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ASYMPTOTICS OF THERMAL DISPERSION IN PERIODIC MEDIA

CLAUDIA TIMOFTE

Department of Mathematics, Faculty of Physics, University of Bucharest, Romania
e-mail: claudia@mat.fizica.unibuc.ro

Using a central limit theorem we will describe a wide class of thermal dispersive phenomena occuring in macrohomogeneous systems. More precisely, we will focus on the diffusions $\boldsymbol{X}_t$ generated by an operator L having periodic coefficients. The central limit theorem asserts that $\lambda^{-\frac{1}{2}}(\boldsymbol{X}_{\lambda t} - \lambda U_0 \overline{\boldsymbol{b}} t)$, $t \geqslant 0$, converges in distribution to Brownian motion as $\lambda \to \infty$. Here $\overline{\boldsymbol{b}}$ is the mean of $\boldsymbol{b}(\boldsymbol{x})$. In the present contribution the functional dependence of the dispersion matrix $\overline{\mathsf{D}}$ of this limiting Brownian motion on the velocity parameter U_0 and the period a is analysed. We will give precise analytical conditions imposed on the geometry of functions b_i which determine the asymptotic behavior of elements $\overline{D}_{ij}$ as functions of U_0. Specific examples are given to illustrate computation of the macroscale coefficients as functions of the comparable microscale data.

Key words: Markov process, central limit theorem, asymptotic dispersion coefficients

1. Introduction

Based upon a rigorous, physico-mathematical description of microtransport processes occuring in heterogeneous systems, macrotransport processes describe a large class of material and non-material dispersive phenomena occuring in macrohomogeneous systems.

Applications of macrotransport theory are presently recognized in numerous fields of scientific and engineering research. In the last years, application of the microcontinuum theory to increasingly complex macrocontinuum systems has emphasized the need for wider theoretical context than that previously provided by the classical microtransport theory.

Various methods have been developed for determination of the macroscale behavior and properties of some heterogeneous complex systems. These include the method of moments, homogenization method, statistical and volume-averaging methods and probabilistic methods based on central limit theorems (see Bensoussan et al., 1978; Brenner and Edwards, 1993; Ene and Paşa, 1987).

Now, we use a probabilistic method based on a central limit theorem for Markov processes to get a macrotransport paradigm for thermal transport phenomena.

Such non-material dispersion phenomena have been considerably less known than the material transport phenomena.

It proves useful to accept the concept of *tracer* for all forms of continuous transport, material or otherwise and, thereby, to establish a Lagrangian description both for material and non-material processes.

The viability of pursuing such a novel Lagrangian perspective in thermal dispersion problems was demonstrated by introducing the notion of a generic conserved tracer entity, called a *thermion* in the case of internal energy transport (see Brenner and Edwards, 1993).

By assuming the existence of such a tracer, we are able to extend the macrotransport theory to cover internal energy transport processes and to establish a macrotransport paradigm for thermal transport phenomena.

The microtransport equation governing the evolution of temperature $T(t, \boldsymbol{x})$ in continuous insulated systems may by represented as

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{1.1}$$

with

$$\boldsymbol{J} = \rho c_p \boldsymbol{U} T - \boldsymbol{K}_T \cdot \nabla T \tag{1.2}$$

The appropriate initial and boundary conditions are imposed upon Eq (1.1).

Note that energy dissipation and kinetic energy contributions are neglected in this microtransport equation.

It is supposed that the thermal properties are nonnegative definite everywhere. Moreover, the thermophysical properties ρ, c_p, K_T, as well as the fluid velocity $\boldsymbol{U}$ are regarded as spatially periodic.

Since inhomogeneities in c_p occur typically only across the phase boundaries, where $\boldsymbol{n} \cdot \boldsymbol{U} = 0$, a term of the form $-T\boldsymbol{U} \cdot \nabla(\rho c_p)$ generally vanishes identically and for this reason will be excluded from our analysis.

Introducing two positive scalars U_0 and a, interpreted as the velocity and spatial scale parameters, respectively, we can express the fluid velocity $\boldsymbol{U}$ in the form

$$\boldsymbol{U}(\boldsymbol{x}) = U_0 \boldsymbol{V}\Big(\frac{\boldsymbol{x}}{a}\Big) \tag{1.3}$$

As we shall see in the next section, we can assume, without loss of generality, that the spatial scale parameter a is held fixed at $a = 1$, while the velocity parameter U_0 varies.

Now, we are especially interested in getting the Fickian approximation of the microtransport equation (1.1). This new partial diferential equation with constant coefficients will govern the evolution of temperature $T(t, \boldsymbol{x})$ for large t.

Here, we begin by considering the state space $\mathcal{T}$ as being $\mathcal{T} = I\!R^3$. In practice, of course, the space will be always bounded by boundaries. One may first construct a Markovian motion that satisfies the equation and then impose the appropriate boundary conditions in order to arrive at the corresponding $p(t; \boldsymbol{y}, \boldsymbol{x})$. This p will satisfy the imposed boundary conditions (see Timofte, 1996)

$$T(t, \boldsymbol{x}) = \int_{\mathcal{T}} (\rho c_p)(\boldsymbol{y}) p(t; \boldsymbol{y}, \boldsymbol{x}) T(0, \boldsymbol{y})\, d\boldsymbol{y} \tag{1.4}$$

we will consider the associate Itô stochastic differential equation and we shall see that $\rho c_p p$ may be interpreted as the conditional probability density of the thermal tracer.

Note that the conditions on coefficients of Eq (1.1) that guarantee the uniqueness and necessary smoothness of the fundamental solution p are assumed throughout.

So, the analysis of the asymptotic behaviour of temperature $T(t, \boldsymbol{x})$ for large t is equivalent to the analysis of the asymptotic behaviour of Markov process $\boldsymbol{X}_t$ defined by the associate Itô stochastic equation for large t.

This connection between the temperature T and the probability density $\rho c_p p$ will allow us to use a central limit theorem to get the desired Fickian approximation of Eq (1.1) and to express the macrotransport coefficients $\overline{\rho c_p}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{K}}_T$ in terms of the prescribed microscale data and the system geometry.

However, in a more general context, $\boldsymbol{X}_t$ can be regarded as diffusion generated by a diferential operator of the form

$$L = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij}(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{k} U_0 b_i(\boldsymbol{x}) \frac{\partial}{\partial x_i} \tag{1.5}$$

the coefficients of which satisfy the following assumptions:

(1) Matrix $[a_{ij}(\boldsymbol{x})]$ is symmetric and positive definite

(2) Functions $a_{ij}(\boldsymbol{x})$ and $b_i(\boldsymbol{x})$ are real valued and periodic, i.e. $a_{ij}(\boldsymbol{x} + \boldsymbol{\nu}) = a_{ij}(\boldsymbol{x})$, $b_i(\boldsymbol{x} + \boldsymbol{\nu}) = b_i(\boldsymbol{x})$, for any $\boldsymbol{x}$ and any vector $\boldsymbol{\nu}$ with integer coordinates

(3) Functions $a_{ij}(\boldsymbol{x})$ have bounded second order derivatives and $b_i(\boldsymbol{x})$ have continuous first order derivatives

(4) U_0 is a real parameter.

Let $(\Omega, \mathcal{A}, P^{\pi'})$ be a probability space in which are defined:

- Random vector $\boldsymbol{X}(0) = \boldsymbol{X}_0$, where $\boldsymbol{X}_0$ is a k-dimensional random vector independent of $\boldsymbol{B}(t)$ and distribution π'
- Standard k-dimensional Brownian motion $\boldsymbol{B}(t) = [B_1(t), ..., B_k(t)]$ which is independent of $\boldsymbol{X}_0$.

Let $\{\boldsymbol{X}(t),\ t \geqslant 0\}$ be the solution (continuous and nonanticipative) to Itô stochastic integral equation

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t U_0 \boldsymbol{b}\Big(\boldsymbol{X}(s)\Big)\, ds + \int_0^t \sigma\Big(\boldsymbol{X}(s)\Big)\, d\boldsymbol{B}(s) \tag{1.6}$$

where $\sigma(\boldsymbol{x})$ is the positive square root of $[a_{ij}(\boldsymbol{x})]$. Periodicity of the coefficients allows us to work in the state space $\mathcal{T}^k = [0,1)^k$ since the coefficients are periodic (see the condition (1.2)), with the process $\dot{\boldsymbol{X}}(t) = \boldsymbol{X}(t)(\text{mod } 1)$ having a transition probability density function $\dot{p}(t;\boldsymbol{x},\boldsymbol{y})$ and an invariant probability density $\pi(\boldsymbol{x})$ on $[0,1)^k$ such that

$$\int_{[0,1)^k} \dot{p}(t;\boldsymbol{x},\boldsymbol{y})\pi(\boldsymbol{x})\, d\boldsymbol{x} = \pi(\boldsymbol{y}) \qquad \text{a.e. on } [0,1)^k \tag{1.7}$$

Let us consider the real Hilbert space $L^2([0,1)^k, \pi)$ with the inner product

$$\langle f, g\rangle = \int_{[0,1)^k} f(\boldsymbol{y})g(\boldsymbol{y})\pi(\boldsymbol{y})\, d\boldsymbol{y} \tag{1.8}$$

and let $\{\dot{T}_t,\ t \geqslant 0\}$ be the strongly continuous semigroup of contractions in this space, defined by

$$(\dot{T}_t f)(\boldsymbol{x}) = \int_{[0,1)^k} \dot{p}(t;\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\, d\boldsymbol{y} \qquad \text{for } \boldsymbol{x} \in [0,1)^k \tag{1.9}$$

The central limit theorem asserts (see Bhattacharya, 1985) that on the assumptions $(1) \div (4)$, no matter what the initial distribution π' is, the stochastic process

$$\Big\{\boldsymbol{Z}_{t,\lambda} = \lambda^{-\frac{1}{2}}(\boldsymbol{X}_{\lambda t} - \lambda U_0 t \bar{\boldsymbol{b}})\ , \qquad t \geqslant 0\Big\} \tag{1.10}$$

converges weakly, as $\lambda \to \infty$, to a Brownian motion with zero drift and the dispersion matrix $\overline{\mathsf{D}} = [\overline{D}_{ij}]$ given by

$$\overline{D}_{ij} = -U_0^2\langle b_i, g_j\rangle - U_0^2\langle b_j, g_i\rangle + \overline{a}_{ij} + \\ + \int_{[0,1)^k} U_0\Big[g_i(\boldsymbol{y})\sum_{r=1}^{k}\frac{\partial}{\partial y_r}\Big(a_{rj}(\boldsymbol{y})\pi(\boldsymbol{y})\Big) + g_j(\boldsymbol{y})\sum_{r=1}^{k}\frac{\partial}{\partial y_r}\Big(a_{ri}(\boldsymbol{y})\pi(\boldsymbol{y})\Big)\Big]\, d\boldsymbol{y} \tag{1.11}$$

where

$$\begin{aligned} \bar{b}_i &= \langle b_i, 1\rangle \qquad & 1 \leqslant i \leqslant k \\ \overline{a}_{ij} &= \langle a_{ij}, 1\rangle \qquad & 1 \leqslant i,j \leqslant k \end{aligned} \tag{1.12}$$

In Eq (1.11), g_i is solution the unique in $\mathcal{D}_{\widehat{A}} \bigcap 1^{\perp}$ of the equation

$$\widehat{A} g_i = b_i - \bar{b}_i \tag{1.13}$$

and $\widehat{A}$ is the infinitesimal generator of the strongly continuous semigroup $\{\dot{T}_t,\ t \geqslant 0\}$ on the domain $\mathcal{D}_{\widehat{A}}$.

We will be especially interested in the functional dependence of the asymptotic dispersion coefficients $\overline{D}_{ij}$ on the velocity and spatial scale parameters. As we shall see in the next section, $\overline{D}_{ij}$ depends only on the product aU_0, the result being in accordance with all the experimental studies that have been done.

We will give precise analytical conditions on the geometry of functions b_i which determine the asymptotic behaviour of the elements $\overline{D}_{ij}$ for large aU_0.

The results of application to thermal dispersion processes in periodic media are shown.

Some examples are given in Section 3 to illustrate the computation of the macrotransport coefficients as functions of the comparable microscale data.

The first examples provide closed-form solutions of the macroscale coefficients, while in the last one, the macroscale coefficients are only shown to exhibit their expected growth as functions of aU_0.

2. Functional dependence of the asymptotic dispersion coefficients on the velocity and spatial scale parameters

In 1989, R.N. Bhattacharya, V.K. Gupta and H.F. Walker showed that in the case of solute dispersion in periodic porous media, the macroscale dispersion matrix $\overline{\mathsf{D}}$ depends only on the product aU_0 (see Bhattacharya et al., 1989).

An extension to cover a more general class of diffusion processes is given by Timofte (1996).

In particular, for the special case of thermal dispersion phenomena, let the large scale dispersion matrix $\overline{\mathsf{D}}$ be denoted by $\overline{\mathsf{D}} = \overline{\mathsf{D}}(a, U_0)$ to indicate its dependence on the spatial scale and velocity parameters. It can be proved (see Timofte, 1996) that if a central limit theorem holds for the solution $\boldsymbol{X}(t)$ of the Itô stochastic equation

$$\begin{aligned} d\boldsymbol{X}(t) &= U_0 \boldsymbol{b}\Big(\frac{\boldsymbol{X}(t)}{a}\Big)dt + \sigma\Big(\frac{\boldsymbol{X}(t)}{a}\Big)d\boldsymbol{B}(t) \\ \boldsymbol{X}(0) &= \boldsymbol{X}_0 \end{aligned} \tag{2.1}$$

then $\overline{\mathsf{D}}$ depends on a and U_0 only through their product aU_0.

In particular

$$\overline{\mathsf{D}}(a, U_0) = \overline{\mathsf{D}}(U_0, a) = \overline{\mathsf{D}}(aU_0, 1) \tag{2.2}$$

This interchangeability of velocity and spatial scale parameters in the large-scale dispersion matrix enables us to assume that the spatial scale parameter a is held fixed at $a = 1$, while the velocity parameter U_0 can vary.

A more precise analysis of the functional dependence of $\overline{D}_{ij}$ on these two parameters can be done in the special case when a_{ij} are constants and b_i are continuously differentiable periodic functions satisfying the condition

$$\operatorname{div} \boldsymbol{b} = 0 \tag{2.3}$$

Taking the period of b_i to be one in each coordinate, we can work on the state space $\mathcal{T} = [0,1)^k$ with the invariant distribution $\pi(\boldsymbol{x}) \equiv 1$.

Let $\mathcal{D}$ denote the following operator

$$\mathcal{D} = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{2.4}$$

In this case, the macrodispersion coefficients are given by

$$\overline{D}_{ij} = a_{ij} - U_0^2 \int_{\mathcal{T}} g_i(\boldsymbol{x})\Big(b_j(\boldsymbol{x}) - \overline{b}_j\Big)\, d\boldsymbol{x} - U_0^2 \int_{\mathcal{T}} g_j(\boldsymbol{x})\Big(b_i(\boldsymbol{x}) - \overline{b}_i\Big)\, d\boldsymbol{x} \tag{2.5}$$

We shall work with the following spaces of complex-valued functions on T

$$\begin{aligned} \boldsymbol{H}^0 &= \Big\{h/\int_{\mathcal{T}} |h(\boldsymbol{x})|^2\, d\boldsymbol{x} < \infty\,,\ \int_{\mathcal{T}} h(\boldsymbol{x})\, d\boldsymbol{x} = 0 \\ &\qquad \text{and } h \text{ satisfies periodic boundary conditions}\Big\} \\ \boldsymbol{H}^1 &= \Big\{h \in \boldsymbol{H}^0/\int_{\mathcal{T}} |\nabla h(\boldsymbol{x})|^2\, d\boldsymbol{x} < \infty\Big\} \qquad (2.6) \\ \boldsymbol{H}^2 &= \Big\{h \in \boldsymbol{H}^1/\int_{\mathcal{T}} \sum_{i,j=1}^{k} \Big|\frac{\partial^2 h(\boldsymbol{x})}{\partial x_i \partial x_j}\Big|^2\, d\boldsymbol{x} < \infty\Big\} \end{aligned}$$

We shall take the following inner product on $\boldsymbol{H}^1$

$$\langle h, u\rangle_1 = \int_{\mathcal{T}} \sum_{i,j=1}^{k} a_{ij} \frac{\partial h(\boldsymbol{x})}{\partial x_i} \frac{\partial \overline{u}(\boldsymbol{x})}{\partial x_j}\, d\boldsymbol{x} \qquad (2.7)$$

for any $h, u \in \boldsymbol{H}^1$.

For a given $f_i \in \boldsymbol{H}^1$, let g_i be a unique solution in $\boldsymbol{H}^2$ to the equation

$$L g_i = f_i \qquad (2.8)$$

Putting

$$E_{ij} = E_{ij}(U_0) = -U_0^2 \int_{\mathcal{T}} g_i(\boldsymbol{x}) f_j(\boldsymbol{x})\, d\boldsymbol{x} \qquad (2.9)$$

we get

$$\overline{D}_{ij} = a_{ij} + E_{ij} + E_{ji} \qquad (2.10)$$

with $f_i = b_i - \overline{b}_i$.

Noting that the operator $\mathcal{D}$ is one to one on $\boldsymbol{H}^2$ onto $\boldsymbol{H}^0$, we can introduce

$$H g(\boldsymbol{x}) = \mathcal{D}^{-1} \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla g(\boldsymbol{x}) \qquad (2.11)$$

As an operator from $\boldsymbol{H}^1$ to itself, H is compact and skew-symmetric and has the eigenfunctions $\{\phi_n\}_{n\geqslant 1}$ corresponding to the eigenvalues $\{\boldsymbol{i}\lambda_n\}_{n\geqslant 1}$ which reveal the following properties:

- λ_n are real and $\lim_{n\to\infty} \lambda_n = 0$
- $\{\phi_n\}_{n\geqslant 1}$ is a complete orthonormal set on $\boldsymbol{H}^1 \bigcap \boldsymbol{N}^{\perp}$, where $\boldsymbol{N}^{\perp}$ is the orthogonal complement of the null space of H in $\boldsymbol{H}^1$

- Each $h \in \boldsymbol{H}^1$ can be represented as

$$h = h_N + \sum_{n=1}^{\infty} \alpha_n \phi_n \tag{2.12}$$

with $h_N \in \boldsymbol{N}$ and $\alpha_n = \langle h, \phi_n \rangle_1$.

If for any $g \in \boldsymbol{H}^2$ and $f \in \boldsymbol{H}^0$ we consider the representations

$$\begin{aligned} g &= g_N + \sum_{n=1}^{\infty} \alpha_n \phi_n \\ \mathcal{D}^{-1} f &= (\mathcal{D}^{-1} f)_N + \sum_{n=1}^{\infty} \beta_n \phi_n \end{aligned} \tag{2.13}$$

we get

$$\begin{aligned} E_{ij}(U_0) &= U_0^2 \Big\{ \langle (\mathcal{D}^{-1} f_i)_N, (\mathcal{D}^{-1} f_j)_N \rangle_1 + \sum_{n=1}^{\infty} \frac{\beta_{in} \overline{\beta}_{jn}}{1 + \boldsymbol{i} U_0 \lambda_n} \Big\} \\ E_{ii}(U_0) &= U_0^2 \Big\{ \| (\mathcal{D}^{-1} f_i)_N \|_1^2 + \sum_{n=1}^{\infty} \frac{|\beta_{in}|^2}{1 + U_0^2 \lambda_n^2} \Big\} \end{aligned} \tag{2.14}$$

Such expressions were given by Bhattacharya et al. (1989) for the special case of solute dispersion in periodic porous media and by Timofte (1996) for a more general class of diffusions generated by a differential operator of the form (1.5), coefficients of which satisfy the conditions:

- a_{ij} are constant and the matrix $[a_{ij}]$ is symmetric and positive definite
- b_i are continuously differentiable periodic functions satisfying the condition $\operatorname{div} \boldsymbol{b} = 0$.

It is obvious that $E_{ij}(U_0) = \mathcal{O}(U_0^2)$ if $\langle (\mathcal{D}^{-1} f_i)_N, (\mathcal{D}^{-1} f_j)_N \rangle_1 \neq 0$ and $E_{ij}(U_0) = o(U_0^2)$ otherwise.

We also note that $\boldsymbol{N}$ is just the null space of $\boldsymbol{b} \cdot \nabla$ in $\boldsymbol{H}^1$.

Proposition 2.1. If $f_i \in \boldsymbol{H}^1 \bigcap \boldsymbol{N}$, then either $f_i = 0$, in which case $E_{ij}(U_0) = 0$ for each j, or $E_{ii}(U_0) = \mathcal{O}(U_0^2)$.

Proof. Since $f_i \in \boldsymbol{H}^1 \bigcap \boldsymbol{N}$, it follows that

$$\langle f_i, \mathcal{D}^{-1} f_i \rangle_1 = - \int_{\mathcal{T}} f_i(\boldsymbol{x})^2 \, d\boldsymbol{x} \tag{2.15}$$

If $f_i = 0$, obviously $E_{ij}(U_0) = 0$, for each j.

If $f_i \neq 0$, then $(\mathcal{D}^{-1} f_i)_N \neq 0$ and finally, $E_{ii}(U_0) = \mathcal{O}(U_0^2)$.

The converse proposition is also true.

As an operator on $\boldsymbol{H}^1$, $\boldsymbol{b} \cdot \nabla$ has the range $\mathcal{R}$ in $\boldsymbol{H}^0$

$$\mathcal{R} = \left\{ f \in \boldsymbol{H}^0 \big/ f = \boldsymbol{b} \cdot \nabla h, \text{ for some } h \in \boldsymbol{H}^1 \right\} \tag{2.16}$$

Theorem 2.2. If $f_i \in \mathcal{R}$, then

$$\lim_{U_0 \to \infty} E_{ii}(U_0) = \|h_i\|_1^2 \tag{2.17}$$

where h_i is the unique element of $\boldsymbol{H}^1 \bigcap \boldsymbol{N}^\perp$ such that $f_i = \boldsymbol{b} \cdot \nabla h_i$.

Also, for $i \neq j$

$$\begin{aligned} E_{ij}(U_0) &= \mathcal{O}(U_0) \simeq U_0 \langle h_i, \mathcal{D}^{-1} f_j \rangle_1 \\ E_{ji}(U_0) &= \mathcal{O}(U_0) \simeq -U_0 \langle \mathcal{D}^{-1} f_j, h_i \rangle_1 \end{aligned} \tag{2.18}$$

for large U_0.

In particular, if the inner products in Eqs (2.18) are zero, then $E_{ij}(U_0)$ and $E_{ji}(U_0)$ are $o(U_0)$.

Proof. Since $f_i \in \mathcal{R}$, it follows that

$$h_i = \sum_{n=1}^{\infty} \gamma_{in} \Phi_n \qquad \mathcal{D}^{-1} f_i = H h_i = \sum_{n=1}^{\infty} \boldsymbol{i} \lambda_n \gamma_{in} \Phi_n \tag{2.19}$$

Then

$$\lim_{U_0 \to \infty} E_{ii}(U_0) = \lim_{U_0 \to \infty} U_0^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2 |\gamma_{in}|^2}{1 + U_0^2 \lambda_n^2} = \sum_{n=1}^{\infty} |\gamma_{in}|^2 = \|h_i\|_1^2 \tag{2.20}$$

For $j \neq i$, we get

$$\begin{aligned} E_{ij}(U_0) &= U_0 \sum_{n=1}^{\infty} \frac{\boldsymbol{i} U_0 \lambda_n \gamma_{in} \overline{\beta}_{jn}}{1 + \boldsymbol{i} U_0 \lambda_n} = \mathcal{O}(U_0) \simeq \\ &\simeq U_0 \sum_{n=1}^{\infty} \gamma_{in} \overline{\beta}_{jn} = U_0 \langle h_i, \mathcal{D}^{-1} f_j \rangle_1 \\ E_{ji}(U_0) &= -U_0 \sum_{n=1}^{\infty} \frac{\boldsymbol{i} U_0 \lambda_n \beta_{jn} \overline{\gamma}_{in}}{1 + \boldsymbol{i} U_0 \lambda_n} = \mathcal{O}(U_0) \simeq \\ &\simeq -U_0 \sum_{n=1}^{\infty} \beta_{jn} \overline{\gamma}_{in} = -U_0 \langle \mathcal{D}^{-1} f_j, h_i \rangle_1 \end{aligned} \tag{2.21}$$

for large U_0.

These results form the extension of those given by Bhattacharya et al. (1989) for the case of solute dispersion in periodic media.

A more precise analysis of the asymptotic behaviour of the dispersion coefficients $\overline{D}_{ij}$ as functions of aU_0 can be done if we make more restrictive assumptions about $\boldsymbol{b}$ (see Timofte, 1996).

3. Applications

As a first example, we shall consider the problem of pure conduction in a layered medium.

The medium is assumed to possess thermophysical properties which vary only in the direction z, i.e.

$$\rho = \rho(z) \qquad c_p = c_p(z) \qquad \boldsymbol{K}_T = K_T(z)\boldsymbol{I} \tag{3.1}$$

and the phenomenological coefficients ρ, c_p and K_T are supposed to be integrable nonnegative definite periodic functions, having the period l_z.

With $\alpha = K_T/(\rho c_p)$, the evolution of the temperature $T(t, \boldsymbol{x})$ will be governed by the following equation

$$\frac{\partial T}{\partial t} = \alpha\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\Big) + \frac{1}{\rho c_p}\frac{dK_T}{dz}\frac{\partial T}{\partial z} \tag{3.2}$$

with the initial condition $T(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$.

Using general formulas given by the above central limit theorem we arrive at the following formulas for the effective volumetric specific heat $\overline{\rho c_p}$ and the effective thermal diffusivity dyadic $\overline{\alpha}$

$$\overline{\rho c_p} = \frac{1}{l_z}\int_0^{l_z} \rho(u) c_p(u)\, du \tag{3.3}$$

and

$$\begin{aligned}
&\overline{\alpha}_{11} = \overline{\alpha}_{22} = \frac{1}{\int_0^{l_z} \rho(u)c_p(u)\, du}\int_0^{l_z} K_T(u)\, du \\
&\overline{\alpha}_{33} = \frac{1}{\int_0^{l_z} \rho(u)c_p(u)\, du}\;\frac{1}{\int_0^{l_z} \frac{1}{K_T(u)}\, du} \\
&\overline{\alpha}_{ij} = 0 \qquad \text{for any } i \neq j, \quad i,j = \overline{1,3}
\end{aligned} \tag{3.4}$$

As a second example, let us consider the problem of internal energy dispersion in a layered periodic porous medium saturated with a viscous incompressible fluid.

Let us suppose that the thermophysical properties ρ, c_p and K_T are constant and the velocity field $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \left[U_0\left(1 + \cos\frac{2\pi z}{l_z}\right), U_0 \cos\frac{2\pi z}{l_z}, 0\right] \tag{3.5}$$

With $\alpha = K_T/(\rho c_p)$, the evolution of the temperature $T(t, \boldsymbol{x})$ will be governed by the following equation

$$\begin{aligned} \frac{\partial T}{\partial t} &= \alpha\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\Big) + \\ &- U_0\Big(1 + \cos\frac{2\pi z}{l_z}\Big)\frac{\partial T}{\partial x} - U_0 \cos\frac{2\pi z}{l_z}\frac{\partial T}{\partial y} \end{aligned} \tag{3.6}$$

with the initial condition $T(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$ imposed.

Obviously

$$\overline{\boldsymbol{U}} = [U_0, 0, 0] \tag{3.7}$$

Using the general formulas given by the above central limit theorem, we have

$$\begin{aligned} &\overline{\alpha}_{11} = \overline{\alpha}_{22} = \alpha + \frac{U_0^2 l_z^2}{8\pi^2\alpha} \qquad && \overline{\alpha}_{33} = \alpha \\ &\overline{\alpha}_{12} = \overline{\alpha}_{21} = \frac{U_0^2 l_z^2}{8\pi^2\alpha} && \overline{\alpha}_{13} = \overline{\alpha}_{31} = \overline{\alpha}_{23} = \overline{\alpha}_{32} = 0 \end{aligned} \tag{3.8}$$

This example proves that the macrotransport coefficients $\overline{\alpha}_{ij}$ grow quadratically with $l_z U_0$.

If we add a non-zero uniform velocity $U_0\overline{\omega}$ in the vertical direction, we get (cf Timofte, 1996)

$$\overline{\boldsymbol{U}} = [U_0, 0, U_0\overline{\omega}] \tag{3.9}$$

and

$$\begin{aligned} &\overline{\alpha}_{11} = \overline{\alpha}_{22} = \alpha + \frac{\alpha l_z^2 U_0^2}{2[(2\pi\alpha)^2 + (U_0 l_z\overline{\omega})^2]} \qquad && \overline{\alpha}_{33} = \alpha \\ &\overline{\alpha}_{12} = \overline{\alpha}_{21} = \frac{\alpha l_z^2 U_0^2}{2[(2\pi\alpha)^2 + (U_0 l_z\overline{\omega})^2]} && \overline{\alpha}_{13} = \overline{\alpha}_{31} = \overline{\alpha}_{23} = \overline{\alpha}_{32} = 0 \end{aligned} \tag{3.10}$$

It is obvious that for small values of $l_z U_0$, coefficients $\overline{\alpha}_{ij}$ depend quadratically on $l_z U_0$. However, as $l_z U_0 \to \infty$, each $\overline{\alpha}_{ij}$ becomes asymptotically constant.

We can also consider the problem of internal energy dispersion in an incompressible viscous fluid moving under laminar flow conditions between two parallel, insulated porous plates separated by a distance h. The upper plate moves at a velocity $\boldsymbol{U}_0$ parallel to it in the $\boldsymbol{x}$-direction. Simultaneously, there exists a uniform flow across the channel (in the negative $\boldsymbol{y}$-direction) at a constant velocity $\boldsymbol{v}_0$.

In this case, the fluid velocity field $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \frac{U_0 y}{h}\boldsymbol{i} - v_0 \boldsymbol{j} \tag{3.11}$$

At $t = 0$, an amount of heat is instantaneously added to our system over some region of the infinite domain between the plates in the form of some initial temperature distribution $T_0(\boldsymbol{x})$.

Assuming that the thermophysical properties ρ, c_p and K_T are constants, evolution of the temperature $T(t, \boldsymbol{x})$ will be governed by the following equation

$$\frac{\partial T}{\partial t} + \frac{U_0 y}{h}\frac{\partial T}{\partial x} - v_0 \frac{\partial T}{\partial y} = \alpha \Delta T \tag{3.12}$$

with the initial condition $T(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$ imposed. In Eq (3.12) $\alpha = K_T/(\rho c_p)$.

Introducing the dimensionless parameter

$$\beta = \frac{v_0 h}{\alpha} \tag{3.13}$$

and considering the incomplete gamma function

$$\gamma(n+1, \beta) = \int_0^\beta \xi^n \exp(-\xi)\, d\xi \qquad n = 0, 1, 2, ... \tag{3.14}$$

the macroscale thermal velocity $\overline{\boldsymbol{U}}$ is given by

$$\overline{\boldsymbol{U}} = \overline{U}\boldsymbol{i} \tag{3.15}$$

where

$$\overline{U} = \frac{U_0 \gamma(2, \beta)}{\beta \gamma(1, \beta)} \tag{3.16}$$

If we consider the mean axial fluid velocity

$$\overline{V} = \frac{U_0}{2} \tag{3.17}$$

we get

$$\frac{\overline{U}}{\overline{\overline{V}}} = \frac{2\gamma(2,\beta)}{\beta\gamma(1,\beta)} \tag{3.18}$$

So, the thermal velocity $\overline{U}$ is different from the mean axial fluid velocity $\overline{V}$.

As the cross flow velocity becomes $v_0 \to 0$, corresponding to the limit $\beta \to 0$, we see that

$$\lim_{\beta\to 0} \frac{\overline{U}}{\overline{\overline{V}}} = 1 \tag{3.19}$$

As $v_0 \to \infty$, then $\beta \to \infty$ and

$$\lim_{\beta\to\infty} \frac{\overline{U}}{\overline{\overline{V}}} = 0 \tag{3.20}$$

Using again the general formulas given by the above central limit theorem, we see that the only component of the effective thermal dispersivity dyadic $\overline{\alpha}$ which is different from zero is

$$\overline{\alpha}_{11} = \alpha + k(\beta)\frac{h^2\overline{V}^2}{\alpha} \tag{3.21}$$

with

$$k(\beta) = \frac{4}{\beta^4}\left[\frac{\gamma(2,\beta)}{\gamma(1,\beta)}\right]^2\left[2\frac{\gamma(2,\beta)}{\gamma(1,\beta)} + 3\frac{\gamma(3,\beta)}{\gamma(2,\beta)}\right] \tag{3.22}$$

The last example deals with the problem of internal energy dispersion in a two-dimensional periodic porous medium saturated with an incompressible viscous fluid having the velocity field $\boldsymbol{U}(\boldsymbol{x}) = U_0\boldsymbol{b}(\boldsymbol{x})$ given by

$$\begin{aligned} b_1(x,y) &= 2 - \cos[2\pi(\sin(2\pi x) - y)] \\ b_2(x,y) &= 2\pi\cos(2\pi x)b_1(x,y) \end{aligned} \tag{3.23}$$

We assume that the spatial scale parameter a is fixed at $a = 1$ and the phenomenological coefficients ρ, c_p and K_T are strictly positive constants.

Obviously, $\overline{b}_1 = 2$ and $\overline{b}_2 = 0$ (cf Timofte, 1996).

For this example, the closed-form solutions of the macrotransport coefficients $\overline{\alpha}_{ij}$ cannot be obtained.

However, the analytical theory developed in Section 2 shows that, as $U_0 \to \infty$, $\overline{\alpha}_{11} = \alpha + \mathcal{O}(U_0^2)$, $\overline{\alpha}_{22} = \alpha + \mathcal{O}(1)$ and $\overline{\alpha}_{12} = \overline{\alpha}_{21} = o(U_0)$.

This example reflects the influence of the geometry of the flow curves on the asymptotic behavior of macrotransport coefficients.

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Asymptotyki dyspersji termicznej w ośrodku periodycznym

Streszczenie

Korzystając z twierdzenia granicznego opisano pewną klasę termicznych zjawisk dyspersyjnych w ośrodkach niejednorodnych. Uwagę zwrócono na dyfuzje $\boldsymbol{X}_t$ generowane przez operator L o współczynnikach periodycznych. Z twierdzenia granicznego wynika, że $\lambda^{-\frac{1}{2}}(\boldsymbol{X}_{\lambda t} - \lambda U_0\bar{\boldsymbol{b}}t)$, $t \geqslant 0$ jest zbieżne w sensie dystrybucyjnym do ruchu Browna, gdy $\lambda \to \infty$. Powyżej $\bar{\boldsymbol{b}}$ jest wartością średnią $\boldsymbol{b}(\boldsymbol{x})$. W pracy analizowano wpływ macierzy dyspersji $\bar{\mathsf{D}}$ granicznego ruchu Browna na parametr prędkości U_0 i okres. Podano ścisłe warunki analityczne dla funkcji b_i, określające asymptotyczny charakter $\bar{\mathsf{D}}$ jako funkcji U_0. Celem zilustrowania obliczeń makrowspółczynników jako funkcji odpowiednich parametrów mikroskopowych podano przykłady szczególne.

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