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STRAPDOWN INERTIAL NAVIGATION SYSTEM. PART 1 – NAVIGATION EQUATIONS

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Almost all aircraft are equipped with the Inertial Navigation Systems. The autonomous Inertial Systems are capable of calculating the navigational parameters of aircraft: position, velocity, and attitude, without external sources of information. In the paper the equations and algorithms of strapdown navigation system are derived and analysed. The research is focused on the calculation methods of the aircraft attitude.

Key words: aircraft, strapdown system, attitude

Notations

I-frame	–	inertial frame of reference
E-frame	–	earth-fixed frame of reference
B-frame	–	body-fixed frame of reference (the same as a measurement unit frame of reference)
T-frame	–	true frame of reference with local-level orientation; x-axis coincidens with local north, y-axis to local east, and z-axis local vertical
Z-frame	–	aircraft-fixed frame of reference
S-frame	–	any other coordinate systems
$\boldsymbol{x}_A$	–	column matrix expressed in the A-frame
$\boldsymbol{\omega}_A^{A-B}$	–	angular velocity vector of the A-frame relative to the B-frame expressed in the A-frame
$\boldsymbol{\varpi}_A^{A\to B}$	–	quaternion of angular velocity vector of the A-frame relative to the B-frame expressed in the A-frame

$\boldsymbol{\Omega}_A^{A\to B}$	–	skew-symmetric matrix formed by the components of the angular velocity vector $\boldsymbol{\omega}_A^{A\to B}$
D_B^A	–	transformation matrix of any vector from the A-frame to the B-frame
$\boldsymbol{\Lambda}_B^A$	–	transformation quaternion of any vector from the A-frame to the B-frame
$\boldsymbol{g}'$	–	gravitational acceleration vector
$\boldsymbol{g}$	–	gravity acceleration vector
$\boldsymbol{a}$	–	specific-force (acceleration) vector
$\boldsymbol{r}_A$	–	position vector expressed in the A-frame
R_N, R_M	–	radius of the curvature of the earth in the east-west and north-south directions, respectively (cf Kayton and Fried, 1976)
e	–	first eccentricity of the reference ellipsoid, $e^2 = (a^2 - b^2)/a^2$
a, b	–	the semi-major and semi-minor axes of the ellipsoid, respectively (cf Wei and Schwarz, 1990)
φ	–	latitude
λ	–	longitude
$(\bar{\cdot})$	–	quaternion conjugate notation
$\circ$	–	quaternions multiplication.

1. Introduction

The first Inertial Navigation System (INS) was developed during World War II by German scientists and V-2 ballistic missiles were equipped with it Pitman (1962). Since then the INS has become a standard equipment in ballistic missiles, almost all modern aircraft, and space ships. Strapdown Inertial Navigation Systems (SDINS) are sophisticated, autonomous, analytic systems for calculating the position, velocity and attitude of the vehicle which they are mounted on. In this system electromechanical gimbal platform is eliminated, and all sensors (accelerometers and gyroscopes) are mounted directly (i.e., strapdown) to the aircraft's fuselage and hence the quantities they measure (angular velocity, specific force) are in a B-frame. This kind of realisation requires knowledge of the specific force components in a navigation co-ordinate system (for example – a true local-level). The transformation from the B-frame to the navigation frame requires good knowledge of the vehicle attitude. The attitude can be found by numeric integration of proper differential equation, which describes the relation between one of the known attitude variables and the measured angular velocity vector.

Testing and verification of SDINS algorithms is a very important phase of the investigation and synthesis of SDINS.

2. Attitude variables

I-frame, E-frame, T-frame, and B-frame have been described in Polish Standards [14] and are summarised in Ortyl (1996).

The attitude of the vehicle relative to the navigation frame can be described by a few sets of variables; the most popular are: transformation matrix $\mathbf{D}_T^B$, quaternion $\mathbf{\Lambda}_T^B$, Cayley-Klein parameters $\mathbf{U}_T^B$, and attitude (Euler) angles Φ, Θ, Ψ. *Three first items (sets) are used to design the SDINS algorithms while* the attitude angles are used to the present the attitude – they have a direct physical interpretation. This procedure requires knowledge of the relation between sets of variables and especially to the attitude angles.

For example, on the base of elements of the transposition matrix $\mathbf{B} = (\mathbf{D}_T^B)^\top$ we can calculate the aircraft attitude angles: pitch, roll, and yaw (see [14]). Because of the ranges of the angles (yaw $0 \leq \Psi < 2\pi$, roll $-\pi < \Phi \leq \pi$) it is profitable to use the following functions $\cot(x/2)$ and $\tan(x/2)$, which are unique in these ranges. For the yaw angle Ψ we use the following relations (Ortyl. 1996)

$$
\begin{aligned}
&\Psi = \begin{cases} 2\text{arccot}\dfrac{\sqrt{1-b_{13}^2}+b_{11}}{b_{12}} & \text{for} \quad b_{12} \neq 0 \\ 0 & \text{for} \quad b_{12} = 0 \;\; \wedge \;\; b_{13} \neq 1 \end{cases} \\
&\Psi(k) = \Psi(k-1) \qquad\qquad \text{for} \quad b_{13} = -1
\end{aligned}
\tag{2.1}
$$

where b_{ij} $(i,j = 1,2,3)$ are the elements of matrix $\mathbf{B}$.

For the roll angle Φ we shall use the relations

$$
\begin{aligned}
&\Phi = 2\arctan\frac{b_{23}}{\sqrt{1-b_{13}^2}+b_{33}} \\
&\Phi = \pi \qquad \text{for} \quad \sqrt{1-b_{13}^2}+b_{33} = 0
\end{aligned}
\tag{2.2}
$$

The pitch angle Θ, whose range is $(-\pi/2, \pi/2)$, is determined from the function $\sin x$ (unique in this range)

$$\Theta = 2\arcsin(-b_{13}) \tag{2.3}$$

On other hand, knowing the attitude angles, we can derive transformation matrix

$$\mathbf{D}_B^T = \begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \begin{array}{c}\sin\Phi\sin\Theta\cos\Psi + \\ -\cos\Phi\sin\Psi\end{array} & \begin{array}{c}\sin\Phi\sin\Theta\sin\Psi + \\ +\cos\Phi\cos\Psi\end{array} & \sin\Phi\cos\Theta \\ \begin{array}{c}\sin\Phi\sin\Psi + \\ +\cos\Phi\sin\Theta\cos\Psi\end{array} & \begin{array}{c}\cos\Phi\sin\Theta\sin\Psi + \\ -\sin\Phi\cos\Psi\end{array} & \cos\Phi\cos\Theta \end{bmatrix} \tag{2.4}$$

The attitude matrix as a function of quaternion $\boldsymbol{\Lambda}$ is given in the following form

$$\mathbf{D}(\Lambda_T^B) = \begin{bmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_1\lambda_3+\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{bmatrix} \tag{2.5}$$

and the quaternion components are related to the attitude angles by

$$\begin{aligned} \lambda_0 &= \cos\frac{\Phi}{2}\cos\frac{\Theta}{2}\cos\frac{\Psi}{2} + \sin\frac{\Phi}{2}\sin\frac{\Theta}{2}\sin\frac{\Psi}{2} \\ \lambda_1 &= \sin\frac{\Phi}{2}\cos\frac{\Theta}{2}\cos\frac{\Psi}{2} - \cos\frac{\Phi}{2}\sin\frac{\Theta}{2}\sin\frac{\Psi}{2} \\ \lambda_2 &= \cos\frac{\Phi}{2}\sin\frac{\Theta}{2}\cos\frac{\Psi}{2} + \sin\frac{\Phi}{2}\cos\frac{\Theta}{2}\sin\frac{\Psi}{2} \\ \lambda_3 &= \cos\frac{\Phi}{2}\cos\frac{\Theta}{2}\sin\frac{\Psi}{2} - \sin\frac{\Phi}{2}\sin\frac{\Theta}{2}\cos\frac{\Psi}{2} \end{aligned} \tag{2.6}$$

Detailed derivations and other relations are presented by Ortyl (1996).

3. SDINS algorithms

The SDINS algorithms consist of the set of equations in which the navigational parameters of the aircraft: position, velocity, and attitude are the variables. The equations with parameters of position and velocity are similar to the well-known equations for the INS with gimbal platform. Therefore, they will be only summarily presented. The attention will be focused on the attitude equations.

3.1. Velocity and position equations

Each accelerometer measures the component of specific force $\boldsymbol{a}_I$ in the B-frame (cf Pinson, 1963), where $\boldsymbol{a}_I$ is the second derivative of position vector $\boldsymbol{r}_I$ in the I-frame minus the gravitational acceleration $\boldsymbol{g}'_I$ in the same frame

$$\ddot{\boldsymbol{r}}_I = \boldsymbol{a}_I + \boldsymbol{g}'_I \tag{3.1}$$

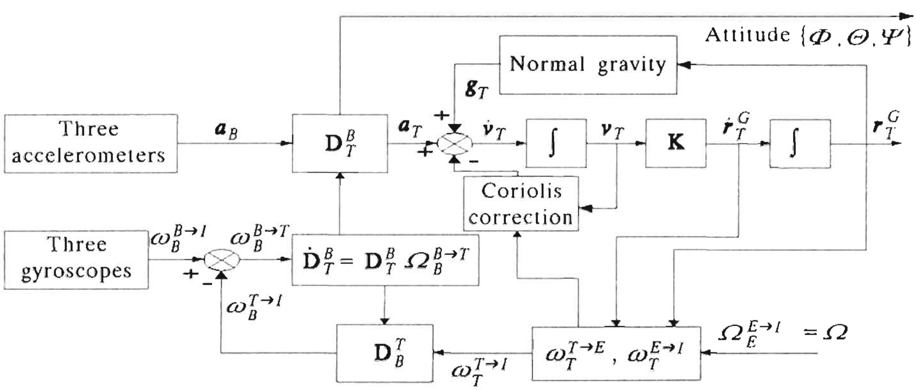

Fig. 1. SDINS realisation in the T-frame using the transformation matrix calculus

The navigation computer uses signals from accelerometers (components of $\boldsymbol{a}_I$) to determine the vector $\boldsymbol{r}_I$, which describes the aircraft's position in the I-frame.

Taking into account the following relation

$$g_T = g'_T - \Omega_T^{E\to I}\Omega_T^{E\to I} r_T \tag{3.2}$$

and the mathematical rules of differentiation with respect to time in the rotating T-frame, we can put the navigation equations of position, velocity, and attitude, respectively in the form:

— in spherical (geographical) coordinates (cf Bar-Itzhack and Goshen-Meskin, 1988)

$$\dot{\boldsymbol{r}}_T^G = \mathbf{K}\boldsymbol{v}_T \tag{3.3}$$

— in Cartesian coordinates

$$
\begin{aligned}
&\dot{\boldsymbol{r}}_T = \boldsymbol{v}_T - \boldsymbol{\Omega}_T^{T\to E}\boldsymbol{r}_T \qquad \text{or} \qquad \dot{\boldsymbol{r}}_E = \mathsf{D}_E^T\boldsymbol{v}_T \\
&\dot{\boldsymbol{v}}_T = \mathsf{D}_T^B\boldsymbol{a}_T - (2\boldsymbol{\Omega}_T^{E\to I} + \boldsymbol{\Omega}_T^{T\to E})\boldsymbol{v}_T + \boldsymbol{g}_T
\end{aligned}
\tag{3.4}
$$

where $\boldsymbol{r}_T^G = [\varphi, \lambda, h]^\top$ is the geographic (spherical) position vector, $\boldsymbol{r}_T = [x, y, z]_T^\top$, $\boldsymbol{r}_E = [x, y, z]_E^\top$ are the Cartesian coordinate vectors in T-frame and E-frame, respectively, $\boldsymbol{v}_T = [v_N, v_E, v_V]^\top$ is the velocity vector in T-frame, while

$$
\begin{aligned}
&\dot{\lambda} = \frac{v_E}{(R_N + h)\cos\varphi} \qquad \dot{\varphi} = \frac{v_N}{R_M + h} \qquad \dot{h} = -v_V \\
&R_M = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}} \approx a\Big[1 + e^2\Big(\frac{3}{2}\sin^2\varphi - 1\Big)\Big] \\
&R_N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \approx a\Big(1 + \frac{1}{2}e^2\sin^2\varphi\Big)
\end{aligned}
\tag{3.5}
$$

K – coefficient matrix

$$
\mathsf{K} = \begin{bmatrix} (R_M + h)^{-1} & 0 & 0 \\ 0 & [(R_N + h)\cos\varphi]^{-1} & 0 \\ 0 & 0 & -1 \end{bmatrix}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\omega_T^{T\to E} = [\rho_N, \rho_E, \rho_V]^\top = [\dot{\lambda}\cos\varphi, -\dot{\varphi}, -\dot{\lambda}\sin\varphi]^\top \\
&\omega_T^{E\to I} = [\Omega_N, \Omega_E, \Omega_V]^\top = [\Omega\cos\varphi, 0, -\Omega\sin\varphi]^\top
\end{aligned}
\tag{3.7}
$$

where $\Omega = 7.292116 \cdot 10^{-5}\,\mathrm{s}^{-1}$ is earth angular speed.

The ideal gravity vector in the T-frame is described as follows

$$
\boldsymbol{g}_T = [0, 0, g_\varphi]^\top \tag{3.8}
$$

where g_φ can be calculated from one of the expressions given in Bar-Itzhack (1997), Wei and Schwarz (1990) and [13], for example

$$
\begin{aligned}
&g_\varphi(h) = g_{0\varphi}\frac{R_\varphi^2}{(R_\varphi + h)^2} \qquad R_\varphi = g_{0\varphi}\frac{2}{g_3 + g_4\cos 2\varphi} \\
&g_{0\varphi} = g_0 + g_1\cos 2\varphi + g_2\cos^2 2\varphi
\end{aligned}
\tag{3.9}
$$

or

$$g_\varphi(h) = g_{0\varphi} + (g_3 + g_4 \cos 2\varphi)h \qquad g_0 = 9.80616\ \mathrm{m/s^2}$$
$$g_1 = g_0 \cdot 0.0026373 \qquad g_2 = g_0 \cdot 0.0000059$$
$$g_3 = 3.085462 \cdot 10^{-6}\ \mathrm{s^{-2}} \qquad g_4 = 2.57 \cdot 10^{-9}\ \mathrm{s^{-2}}$$

where R_φ is the nominal earth radius on the latitude φ.

3.2. Attitude equations

3.2.1. *Transformation matrix*

Differential equation which describes the relation between transformation matrix and measured angular velocity vector is as follows (cf Ortyl, 1996)

$$\dot{\mathbf{D}}_I^B = \mathbf{D}_I^B \boldsymbol{\Omega}_B^{B\to I} \tag{3.10}$$

We take into consideration another T-frame rotating relative to the I-frame at angular velocity $\omega_T^{T\to I} = \omega_T^{T\to E} + \omega_T^{E\to I}$. According to Eq (3.10) we can write

$$\boldsymbol{\Omega}_B^{B\to I} = \mathbf{D}_B^I \dot{\mathbf{D}}_I^B \qquad \boldsymbol{\Omega}_T^{T\to I} = \mathbf{D}_T^I \dot{\mathbf{D}}_I^T \tag{3.11}$$

Transformation matrix which transforms vectors from the B-frame to the T-frame is described as the product

$$\mathbf{D}_T^B = \mathbf{D}_T^I \mathbf{D}_I^B \tag{3.12}$$

The differentiation of Eq (3.12) yields

$$\dot{\mathbf{D}}_T^B = \mathbf{D}_T^I \dot{\mathbf{D}}_I^B + \dot{\mathbf{D}}_T^I \mathbf{D}_I^B \tag{3.13}$$

Transformation matrix should satisfy the orthogonal conditions

$$\begin{aligned} \mathbf{D}_I^T \mathbf{D}_T^I &= \mathbf{D}_T^I \mathbf{D}_I^T = \mathbf{I} \\ \mathbf{D}_I^B \mathbf{D}_B^I &= \mathbf{D}_B^I \mathbf{D}_I^B = \mathbf{I} \end{aligned} \tag{3.14}$$

Substituting Eqs (3.14) into Eq (3.13) we have

$$\dot{\mathbf{D}}_T^B = \mathbf{D}_T^I \mathbf{D}_I^B \mathbf{D}_B^I \dot{\mathbf{D}}_I^B + \dot{\mathbf{D}}_T^I \mathbf{D}_I^T \mathbf{D}_T^I \mathbf{D}_I^B = \mathbf{D}_T^B (\mathbf{D}_B^I \dot{\mathbf{D}}_I^B) + (\dot{\mathbf{D}}_T^I \mathbf{D}_I^T) \mathbf{D}_T^B \tag{3.15}$$

and using (3.11) we obtain

$$\dot{\mathbf{D}}_T^B = \mathbf{D}_T^B \boldsymbol{\Omega}_B^{B\to I} + (\boldsymbol{\Omega}_T^{T\to I})^\top \mathbf{D}_T^B = \mathbf{D}_T^B \boldsymbol{\Omega}_B^{B\to I} - \boldsymbol{\Omega}_T^{T\to I} \mathbf{D}_T^B \tag{3.16}$$

where the following relation is taken into account $(\Omega_T^{T\to I})^\top = -\Omega_T^{T\to I}$.

Last equation can be rewritten to the following notation (only for orthogonal matrix)

$$\dot{\mathbf{D}}_T^B = \mathbf{D}_T^B\left(\mathbf{\Omega}_B^{B\to I} - \mathbf{\Omega}_B^{T\to I}\right) \tag{3.17}$$

Using Eqs (3.3), $(3.4)_2$ and (3.17) we can obtain a flowchart of the SDINS realisation in the T-frame with transformation matrix calculus (see Fig.1).

3.2.2. Quaternions

Differential equation which represents the relation between quaternions and measured angular velocity vector as follows (cf Braniec and Shmiglevskiĭ, 1973; Friedland, 1978; Ortyl, 1996)

$$\dot{\mathbf{\Lambda}}_I^B(t) = \frac{1}{2}\overline{\mathbf{M}}(\omega_B^{B\to I})\mathbf{\Lambda}_I^B(t) \tag{3.18}$$

or using quaternions multiplication it is

$$\dot{\mathbf{\Lambda}}_I^B(t) = \frac{1}{2}\mathbf{\Lambda}_I^B(t) \circ \boldsymbol{\varpi}_B^{B\to I} \tag{3.19}$$

where $\mathbf{\Lambda}_I^B(t) = [\lambda_0, \lambda_1, \lambda_2, \lambda_3]^\top$ – quaternion, which describes the relation between the B-frame and the I-frame and norm which should satisfy the condition $\|\mathbf{\Lambda}\| = 1$; $\overline{\mathbf{M}}(\omega_B^{B\to I})$ – skew-symmetric matrix with dimensions 4×4 and

$$\overline{\mathbf{M}}(\omega_B^{B\to I}) = \begin{bmatrix} 0 & -(\omega_B^{B\to I})^\top \\ \omega_B^{B\to I} & (\Omega_B^{B\to I})^\top \end{bmatrix}$$

$\omega_B^{B\to I} = [p, q, r]^\top$ – measured angular velocity vector; $\boldsymbol{\varpi}_B^{B\to I} = [0, p, q, r]^\top$ – quaternion formed by the components of the measured angular velocity vector; Ω_B^{B-I} – skew-symmetric matrix formed by the components of the measured angular velocity vector and

$$\mathbf{\Omega}_B^{B\to I} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Eq (3.18) is linear in function of quaternion and angular velocity vector, so we can then write (cf Friedland, 1978)

$$\mathbf{\Lambda}(t) = \frac{1}{2}\mathbf{Q}(\mathbf{\Lambda})\omega(t) \tag{3.20}$$

where

$$\mathbf{Q}(\Lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} \tag{3.21}$$

We take into consideration the another T-frame, rotating relative to I-frame with angular velocity

$$\boldsymbol{\omega}_T^{T-I} = [\omega_N, \omega_E, \omega_V]^\top = \boldsymbol{\omega}_T^{T-E} + \boldsymbol{\omega}_T^{E-I} \tag{3.22}$$

where $\boldsymbol{\omega}_T^{T-E}$, $\boldsymbol{\omega}_T^{E-I}$ are given by Eqs (3.7), respectively.

According to relation (3.19) we can write

$$\frac{1}{2}\boldsymbol{\varpi}_B^{B-I} = \overline{\boldsymbol{\Lambda}}_I^B \circ \dot{\boldsymbol{\Lambda}}_I^B \tag{3.23}$$

and

$$\dot{\boldsymbol{\Lambda}}_I^T = \frac{1}{2}\boldsymbol{\Lambda}_I^T \boldsymbol{\varpi}_T^{T-I} \quad \Rightarrow \quad \frac{1}{2}\boldsymbol{\varpi}_T^{T-I} = \overline{\boldsymbol{\Lambda}}_I^T \circ \dot{\boldsymbol{\Lambda}}_I^T \tag{3.24}$$

Defining the two-side conjugate quaternion of (3.24) and taking in mind that for normalised quaternion the inverse quaternion is equal to the conjugate one, we have

$$\frac{1}{2}\overline{\boldsymbol{\varpi}}_T^{T-I} = -\frac{1}{2}\boldsymbol{\varpi}_T^{T-I} = \overline{\dot{\boldsymbol{\Lambda}}}_I^T \circ \boldsymbol{\Lambda}_I^T = \dot{\boldsymbol{\Lambda}}_T^I \circ \overline{\boldsymbol{\Lambda}}_T^I \tag{3.25}$$

The transformation quaternion between B-frame and T-frame is as follows (see Fig.2)

$$\boldsymbol{\Lambda}_T^B = \overline{\boldsymbol{\Lambda}}_I^T \circ \boldsymbol{\Lambda}_I^B = \boldsymbol{\Lambda}_T^I \circ \boldsymbol{\Lambda}_I^B \tag{3.26}$$

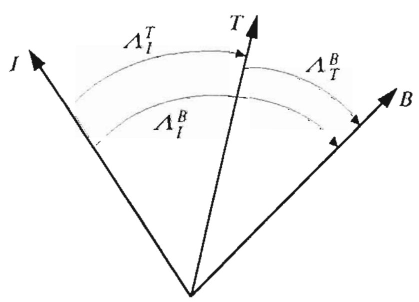

Fig. 2. The relations between frames I, T and B

Differentiation of (3.26) leads to

$$\dot{\boldsymbol{\Lambda}}_T^B = \dot{\boldsymbol{\Lambda}}_T^I \circ \boldsymbol{\Lambda}_I^B + \boldsymbol{\Lambda}_T^I \circ \dot{\boldsymbol{\Lambda}}_I^B \tag{3.27}$$

The norm of quaternion should satisfy the conditions

$$
\begin{aligned}
||\Lambda_T^I|| &= \overline{\Lambda}_T^I \circ \Lambda_T^I = \Lambda_T^I \circ \overline{\Lambda}_T^I = 1 \\
||\Lambda_I^B|| &= \overline{\Lambda}_I^B \circ \Lambda_I^B = \Lambda_I^B \circ \overline{\Lambda}_I^B = 1
\end{aligned}
\tag{3.28}
$$

Substituting Eqs (3.28) into Eq (3.27) and utilising Eq (3.26) we obtain the relation

$$
\dot{\Lambda}_T^B = (\dot{\Lambda}_T^I \circ \overline{\Lambda}_T^I) \circ \Lambda_T^B + \Lambda_T^B \circ (\overline{\Lambda}_I^B \circ \dot{\Lambda}_I^B)
\tag{3.29}
$$

Replacing the terms in brackets by (3.23) and (3.25) we obtain

$$
\dot{\Lambda}_T^B = \frac{1}{2}\Lambda_T^B \circ \varpi_B^{B\to I} - \frac{1}{2}\varpi_T^{T\to I} \circ \Lambda_T^B
\tag{3.30}
$$

To solve the above differential equation some its transformation should be done to avoid the quaternion subtraction. It is realised by the replacement of the quaternions by proper quaternion matrices. This way the quaternion multiplication is replaced by the matrix multiplication

$$
\dot{\Lambda}_T^B = \frac{1}{2}\overline{\mathsf{M}}(\omega_B^{B\to I})\Lambda_T^B - \frac{1}{2}\overline{\mathsf{M}}(\omega_T^{T\to I})\Lambda_T^B = \frac{1}{2}\Big[\overline{\mathsf{M}}(\omega_B^{B\to I}) - \mathsf{M}(\omega_T^{T\to I})\Big]\Lambda_T^B
\tag{3.31}
$$

where

$$
\mathsf{M}(\omega_T^{T\to I}) = \begin{bmatrix} 0 & -(\omega_T^{T\to I})^\top \\ \boldsymbol{\omega}_T^{T\to I} & \boldsymbol{\Omega}_T^{T\to I} \end{bmatrix}
\tag{3.32}
$$

A few methods for the solution of Eq (3.31) will be presented below. Now we note that it can be realised directly by the relation (3.26). To do that one should solve the differential equations (3.19) and (3.24) and next multiply the quaternions according to relation (3.26). The possibility of calculation of the T-frame attitude relative to the I-frame is an additional advantage of the above method. In the same way the measurement frame attitude relative to any navigational rotating frame (for example – the I-frame) motion of which is described by analytical relations can be calculated. The fundamentals of quaternion algebra which are useful in the mentioned method are given by Braniec and Shmiglevskiĭ (1973), Ortyl (1996).

Using Eqs (3.3), $(3.4)_2$, (3.19), (3.24), and (3.26) we can obtain the flowchart of SDINS realisation in the T-frame with quaternion calculus (Fig.3).

4. Integration of attitude differential equations

As it was earlier mentioned the main difference between Gimbal Inertial Navigation System and Strapdown Inertial Navigation System is connected

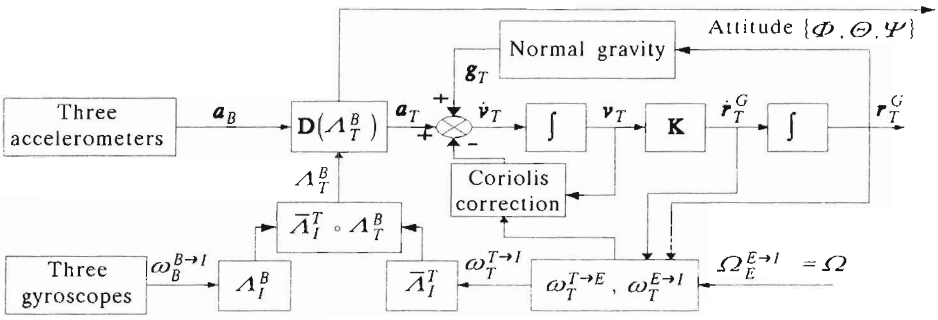

Fig. 3. SDINS realisation in the T-frame with quaternion calculus

with the modelling of the navigation co-ordinate system (in our case – the T-frame). It is realised in the SDINS by numeric integration of a chosen form of the attitude equation (3.17), (3.31) or (3.24), (3.19), and (3.26). The difference equation is obtained in each of these two cases but the calculation of the fundamental (transition) matrix or its equivalent is different in these two cases.

Assuming the constant coefficients within the integration step the standard solution of Eqs (3.17) is as follows (cf Ortyl, 1996)

$$\mathbf{D}(t) = \mathbf{D}(t-\tau)\exp\Xi \tag{4.1}$$

where

$\mathbf{D}(t-\tau)$	–	transformation matrix in the time $t-\tau$
$\exp\Xi$	–	transition matrix; $\exp\Xi = \mathbf{D}_n(t-\tau;t)$
Ξ	–	skew-symmetric matrix of rotation angles which are the integrals of the angular velocity vector elements (coefficients in Eq (3.17)).

The transition matrix in the standard solution is expanded into a Taylor series with n components

$$\mathbf{D}_n = \mathbf{I} + S_n^c\Xi + C_n'^c\Xi^2 \tag{4.2}$$

where S_n^c, C_n^c – Taylor series coefficients, for example for $n=3$, they are respectively $1-\xi_0^2/6$ and $1/2$, where ξ_0 is a resultant rotating angle during interval of integration and $\xi_0^2 = \sqrt{\xi_x^2+\xi_y^2+\xi_z^2}$.

The adequate standard solution of Eq (3.31) is as follows

$$\mathbf{\Lambda}(t) = \frac{1}{2}\mathbf{M}[\Lambda(t-\tau)]\mathbf{N}_n(t-\tau;t) \tag{4.3}$$

where

$\mathbf{N}_n(t-\tau;t)$ – correction quaternion during the step of integration $(t-\tau;t)$ corresponding to the transition matrix

$$\mathbf{N}_n(t-\tau;t) = \exp\Big(\frac{1}{2}\Xi^k\Big) = \Big[C_n^k, \xi_x S_n^k, \xi_y S_n^k, \xi_z S_n^k\Big]^\top$$

$\mathbf{M}[\Lambda(t-\tau)]$ – matrix of dimension 4×4 formed by the components of quaternion $\mathbf{\Lambda}(t-\tau)$

$$\mathbf{M}[\Lambda(t-\tau)] = \begin{bmatrix} \lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix}$$

$\Xi^k(t-\tau;t)$ – quaternion of the rotation angle, $\Xi^k(t-\tau;t) = \int_{t-\tau}^{t} \boldsymbol{\varpi}\, d\nu$

S_n^k, C_n^k – coefficients of the Taylor series, for example for $n=3$ they are $0.5(1-\xi_0^2/24)$ and $1-\xi_0^2/8$, respectively.

For details the reader is referred to Ortyl (1996).

After a few integration steps of the differential equations the transformation matrix loses its orthogonality and the quaternion changes its norm. To avoid these problems the periodical check and compensation of the errors should be performed to restore the orthogonality or the norm. Substituting other approximation, for example the Pade approximation of the transition matrix, for the slowly convergent Taylor series is the another solution of the above problem.

The simple algorithm of the transformation matrix orthogonalization has the form (cf Bar-Itzhack and Goshen-Meskin, 1988)

$$\widehat{\mathbf{D}} = \mathbf{D}\Big[\frac{3}{2}\mathbf{I} - \frac{1}{2}\mathbf{D}^\top\mathbf{D}\Big] \tag{4.4}$$

while the algorithm of the quaternion normalisation is as follows (cf Bar-Itzhack, 1971)

$$\widehat{\lambda}_i = \frac{\lambda_i}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} \qquad i=0,1,2,3 \tag{4.5}$$

According to (4.5) the quaternion normalisation means scaling it to the vector the arrow of which moves on the hypersphere with unit radius.

In the diagonal Pade approximation the transition matrix $\exp \Xi$ is replaced by the product of the matrix polynomials (cf Hyslop, 1987; Ortyl, 1996)

$$\mathbf{D}_{2n} = \Big[a_0\mathbf{I} + \ldots + a_n\Xi^n\Big]\Big[\mathbf{I} + \ldots + b_n\Xi^n\Big]^{-1} \tag{4.6}$$

For $n = 1$, the coefficients of Pade approximation are $a_0 = 1$, $a_1 = 1/2$, $b_1 = -1/2$, respectively. Thus, after the matrix inversion (4.6) has the form

$$\mathbf{D}_2^{Pade1} = \mathbf{I} + S_2^{Pc}\Big(\Xi + \frac{1}{2}\Xi^2\Big) \tag{4.7}$$

where $S_2^{Pc} = (1 + \xi_0^2/4)^{-1}$.

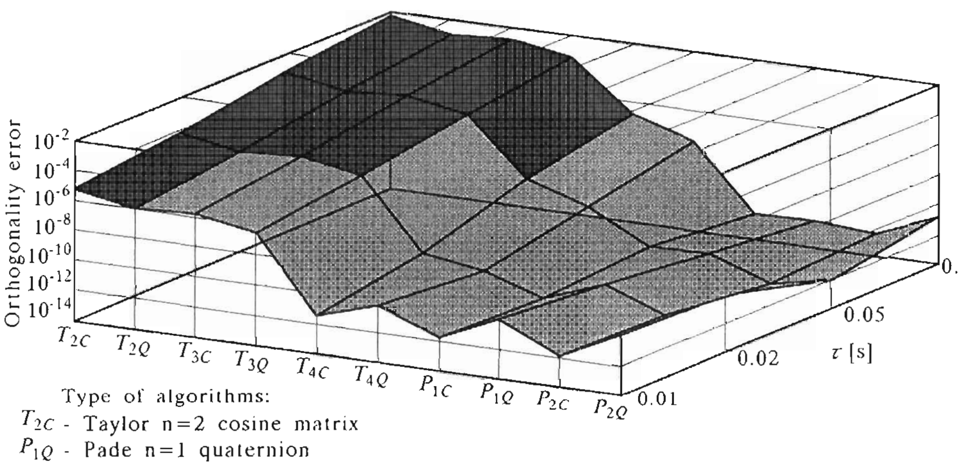

Fig. 4. Orthogonality errors after 1hr: T – Taylor series, C – transformation matrix, P – Pade approximation, Q – quaternion, n – algorithm rank

The Pade approximation can be only applied to the unitary matrices and can not be applied directly to the quaternion calculations. To omit that problem the Caley-Klein parameters are used instead of the quaternion parameters. The Caley-Klein parameters forms the unitary matrix with complex elements and these elements are the combination of the quaternion parameters. For $n = 1$ after many calculations we can find the proper quaternion of the correction in the form

$$\mathbf{N}_2^{Pade1} = \Big[C_2^{Pk}, S_2^{Pk}\xi_x, S_2^{Pk}\xi_y, S_2^{Pk}\xi_z\Big] \tag{4.8}$$

where

$$C_2^{Pk} = \frac{16 - \xi_0^2}{16 + \xi_0^2} \qquad\qquad S_2^{Pk} = \frac{8}{16 + \xi_0^2}$$

Several algorithms of the attitude calculation can be generated using the above presented procedure. These algorithms differs from each other in: mathematical notation, degree of reduction, method of error correction, and chosen integration time step. It is important to choose the best algorithm for the designed navigation system. To make a choice several simulations with different models and algorithms should be carried out.

The influence of a few algorithms on the orthogonality for different integration time steps after 1 h operation is presented in Fig.4. It is seen that algorithms with the Pade approximation reveal the smallest errors for all simulated integration time steps and correction of the orthogonality error can be omitted.

5. Conclusions

The SDINS equations for calculation of the navigation parameters of the aircraft: velocity, position, and attitude have been presented in the paper. The "standard" attitude equations of SDINS employing the transformation matrix and the quaternions with the Taylor series expansion have been compared with the algorithms employing the Pade approximation for both the mentioned notations.

For the manoeuvrable aircraft the equations based on the Pade approximation should be used in design of navigation system, simulations of its behaviour, derivation of navigation error equations and for further studies.

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Inercjany bezkardanowy system nawigacji. Część 1 – Równania nawigacji

Streszczenie

Prawie każdy statek powietrzny jest wyposażony w inercjalny system nawigacji (INS). Autonomiczny inercjalny system nawigacji jest w stanie wyznaczyć parametry nawigacyjne: pozycję, prędkość i orientację przestrzenną bez korzystania z zewnętrznych źródeł informacji. W niniejszej pracy wyprowadzono i dokonano analizy algorytmów Inercjalnego Bezkardanowego Systemu Nawigacji. Szczególną uwagę poświęcono metodom wyznaczenia orientacji przestrzennej SP.

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