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FATIGUE DAMAGE AND RELIABILITY ASSESSMENT OF CEMENTED HIP PROSTHESIS

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Continuum damage mechanics model is used in the finite element calculation of fatigue process in a cemented hip prosthesis. The cement mantle is assumed to be subjected to the fatigue damage resulting in evolution of the damage-induced material anisotropy and leading, eventually, to loosening of the prosthesis. Fatigue parameters and loads are assumed to be random variables. The reliability analysis is carried out for a linear approximation of the limit state function and, eventually, some corrections are presented to improve the fatigue lifetime probability distribution assessment.

Key words: hip prosthesis, fatigue damage, material anisotropy, reliability

1. Introduction

In total hip arthroplasty an implant is often fixed into the marrow cavity of bones using a poly-methylmethacrylite called a bone cement. It forms a mantle around the prosthesis. The mantle should assure the implant fixation to be durable and reliable. However, forces acting upon the prosthesis due to the human activity generate complex multiaxial stresses varying in time and resulting in deterioration of the endurance of the fixation due to cement-stem debonding and development of fatigue damage process in the cement mantle. It leads to loosening of the implants and requires replacement.

Modelling of the damage accumulation in cemented hip prostheses involves many uncertainties associated with the initial conditions, material properties, geometry and loading. Their random nature can be modelled by parameters assumed to be random variables or functions. Thus, the fatigue damage appears to be a stochastic process and the time to implant loosening is a random variable. The reliability of the prosthesis defined as the probability of loosening at a given time depends on the probabilistic features of the random quantities characterising the quality of materials and surgical treatment, level of conservation of technological regimes and the variety of the patient activity.

In the present contribution the loading and fatigue resistance of bone cement are assumed to be random. The damage evolves randomly in the volume of the cement mantle and in the both stem-cement and bone-cement interfaces. It results in formation of the pathways along which some wear particles can migrate from the joint space to the fractured sites (Anthony et al., 1990). These wear particles activate the bio-degradation of bone and accelerate the loosening. Therefore, the prosthesis lifetime depends substantially on the monotonically increasing process of damage in the cement mantle and in the interfaces. Due to the random fatigue resistance properties of material the damage process reaches a critical level at a random instant. With an appropriate definition of the critical damage this event is considered as the failure. Its probability means the probability of the prosthesis failure and is an increasing function of time.

2. Mechanical foundations

Verdonschot and Huiskes (1997) applied a model based on the theory of continuum damage mechanics to fatigue lifetime assessment of cemented hip prosthesis. In the three-dimensional finite element analysis a tensorial variable $\mathbf{D}$ is introduced to describe the amount of accumulated damage in multiple directions at every integration point ip and, eventually, the cracking leading to loosening of the implant. In what follows, the main features of the approach are sketched and some innovations accounting for an interaction of the damage intensity and damage rate resulting in a progressive damage growth model are proposed.

In the FEM the bone, cement and steel prosthesis are appropriately modelled by isoparametric elements. Some gap-elements are introduced to simulate the debonded stem-cement interface. All materials are considered to be elastic, initially isotropic with deterministic Young moduli. Loads acting on the prosthesis head and on the greater trochanter simulate the loading generated during the stance phase of walking. One parameter load description is admitted.

2.1. One-dimensional fatigue damage evolution model of bone cement

Fatigue experiments performed on bone cement specimens provide mostly a relation between the stress amplitude $\Delta\sigma$ and the number of cycles to failure N_F. Davies et al. (1987) proposed such a relation in the form of the S-N (Wöhler) curve

$$\log N_F = -m \log \Delta\sigma + \log c_0 \tag{2.1}$$

with the material parameters $m = 4.68$ and $\log c_0 = 8.77$ ($c_0 = 5.89 \cdot 10^8$) for the uncentrifuged bone cement. The stress amplitude $\Delta\sigma$ means the nominal stress amplitude and is defined as the ratio of the load amplitude to the nominal (initial) area of the specimen. Eq (2.1) defines also the number of cycles when a specimen fails due to the damage process undergoing in the material. The effective stress amplitude in the specimen $\Delta\widetilde{\sigma}$ can be determined in the following form

$$\Delta\widetilde{\sigma} = \frac{\Delta\sigma}{1-D}$$

where $0 \leqslant D \leqslant 1$ denotes the damage parameter so that $D = 0$ for undamaged material and $D = 1$ at failure. The damage parameter can be related to the effective area of a section, current stiffness of an element, local hardness, etc. Independently of the physical interpretation the damage parameter is monotonic increasing in the course of cyclic loading. Since the experimental evidence on the fatigue behaviour of bone cement is rather scant the formula for the damage parameter growth rate $\dot{D}$ can be only postulated. It is assumed that $\dot{D}$ depends on the current effective stress amplitude $\Delta\widetilde{\sigma}$. The evolution equation of the damage parameter can be defined in the following form

$$\dot{D} = c\Delta\widetilde{\sigma}^m = c\Big(\frac{\Delta\sigma}{1-D}\Big)^m \tag{2.2}$$

where $c = [(m+1)c_0]^{-1}$ results from the assumption of the S-N curve (Eq (2.1)), as the failure condition. Integrating Eq (2.2) over the time t, from $t = 0$ when $D = 0$ up to the failure $t = T_F$, when $D = 1$ and assuming one cycle as a time unit, i.e. $T_F = N_F$, the S-N curve is restored in the non-logarithmic form as follows

$$N_F = \frac{1}{(m+1)c}\Delta\sigma^{-m} = c_0\Delta\sigma^{-m}$$

Integrating the damage parameter evolution equation (2.2) over the time interval $n \in [0, N_1]$ provided that $D = D_0$ at $n = 0$ the damage intensity

$D = D_1$ at $n = N_1$ or the number of stress cycles $n = N_1$ that augments the damage intensity up to $D = D_1$, can be calculated as follows

$$D_1 = 1 - \sqrt[m+1]{(1-D_0)^{m+1} - \frac{\Delta\sigma^m}{c_0} N_1}$$

$$N_1 = c_0 \Delta\sigma^{-m} (1-D_0)^{m+1} \Big[1 - \Big(\frac{1-D_1}{1-D_0}\Big)^{m+1}\Big] \tag{2.3}$$

2.2. Three-dimensional damage-induced material anisotropy

Let us assume an initial damage intensity in a material element to be given by the damage tensor $\mathbf{D}$. The presence of the so-defined damage implicates some anisotropy of the initially undamaged isotropic material element. The principal directions of $\mathbf{D}$ determine the directions of the material orthotropy and the principal values D_1, D_2, D_3, affect the initial compliance $\mathbf{S}(\mathbf{0})$ and stiffness $\mathbf{C}(\mathbf{0})$ tensors. The stress-strain relation in the damaged material becomes (cf Lu and Chow, 1990)

$$\boldsymbol{\varepsilon} = \mathbf{S}(\mathbf{D}) : \boldsymbol{\sigma} = \widetilde{\mathbf{S}} : \boldsymbol{\sigma} \qquad \text{or} \qquad \boldsymbol{\sigma} = \mathbf{C}(\mathbf{D}) : \boldsymbol{\varepsilon} = \widetilde{\mathbf{C}} : \boldsymbol{\varepsilon}$$

where $\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{C}}$ denote, respectively, the compliance and stiffness tensors of the damaged material. The effective stress tensor $\widetilde{\boldsymbol{\sigma}}$ is also related to the existing damage by means of a damage effect tensor $\mathbf{M}(\mathbf{D})$ being a fourth rank symmetric tensor. Adopting the Voigt notation

$$[\boldsymbol{\sigma}] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^\top$$

$$[\boldsymbol{\varepsilon}] = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}]^\top$$

the damage effect tensor can be written in the principal co-ordinate system as follows

$$
\mathsf{M}(\mathsf{D}) = \begin{bmatrix}
\frac{1}{\widetilde{D}_1} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\widetilde{D}_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\widetilde{D}_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{\widetilde{D}_2\widetilde{D}_3}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{\widetilde{D}_3\widetilde{D}_1}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\widetilde{D}_1\widetilde{D}_2}}
\end{bmatrix}
$$

where $\widetilde{D}_i = 1 - D_i$ and the effective stress tensor is given by

$$
\widetilde{\boldsymbol{\sigma}} = \mathsf{M}(\mathsf{D}) : \boldsymbol{\sigma} \tag{2.4}
$$

Admitting the complementary elastic energy equivalence hypothesis

$$
W^e(\boldsymbol{\sigma}, \mathsf{D}) = W^e(\widetilde{\boldsymbol{\sigma}})
$$

which assumes the complementary elastic energy of the *damaged* material $2W^e(\boldsymbol{\sigma}, \mathsf{D}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \mathsf{S}(\mathsf{D}) : \boldsymbol{\sigma} = \boldsymbol{\sigma} : \widetilde{\mathsf{S}} : \boldsymbol{\sigma}$ to be equal to the corresponding complementary elastic energy of the *undamaged* material $2W^e(\widetilde{\boldsymbol{\sigma}}) = \widetilde{\boldsymbol{\sigma}} : \widetilde{\boldsymbol{\varepsilon}} = \widetilde{\boldsymbol{\sigma}} : \mathsf{S}(\mathbf{0}) : \widetilde{\boldsymbol{\sigma}}$ except that the stress variable is replaced by the *effective* stress (2.4) (cf Cordebois and Sidoroff, 1982), the damaged elastic compliance tensor $\widetilde{\mathsf{S}} = \mathsf{S}(\mathsf{D})$, can be obtained as follows

$$
\mathsf{S}(\mathsf{D}) = \mathsf{M}(\mathsf{D})^{\top} : \mathsf{S}(\mathbf{0}) : \mathsf{M}(\mathsf{D}) \tag{2.5}
$$

Since the undamaged elastic compliance matrix $\mathsf{S}(\mathbf{0})$ for an isotropic material is

$$
\mathsf{S}(\mathbf{0}) = \frac{1}{E}\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
$$

the compliance matrix $\widetilde{\mathsf{S}} = \mathsf{S}(\mathsf{D})$ for the damaged material can be calculated from Eq (2.5) and takes the following form

$$\widetilde{\mathsf{S}} = \mathsf{S}(\mathsf{D}) = \frac{1}{E}\begin{bmatrix} \dfrac{1}{\widetilde{D}_1^2} & \dfrac{-\nu}{\widetilde{D}_1\widetilde{D}_2} & \dfrac{-\nu}{\widetilde{D}_1\widetilde{D}_3} & 0 & 0 & 0 \\ \dfrac{-\nu}{\widetilde{D}_2\widetilde{D}_1} & \dfrac{1}{\widetilde{D}_2^2} & \dfrac{-\nu}{\widetilde{D}_2\widetilde{D}_3} & 0 & 0 & 0 \\ \dfrac{-\nu}{\widetilde{D}_3\widetilde{D}_1} & \dfrac{-\nu}{\widetilde{D}_3\widetilde{D}_2} & \dfrac{1}{\widetilde{D}_3^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{2(1+\nu)}{\widetilde{D}_2\widetilde{D}_3} & 0 & 0 \\ 070 & 0 & 0 & \dfrac{2(1+\nu)}{\widetilde{D}_3\widetilde{D}_1} & 0 & \\ 0 & 0 & 0 & 0 & 0 & \dfrac{2(1+\nu)}{\widetilde{D}_1\widetilde{D}_2} \end{bmatrix} \tag{2.6}$$

The stiffness matrix of the damaged material as the inverse of the compliance matrix $\widetilde{\mathsf{C}} = \mathsf{C}(\mathsf{D}) = \mathsf{S}^{-1}(\mathsf{D}) = \widetilde{\mathsf{C}}^{-1}$ can be written as follows

$$\widetilde{\mathsf{C}} = \mathsf{C}(\mathsf{D}) = \tag{2.7}$$

$$= G\begin{bmatrix} \dfrac{2(1-\nu)\widetilde{D}_1^2}{1-2\nu} & \dfrac{2\nu\widetilde{D}_1\widetilde{D}_2}{1-2\nu} & \dfrac{2\nu\widetilde{D}_1\widetilde{D}_3}{1-2\nu} & 0 & 0 & 0 \\ \dfrac{2\nu\widetilde{D}_2\widetilde{D}_1}{1-2\nu} & \dfrac{2(1-\nu)\widetilde{D}_2^2}{1-2\nu} & \dfrac{2\nu\widetilde{D}_2\widetilde{D}_3}{1-2\nu} & 0 & 0 & 0 \\ \dfrac{2\nu\widetilde{D}_3\widetilde{D}_1}{1-2\nu} & \dfrac{2\nu\widetilde{D}_3\widetilde{D}_2}{1-2\nu} & \dfrac{2(1-\nu)\widetilde{D}_3^2}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{D}_2\widetilde{D}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{D}_3\widetilde{D}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{D}_1\widetilde{D}_2 \end{bmatrix}$$

where $G = E/[2(1+\nu)]$ is the shear modulus of the initially isotropic material.

2.3. Three-dimensional fatigue damage failure

It is assumed that the current damage due to a load cycle is governed by the amplitudes $\Delta \boldsymbol{s}$ of the principal stresses $\boldsymbol{s} = [s_1, s_2, s_3]$, generated by the loading in a material element. The damage existing in the structure is defined by the damage tensor field $\mathsf{D}(\boldsymbol{x})$. Thus, the damage induces a material non-homogeneity and anisotropy described by the compliance $\widetilde{\mathsf{S}}(\boldsymbol{x}) = \mathsf{S}(\mathsf{D};\boldsymbol{x})$ or stiffness $\widetilde{\mathsf{C}}(\boldsymbol{x}) = \mathsf{C}(\mathsf{D};\boldsymbol{x})$ tensor field that have to be taken into account while looking for the stresses generated by a maximum and minimum of the loading cycle. It is obvious that the solution for a non-homogeneous anisotropic structure has to be found numerically, e.g. by using the finite element method (FEM). Assuming that the principal directions of the stress tensor do not change during the following load cycles the solutions yield the amplitudes of stresses $\Delta\boldsymbol{\sigma}^{ip} = \boldsymbol{\sigma}^{+,ip} - \boldsymbol{\sigma}^{-,ip}$ in every integration point ip and, eventually, the amplitudes of the principal stresses $\Delta\boldsymbol{s}^{ip} = [\Delta s_1^{ip}, \Delta s_2^{ip}, \Delta s_3^{ip}]$ for the ongoing loading history. All calculated stresses are nominal for material with the damage intensity defined by the damage tensor D^{ip}. Thus, the new damage develops on the planes perpendicular to the stress principal directions. In order to take into account the existing damage D^{ip} and to employ the damage evolution equation (2.2) the equivalent nominal stress amplitude $\Delta s_i^{ip,eq}$ corresponding to the undamaged material has to be calculated as

$$\Delta s_i^{ip,eq} = \Delta s_i^{ip}(1 - D_{ii}^{ip})$$

where D_{ii}^{ip} denotes the ii diagonal component of the existing damage tensor transformed into the principal stress co-ordinate system at the ip integration point.

The number of cycles to failure N_i^{ip} for every principal stress direction $i = 1, 2, 3$ can be calculated from the failure criterion $(2.3)_2$ with the equivalent nominal $\Delta s_i^{ip,eq}$ or calculated nominal Δs_i^{ip} stress amplitude, and with the damage intensity parameter D_{ii}^{ip}, i.e.

$$N_i^{ip} = c_0\Big(\frac{\Delta s_i^{ip,eq}}{1 - D_{ii}^{ip}}\Big)^{-m}(1 - D_{ii}^{ip}) = c_0(\Delta s_i^{ip})^{-m}(1 - D_{ii}^{ip})$$

The direction i_F^{ip} for which the number of cycles is the smallest one, i.e.

$$N_{i_F}^{ip} = \min_i N_i^{ip}$$

is assumed to be critical. Thus, on the plane perpendicular to the i_Fth axis the damage failure criterion is fulfilled after the $N_{i_F}^{ip}$ load cycles.

2.4. Damage-induced updating in the FEM procedure

The analysis described above is performed for all integration points involved in the FEM procedure. The integration point ip_F for which the number of cycles to failure $N_{i_F}^{ip}$ is the smallest one, i.e.

$$N_{i_F}^{ip_F} = \min_{ip} N_{i_F}^{ip} \tag{2.8}$$

is considered to be critical in the current calculation step and a damage option will be introduced at it.

Before going on with further calculation the increment of damage due to the $N_{i_F}^{ip}$ load cycles is added to the existing damage at all integration points. Hence, the diagonal components D_{ii}^{ip} of the existing damage tensor $\mathbf{D}^{ip}$ in the principal stress co-ordinate system at any integration point ip are updated to $D_{ii}^{ip,up}$ according to Eq $(2.3)_1$ that takes the following form

$$\begin{aligned} D_{ii}^{ip,up} &= 1 - \sqrt[m+1]{(1 - D_{ii}^{ip})^{m+1} - \frac{(\Delta s_i^{ip,eq})^m}{c_0} N_{i_F}^{ip_F}} = \\ &= 1 - (1 - D_{ii}^{ip}) \sqrt[m+1]{1 - \frac{(\Delta s_i^{ip})^m}{c_0(1 - D_{ii}^{ip})} N_{i_F}^{ip_F}} \end{aligned} \tag{2.9}$$

Eq (2.8) determines an integration point ip_F, where a diagonal component of the damage tensor $D_{i_F i_F}^{ip_F}$ represented in the principal stress co-ordinate system reaches its critical value $D_{i_F i_F}^{ip_F} = 1$ after $N_{i_F}^{ip_F}$ load cycles in the current calculation step. In order to model this event the cracking option offered in the MARK numerical FEM code (MARK Analysis Corporation, Palo Alto, CA) was employed in Verdonschot and Huiskes (1997). This option assumes a crack perpendicular to the critical principal direction. As a consequence it nullifies the material stiffness for tensile and shear in this direction and retains it while the crack is forced to be closed due to compressive stresses.

Moreover, in order to accelerate the calculation and to assure the results to be conservative the cracking option is applied at the integration points and for the principal directions of the updated damage tensor for which the corresponding principal values D_i^{ip} are not smaller than a given critical value D_c smaller than one, i.e.

$$D_i^{ip} \geqslant D_c$$

while $D_c = 0.95$ is assumed. Also in this paper such an approach is recommended in modelling of the fatigue damaging process of the bone cement.

The damage tensors $\mathbf{D}^{ip,up}$ are updated at every integration point ip, see Eq (2.9), over the cement mantle. In order to proceed with finite element calculation the mechanical properties of elements have to be defined. The updated damage intensity of the element is proposed to be determined as an average of the updated damage over the integration points belonging to the element $ip_j = 1, 2, ..., IP_j$, where IP_j is the number of the integration points in the jth element. Thus, in each finite element the updated damage tensors $\mathbf{D}^{ip_j,up}$ at the integration points have to be transformed into a common coordinate system and their components $\overline{D}_{kl}^{j,up}$ are then averaged as follows

$$\overline{\mathbf{D}}_{kl}^{j,up} = \frac{1}{IP_j} \sum_{ip_j=1}^{IP_j} D_{kl}^{ip_j,up} \tag{2.10}$$

The principal directions of the averaged updated damage tensors $\overline{\mathbf{D}}^{j,up}$ define the new orthotropy directions of the damaged material in every element of the finite element mesh. The compliance or stiffness matrix of an element involves the principal values $\overline{D}_i^{j,up}$ of the damage tensor $\overline{\mathbf{D}}^{j,up}$, see Eqs (2.6) or (2.7), respectively. Thus, substituting $\overline{D}_i^{j,up}$'s for D_i's into Eqs (2.6) or (2.7) the updated damage-induced compliance or stiffness matrix is accomplished.

Repeating this procedure for all elements the damage-induced anisotropy and non-homogeneity are updated for the next step of the FEM calculation. Now, the calculation is performed as proposed in Sections 2.3 and 2.4 with the assumption that the damage tensor $\mathbf{D}^{ip}$ describing there the existing damage at an integration point ip is equal to the respective averaged tensor $\overline{\mathbf{D}}^{j,up}$, i.e. $\mathbf{D}^{ip} = \overline{\mathbf{D}}^{j,up}$ for all ip's in the jth element.

2.5. Global damage criteria

Different damage measures may be formulated to define some criteria of a critical stage of the prosthesis and determining its lifetime N_F counted in the load cycle number. In Verdonschot and Huiskes (1997) defined the following damage measures:

- Total damage $D_{tot}(n)$ accumulated in the cement mantle

$$D_{tot}(n) = \sum_{ip=1}^{IP} \sum_{i=1}^{3} D_i^{ip}(n) \tag{2.11}$$

- Total number of cracks N_{crack} defined as

$$N_{crack}(n) = \sum_{ip=1}^{IP} \sum_{i=1}^{3} H\Big(D_i^{ip}(n) - D_c\Big) \tag{2.12}$$

where $H(\cdot)$ is the Heaviside function and n denotes the number of load cycles elapsed.

Admitting some critical values of the total damage $d_{tot,F}$ or number of cracks $n_{crack,F}$ the number of load cycles to failure $n_{F,D}$ or $n_{F,N}$ can be determined from the failure criterion of the following type

$$D(n_F) = d_F \tag{2.13}$$

where $D(n_F)$ denotes any damage measure, e.g. Eeqs (2.11) or (2.12), and n_F denotes the corresponding lifetime $n_{F,D}$ or $n_{F,N}$, respectively.

3. Reliability formulation

The material parameters of bone and cement, some geometrical parameters and the loading itself are of random nature and should be considered as random quantities. However, the complexity of the mechanical problem itself does not allow us to introduce very complex stochastic model without making it impractical. In the present contribution the fatigue strength parameters c_0 and m, see Eq (2.1), as well as the load parameter λ are assumed to be the random variables C_0, M and Λ, only. They are called the basic variables and compose a random vector $\boldsymbol{X} = [X_1, X_2, X_3] = [C_0, M, \Lambda]$ with known joint probability density function $f_X(\boldsymbol{x})$. Thus, any damage measure becomes a function of the random vector $\boldsymbol{X}$ for any number of cycles and the failure criterion, see Eq (2.13), defines a boundary $D(\boldsymbol{x}, n) = d_F$, between the safe $A_S(n) = \{\boldsymbol{x} : \ D(\boldsymbol{x}, n) \leqslant d_F\}$ and failure $A_F(n) = \{\boldsymbol{x} : \ D(\boldsymbol{x}, n) > d_F\}$, domains in the space $\mathcal{X}$ of samples of the basic variables $\boldsymbol{x} = [x_1, x_2, x_3]$. Hence, the failure at a given number of load cycles n_F is a random event with the probability $P_F(n_F, d_F)$ defined as follows

$$P_F(n_F, d_F) = \mathbb{P}\Big[D(\boldsymbol{X}, n_F) > d_F\Big] = \int\limits_{A_F(n_F)} f_X(\boldsymbol{x})\, d\boldsymbol{x} \tag{3.1}$$

where $\mathbb{P}[A]$ denotes the probability of the random event A. Since the damage measures are increasing functions of the number of fatigue cycles Eq (3.1) defines also the probability distribution $F_{N_F}(n|d_F)$ of the random fatigue lifetime $N_F(\boldsymbol{X}, d_F)$ for a given critical damage intensity d_F, i.e.

$$F_{N_F}(n|d_F) = \mathbb{P}\Big[N_F(\boldsymbol{X}, d_F) \leqslant n\Big] = \mathbb{P}\Big[D(\boldsymbol{X}, n_F) > d_F\Big] \tag{3.2}$$

It is obvious that numerical multidimensional integration in Eq (3.1) is impractical to be carried out because any verification of a point $\boldsymbol{x}$ whether or not it belongs to the integration domain (failure domain) $A_F(n_F)$ would require a complete solution of the very time-consuming fatigue problem discussed in the previous section. Therefore, a method with a minimal number of fatigue problem calculations has to be searched for to efficiently provide an estimate of the fatigue lifetime probability distribution function in the range of very small probability values. This range is the most interesting for the reliability and, eventually, usefulness assessment of the prosthesis.

4. Reliability assessment

The method developed by Y.T. Wu (Wu et al., 1989) and discussed by Wisching et al. (1991) seems to be especially useful in assessment of the probability distribution of the fatigue lifetime for limit state functions resulting from the involved finite element analysis. The point is that the number of cycles $N_F(\boldsymbol{x}, d_F)$ defined in the space $\mathcal{X}$ of samples of basic variables $\boldsymbol{x}$ is transformed into the space $\mathcal{U}$ of standard Gaussian variables U_k. Such a transformation denoted here as

$$u_i = \Phi^{-1}\Big[H_i(x_i|x_1, x_2, ..., x_{i-1})\Big] = T_i(\boldsymbol{x})$$

involves the standard Gaussian probability distributions $\Phi(u)$ and conditional probability distributions $H_i(x_i|x_1, x_2, ..., x_{i-1})$ of the basic random variables x_i. It was proposed by Rosenblatt (1952) and is a standard procedure in reliability analysis, cf Hohenbichler and Rackwitz (1981).

A linear approximation of the number of cycles $N_F(\boldsymbol{x}, d_F)$ in the space $\mathcal{U}$ is looked for. Thus, we need the $K+1$ solutions of the fatigue problem defined in the previous sections where K denotes the dimension of the basic variable vector $\boldsymbol{X}$ ($K = 3$ in the present contribution). It is proposed to look for the solutions in $\boldsymbol{u}_0 = \boldsymbol{T}(\widehat{\boldsymbol{X}}) = \boldsymbol{T}(\widehat{X}_1, ..., \widehat{X}_k, ..., \widehat{X}_K)$, where $\widehat{X}_i$ denotes the median of X_i, and in the K values of $\boldsymbol{x}_k = \boldsymbol{T}^{-1}(\boldsymbol{u}_k)$ shifted with respect to $\boldsymbol{u}_0$, e.g. $\boldsymbol{u}_k = [u_{0,1}, ..., u_{0,k} \pm 0.1, ..., u_{0,K}]$. It is easily seen that for independent basic variables $\boldsymbol{u}_0 = \boldsymbol{0}$. The sign "+" is chosen for the components k corresponding to loading and the sign "-" for those corresponding to the material strength. The linear approximation of the lifetime in $\mathcal{U}$

$$\widetilde{N}_F(\boldsymbol{u}, d_F) = N_F(\widehat{\boldsymbol{X}}, d_F) + \sum_{k=1}^{K} \frac{n_F(\boldsymbol{x}_k, d_F) - N_F(\widehat{\boldsymbol{X}}, d_F)}{\pm 0.1}(u_k - u_{0,k}) =$$

$$= N_F(\widehat{\boldsymbol{X}}, d_F) + \sum_{k=1}^{K} \delta_k (u_k - u_{0,k}) \tag{4.1}$$

is admitted in the limit state function $\widetilde{N}_F(\boldsymbol{x}, d_F) - n_j = 0$, see Eq (3.2), used in calculation of the probability that the number of load cycles to failure is smaller than a given number n_j, $j = 1, 2, ..., J$. Due to axisymmetry of the Gaussian probability measure in the $\mathcal{U}$ space the probability can be estimated as $P_F(n_j, d_F) \approx \Phi(-\beta_j)$, where β_j is the reliability index defined as the shortest distance between the linear approximation of the limit state function in $\mathcal{U}$ and the origin. For the approximation (4.1) the reliability index can be determined as

$$\beta_j = -\frac{N_F(\widehat{\boldsymbol{X}}, d_F) - n_j - \sum_{k=1}^{K} \delta_k u_{0,k}}{\sqrt{\sum_{k=1}^{K} \delta_k^2}}$$

The point

$$\boldsymbol{u}_j^* = \beta_j \frac{\boldsymbol{\delta}}{\sqrt{\sum_{k=1}^{K} \delta_k^2}}$$

belonging to the transformed limit state function and closest to the origin in $\mathcal{U}$ is called the design point. In order to improve the probability estimate the fatigue problem is again solved for the point $\boldsymbol{x}_j^* = \boldsymbol{T}(\boldsymbol{u}_j^*)$ that is the counterpart of $\boldsymbol{u}_j^*$ in the original space $\mathcal{X}$. The solution provides a corrected number of cycles to failure, i.e. $N_F(\boldsymbol{x}_j^*, d_F) = n_j^*$ corresponding to the previously calculated probability. Thus, the probability $\Phi(-\beta_j)$ is now admitted as the value of the lifetime probability distribution function (3.2) for the n_j^* load cycles, i.e.

$$F_{N_F}(n_j^* | d_F) \approx \Phi(-\beta_j)$$

Repeating the reliability analysis for J various numbers of load cycles n_j and solving the fatigue problem J times for every $\boldsymbol{x}_j^*$, i.e. for every retransformed design point $\boldsymbol{u}_j^*$, we obtain J values of the lifetime probability distribution or, equivalently, of the reliability index $\beta_j = \beta(n_j^* | d_F)$. Thus, the $K + J + 1$ solutions to the fatigue problem are necessary to estimate the relationship $\beta = \beta(n | d_F)$ between the reliability index β and the number of load cycles n over a segment of the load cycle number. A polynomial fit $\beta(n | d_F) \approx \sum_{q=0}^{Q} a_q n^q$, with a_q calculated from several approximate solutions appears usually to be a very good approximation of the relationship.

5. Concluding remarks

The fatigue damage analysis presented in the paper is based on the approach proposed by Verdonschot and Huiskes (1997). Some simplifying assumptions are introduced. They allow us to take into consideration the evolution of the damage-induced material anisotropy and non-homogeneity developing in the course of fatigue loading process. For successive calculation steps the stiffness of the cement mantle is related to the increasing damage intensity and updated. The damage evolution equation is also assumed to account for an increase of the fatigue damage growth rate due to the increase of the damage intensity itself. It is expected that the proposed modifications will alter the numerical results presented in Verdonschot's and Huiskes' (1997) paper where an asymptotically ceasing character of the total fatigue damage process was obtained. Decrease of the fatigue damage rate while no additional effect like bone remodelling, load sequence interaction, material hardening are taken into account seems rather surprising since most observations collected in fatigue experiments show an exploding character of the fatigue damage process. Now, the numerical calculations are being in progress. The first preliminary results, however, have already showed that the modifications significantly affect the fatigue damage process calculation. After accomplishing the computation the results will be published elsewhere.

The complex numerical procedure inhibits the application of any advanced reliability algorithms. The proposed reliability calculations are also extremely reduced by application of a linear approximation of the limit condition in the standard Gaussian probability space. It allows us to carry out the reliability calculations almost analytically without employing any reliability software.

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References

1. ANTHONY P.P., GIE G.A., HOWIE C.R., LING R.S.M., 1990, Localised Endosteal Bone Analysis in Relation to the Femoral Components of Cemented Total Hip Arthroplasties, *Journal of Bone Joint and Surgery*, **72-B**, 971-979

2. CORDEBOIS J.P., SIDOROFF F., 1982, Endommagement anisotrope en élasticité et plasticité, *Journal de Mécanique Théoretique et Appliquée*, Numéro spécial, 45-60

3. DAVIES J.P., BURKE D.W., O'CONNOR D.O., HARRIS W.H., 1987, Comparison of the Fatigue Characteristics of Centrifuged and Uncentrifuged Simplex P Bone Cement, *J. Orthop. Res.*, **5**, 366-371

4. HOHENBICHLER M., RACKWITZ R., 1981, Non-Normal Dependent Vectors in Structural Safety, *Journal of Engineering Mechanics, ASCE*, **107**, 6, 1227-1249

5. LU T.J., CHOW C.L., 1990). On Constitutive Equations of Inelastic Solids with Anisotropic Damage, *Theoretical and Applied Fracture Mechanics*, **14**, 187-218

6. MADSEN O.H., KRENK S., LIND N.C., 1986, *Methods of Structural Safety*, Prentience Hall, Englewoods Cliffs

7. ROSENBLATT M., 1952, Remarks on a Multivariate Transformations, *Ann. Math. Stat.*, **23**, 470-472

8. VERDONSCHOT N., HUISKES R., 1997, The Effects of Cement-Stem Debonding in THA on the Long-Term Failure Probability of Cement, *Journal of Biomechanics*, **30**, 8, 795-802

9. WIRSCHING P.H., TORNG T.Y., MARTIN W.S., 1991, Advanced Fatigue Reliability Analysis, *Int. J. Fatigue*, **13**, 5, 389-394

10. WU Y.T., MILLWATER H.R., CRUSE T.A., 1989, An Advanced Probability Structural Analysis Method for Implicit Performance Functions, *30th Structures, Structural Dynamics and Materials Conference*, Mobile, Alabama

Szacowanie uszkodzeń zmęczeniowych i niezawodności cementowanej protezy biodrowej

Streszczenie

Do modelowania procesu zniszczenia zmęczeniowego cementowanej protezy biodrowej zastosowano metodę elementów skończonych oraz kontynualny model mechaniki zniszczenia. Otoczka cementowa tworząca zamocowanie protezy narażona jest na uszkodzenia zmęczeniowe, które powodują powstanie i ewolucję anizotropii materiału oraz prowadzą do obluzowania protezy. Parametry materiałowe charakteryzujące proces zmęczenia oraz obciążenia przyjęto jako zmienne losowe. W analizie niezawodności przyjęto liniową aproksymację warunku granicznego określającego przydatność protezy i zaproponowano pewne modyfikacje pozwalające efektywnie szacować rozkład prawdopodobieństwa czasu życia protezy.

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