

inequalities (5.11) are satisfied for $j = m + k$, where $k = 1, 2, ...$ For $j = m - k$, where $k = 1, 2, ..., m - 1$, similar conditions for the parameter ε are obtained

$$k(2m - k) \geqslant 2\varepsilon \tag{5.15}$$

It is interesting that Eqs (5.12) and (5.14) can also be derived from the requirement of zero discriminants of quadratic equations (5.7) with neglecting $\Delta\gamma_j$ (large j). In fact, this connection of relations (5.12) and (5.14) with the bifurcation points of Eq (5.7) distinguishes them from other linear possibilities

$$\delta = m^2 \pm u\varepsilon \tag{5.16}$$

where u is properly chosen constant parameter. The choice of Eqs (5.12) and (5.14) means that the infinitesimal change of δ may cause a loss of the positive value for a discriminant of the one of γ coefficient. A symptom which points that conditions of real values for γ definitively cannot be satisfied and that the original parameters indeed were placed upon the border. The concept of strongly stable systems according to which small variations of parameters of ME should not change the stability of trivial solution (Arnold, 1974, 1975) should be noted here. In other words, small variations should not radically change the properties of the solution considered if the original parameters do not belong to a border of instability in the Ince-Strutt diagram.

Following this reasoning we accept the bifurcation philosophy and postulate that for small j

$$\Big(\delta - j^2 - \frac{\varepsilon}{2}\Delta\gamma_j\Big)^2 = \varepsilon^2 \tag{5.17}$$

Now we examine the consequences of that hypothesis in the simplest case $j = 2$. We get then

$$\Big[\delta - 4 - \frac{\varepsilon}{2}(\gamma_2 - \gamma_1)\Big]^2 = \varepsilon^2$$

From (5.4) and (5.1)

$$\begin{aligned}
\gamma_1 &= \alpha_1 - \frac{2(1-\delta)}{\varepsilon} = -\frac{\varepsilon}{\delta} - \frac{2(1-\delta)}{\varepsilon} \\
\gamma_2 &= \frac{2(\delta-4)}{\varepsilon} + \alpha_2 = \frac{2(\delta-4)}{\varepsilon} + \frac{\varepsilon\delta}{2\delta(1-\delta)+\varepsilon^2}
\end{aligned} \tag{5.18}$$

Substituting these formulas into Eq (5.17) we get, according to the above hypothesis the equation describing the stability borders regions for small j

$$\delta - 4 - \frac{\varepsilon}{2}\Big[\frac{2(\delta-4)}{\varepsilon} + \frac{\varepsilon\delta}{2\delta(1-\delta)+\varepsilon^2} + \frac{\varepsilon}{\delta} + \frac{2(1-\delta)}{\varepsilon}\Big] \tag{5.19}$$

Hence, for small values of ε

$$\delta = \pm\varepsilon + 1 \tag{5.20}$$

This result is similar to results (5.12) and (5.14) which were confirmed in another way for large j. Unfortunately, it is not in agreement with the standard results based on the method of nullification of the sequel of determinants related to approximated closed equations obtained from Eq (3.3) (cf, e.g., Bolotin, 1956; Jordan and Smith, 1987). The above results can be referred to even solutions to the MEs with period 2π. Odd solutions, with the same period, do not yield new results because they lead to the relations

$$\delta = (m+1)^2 \pm \varepsilon \tag{5.21}$$

for $m = 0, 1, 2, \ldots$. The same may be said about even and odd solutions to the MEs with period 4π, but now

$$\delta = \frac{m^2}{4} \pm \varepsilon \tag{5.22}$$

for $m = 1, 3, 5, \ldots$. This results from the following formulae

$$a_j = b_j = -\frac{\varepsilon}{2} \qquad\qquad c_j = \delta - \frac{j^2}{4} \tag{5.23}$$

(cf Bolotin, 1956; Mierkin, 1987). From the above it follows that the extended instability regions of Ince-Strutt diagrams are represented by the areas, the edges of which lie above the δ axis exactly at these points which were described by the monodromy matrix theory (Arnold, 1975, 1976)

$$\delta = \begin{cases} m^2 & \text{for} \quad m = 1,2,3,... \\ \dfrac{m^2}{2} & \text{for} \quad m = 1,3,5,... \end{cases} \tag{5.24}$$

6. Final remarks

In this description of parametrical resonance (unbounded solutions at particular values of parameters (Mierkin, 1987; Kudrewicz, 1996; Salama and Chen, 1973) the attention has been focused on the properties of Fourier coefficients (variables), representing periodic solutions for T and $2T$ periods. An essensial factor in our analysis was the introduction of new α variables due to which the old variables y are presented as their products (carry out iteration of (4.5)). Using of these variables uniform convergence of the considered Fourier series was expressed in terms of separated inequalities, see Eq (4.6). The absolute values of these variables, even for a uniformly convergent Fourier series, are only bounded below. The absolute values of *gamma* variables have both upper and lower bounds. This is an important factor in construction of the approximation formulas given in Section 5.

The α coefficients and the γ variables corresponded to variables y satisfy infinite chains of equations, which are not broken (cut-off) at any point; i.e., we separate successfully the problem of searching for untrivial solutions to Eqs (4.1) (banal in this case) from the problem of searching for values of the parameters ε and δ related to parametrical resonance. This also means that singular approximations were avoided.

The main assumption of the paper that the periodic solutions to MEs with periods 2π and 4π can be well approximated or exactly expressed by uniformly convergent Fourier series (3.1) can be weakened by the requirement that averaged solutions satisfying the same equations be considered. It is also true that all other solutions (unperiodic or with other periods) can not satisfy that condition because of the Gibbs phenomenon.

We may say in recapitulation that the traditonal approach to the parametrical resonance based on the disconnection of infinite chains of equations for the Fourier coefficients has been substituted by a new approach in which the disconnection (cut-off) is replaced by the idea of uniform convergence of the corresponding Fourier series and the bifurcation concept. Both methods lead to identical results for ε tending to zero. For other cases there arise differences. These differences, particularly for low order parametrical resonances, could be the subject of experimental and computational verification (see Arnold, 1974. 1975).

We believe that additional insights into parametrical resonance may result from taking into account statistical descripton of perturbative factors (Tylikowski, 1991), non-linearity and the possibility of its realization in electrical systems (Kudrewicz, 1996).

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Niesingularny opis rezonansu parametrycznego

Sstreszczenie

Na przykładzie równania Mathieu przedstawiamy niesingularną metodę konstrukcji diagramów Ince'a-Strutta, które obrazują na płaszczyźnie F parametrów równania obszary stabilności i niestabilności. Niesingularność metody została osiągnięta dzięki nie obrywaniu nieskończonych łańcuchów równań dla współczynników Fouriera. Pozwala to na zachowanie pierwotnego charakteru rozważanych równań (zagadnienie początkowe). Przedstawione rozważania sugerują poszerzenie obszarów niestabilności.

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