JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
3, **36**, 1998

HOMOGENISATION THEORIES APPLIED TO POROUS MEDIA MECHANICS

DARIUSZ ŁYDŻBA

Institute of Geotechnics and Hydrotechnics, Wrocław University of Technology
e-mail: lydzba@i10.igh.pwr.wroc.pl

The present paper surveys and compares methods of homogenisation applied to porous media mechanics. In particular, it details the methods: volumetric averaging, averaging by weight, self-consistent, asymptotic homogenisation method for periodic structures as well as two-scale convergence. The closure hypothesis for a representative volume element has been discussed.

Key words: porous media, representative volume element, homogenisation

1. Introduction

It is obvious that the mathematical description of physical processes occurring in a real material depends on the observation scale. Hence, while a sample of given material in a natural observation scale might be treated as homogeneous, microscopically it is clearly heterogeneous. Thus, a description of such a material in the natural observation scale is, within the framework of continuous media mechanics, a certain approximation, which also refers to any such-scale investigation into the constitutive behaviour of the medium, is in fact a relationship between averaged physical fields observed during the experiment.

In case we consider a porous material – we mostly deal with a multiphase medium. This medium is built of the solid phase, making up the skeleton, and empty spaces joined together or not, filled with a liquid, a gas or a liquid and gas at the same time. In addition, the phase separation surfaces display discontinuous step-like changes of at least one parameter of the medium.

When analysing such media, two different mathematical modelling approaches are possible. The first, the so-called macroscopic one: the distances of the

order of pore diameters are ignored, and only macroscopic distances are taken into account. What we look for in that case is the mathematical description of the hypothetical homogeneous continuous medium, in which each point of the space is occupied simultaneously by all the medium phases (components) treated as continuums. The laws of conservation are introduced at the macroscopic level analogous to the laws of equilibrium for a single-phase continuous medium, supplemented with additional expressions that take into account interactions between the phases. The constitutive equations are postulated on the basis of the analysis of the second law of thermodynamics. Such a way of modelling is called the theory of mixture. A survey of these theories was shown by, among others (Bedford and Drumheller, 1983).

The other kind of modelling is the microscopic approach: each distinct component of the mixture, at the level of heterogeneity, is treated as a continuous medium with its own constitutive equation, the laws of conservation and the boundary conditions at the boundary of phase separation. Obviously, such a way of modelling is only possible when the specific dimension of a particular component agrees with the basic hypothesis of continuum mechanics, i.e. it is much larger than the molecular distances. Then the microscopic level equations are averaged to obtain macroscopic equations. The effect of this procedure, i.e. the passage with a mathematical description from the pores scale (heterogeneity) to the macro-scale, is the equivalent macroscopic description of the substitute *hypothetical homogeneous continuous medium*. Such a procedure is referred to as the method of homogenisation (Sanchez-Palencia, 1980, 1987).

The common assumption of the two way of modelling mentioned above, i.e. the theory of mixture and the theory of homogenisation, is the postulate about a possibility of defining the so-called *statistically homogeneous specimen* or Representative Volume Element (RVE) in the medium under consideration. What we understand by this notion is the smallest volume of the considered medium, which contains all the information necessary for the complete description of the structure and properties of the entire material (de Buhan and Taliercio, 1991). Following modelling, either according to the theory of mixtures or the theory of homogenisation, the representative medium volume, exhibiting a multiphase composition and a heterogeneous structure, is, in the hypothetical homogeneous medium, assigned a homogeneous structure in which each point is occupied by all the phases simultaneously (Fig.1).

Definitely, the assumption about the statistical homogeneity of the material and the choice of the basic representative element is an extremely complicated matter, which can be illustrated by, for instance, the effect of sample size in lab tests on material strength. However, further on we assume that the

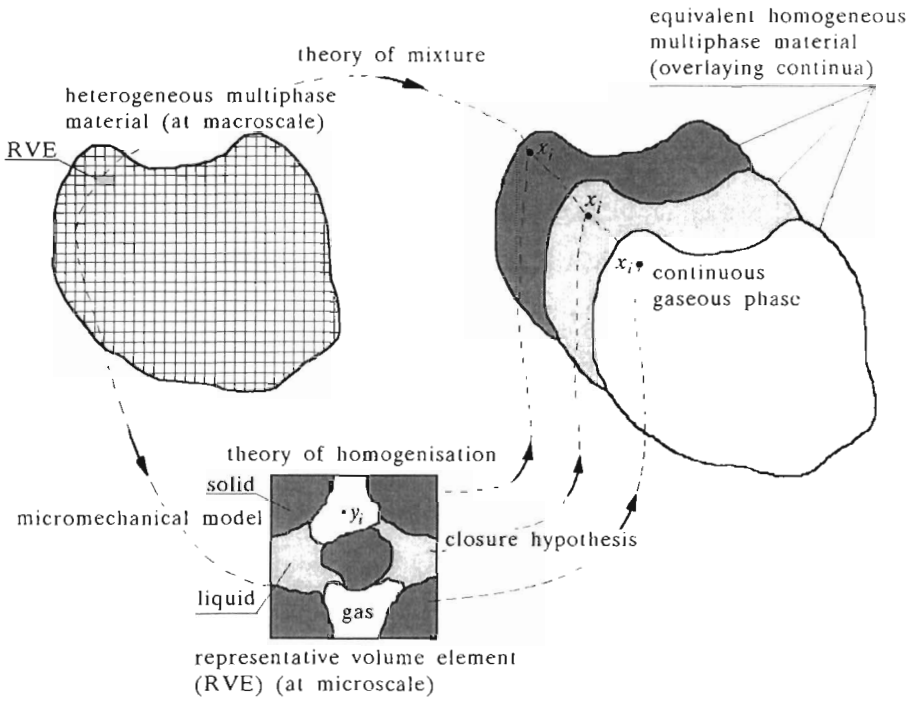

Fig. 1. Schematic view at modelling approaches applied to porous media mechanics

choice of such a basic unit (RVE) is feasible.

In case we seek a mathematical description within the frame of the mixture theory, the detailed knowledge of this basic element of the medium is not necessary whereas in the case of the homogenisation theory – it is (Fig.1). The choice of this element (i.e. RVE) and the way it is modelled make up the basic difference between different variants of the homogenisation method (see e.g. Hashin, 1983; Hill, 1965; Suquet, 1987).

The present paper is aimed at reviewing and discussing different techniques of homogenisation applied to porous media mechanics. Special emphasis has been laid on methods used in the mechanics of multiphase media, the form of the so-called closure hypothesis, i.e. boundary conditions formulated at the boundary of RVE and arriving at the local description of a multiphase medium. The presentation has been confined to deterministic methods only, i.e. those where RVE geometry and properties are determined unanimously. Homogenisation methods for random media are presented by, for instance, Kröner (1972), Sobczyk (1982), Rubinstain and Torquato (1989), Emeriault et al. (1996).

2. Homogenisation process – general remarks

According to what we said above, the objective of the homogenisation method is to find the equivalent macroscopic description of a process under study when its micro-scale description is known entirely. Hence, the following data are to be available in this scale (micro):

- equilibrium equations for each phase (each component) of the system,
- boundary condition at the phase separation boundary,
- constitutive relationships including the parameters,
- geometry.

In the macroscopic scale (natural scale of observation), the homogenisation process is to yield (Auriault, 1991):

- the equilibrium equations,
- the constitutive relationships and the effective parameters,
- the localisation law, i.e. the relationship which allows one to determine all of the physical fields on the microscopic level when macroscopic physical fields are known.

The condition necessary for homogenisation to be possible is, certainly, a small size of RVE when compared to the size of the porous medium in question.

2.1. Micro- and macroscopic variables

Since, following the process of homogenisation, the discrete structure of RVE is identified, in the natural observation scale, by a material point with the spatial co-ordinate x, in the process of the passage with a mathematical description from the heterogeneity to the macroscopic scale, two distinct families of physical variables should be distinguished, i.e. macroscopic variables, describing the state of the homogeneous medium the properties of which we are looking for and microscopic variables – describing the state of the medium within RVE (Fig.2).

Macroscopic variables are typically represented by averaged microscopic variables, in general by the volumetric means calculated in the RVE. Thus, for instance, if $\rho(y)$ represents the medium material point density in the micro-scale, its related macroscopic variable $\langle\rho\rangle$ is determined by

$$\langle\rho\rangle = \frac{1}{\|\Omega\|}\int_{\Omega}\rho(y)\,d\Omega \tag{2.1}$$

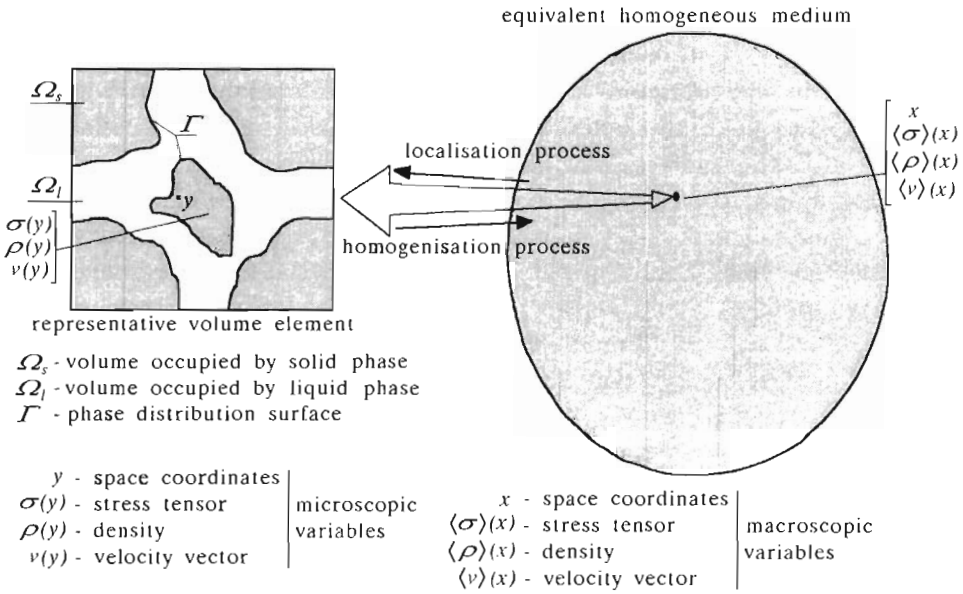

Fig. 2. Two families of physical variables used in a homogenisation theory

where $\|\Omega\|$ is the measure of RVE volume.

Referring to density, the above dependence properly defines the macroscopic variable in the sense of physics. A problem appears when the volumetric mean from a given microscopic variable fails to correspond to the classical meaning of this physical variable. This can be exemplified by stress tensor. The macroscopic variable should represent the value of force per unit area, hence it should be a stress mean value, in the micro-scale, calculated per unit area – not a volumetric mean. The situation is analogous when we consider the velocity of liquid filtration through a porous medium. The macroscopic value should represent the flux, hence it should be calculated, like stress tensor, as an area mean.

The correct process of homogenisation should lead to the macroscopic description expressed by the respective, from the physics viewpoint, macroscopic variables.

2.2. Local description – the role of the material intrinsic structure

In the case of liquid-saturated porous media, the formulation of the micromechanical issue (local description) which, when solved after averaging, is supposed to bring about the macroscopic description not only depends on the kind

of fluid filling the porous space (e.g. chemical activity with the solid constituting the skeleton), but also on the intrinsic structure of the porous medium. Mostly (and in fact commonly), when formulating a local model it is assumed that the porous medium skeleton is built of an elastic material while the liquid which fills the pores is described by means of the Navier-Stokes equations for a barotropic viscous Newtonian liquid (see e.g. Gilbert, 1990; Nigmatulin, 1981; Sanchez-Palencia, 1974; Whitaker, 1986). Nevertheless, the two examples presented below point out the role of the material intrinsic structure in the behaviour of the fluid in the interstitial space, hence its mathematical description and the way of modelling on the local level.

Let us analyse the behaviour of a gas in the pore space with a given pore diameter, d_a. Taking into account the value δ of the mean free path of a gas molecule, i.e. the average distance the molecules travel between two successive collisions, we can distinguish at least three extreme cases of gas behaviour:

- $d_a/\delta \gg 1$ – gas behaviour in the pore space is solely dominated by collisions of gas molecules between one another. From the mechanical viewpoint, the gas can be modelled as a barotropic viscous Newtonian liquid

- $d_a/\delta \ll 1$ – in this case, the gas flow is determined by collisions of gas molecules with the surface of solid. Certainly, the basic postulate of continuum mechanics is not satisfied in this case. Mathematical modelling of the flow can be performed following statistical mechanics or phenomenologically, by treating the solid with the gas as a mixture in which the flow process is governed by, e.g. the Fick molecular diffusion

- $d_a/\delta \approx 1$ – the collision between a gas molecule and the surface of the solid occur equally often. Then the so-called dusty gas model (Kaviany, 1991), obtained within the modified kinetic theory of gas is used for description.

Below we have shown the values of the mean free path for three kinds of gas (calculated under the so-called normal conditions, i.e. atmospheric pressure, temperature $T = 294°\mathrm{K}$) (Łydżba, 1996): carbon dioxide $\delta = 57.4\,\mathrm{nm}$, steam $\delta = 43\,\mathrm{nm}$, nitrogen $\delta = 67.7\,\mathrm{nm}$.

The other example of the role played by the pore size in terms of, this time, the behaviour of a liquid is provided by the laboratory investigation results obtained by Gee et al. (1990). These studies were aimed at determining the nature of the continual-to-molecular behaviour transition of a thin layer of a fluid between two solid plates. It was noticed that, depending on the layer

thickness, the fluid behaves like either a viscous Newtonian liquid or a solid. It was also found out that the fluid layers behaving like a solid, subjected the shear stress critical value tend to behave like a ductile solid subject to plastic deformation.

The two examples of interstitial fluid behaviour we discussed above clearly imply that the local description formulated for a given porous medium must also be adequate to its intrinsic structure. Admittedly, many porous media exhibit the so-called hierarchical structure (Kisiel, 1982; Lasoń, 1988). This is illustrated by the intrinsic structure of silty soils presented in Fig.3. In this case, instead of two scales of observation, at least three are distinguished: micro-scale, meso-scale and macro-scale. The process of homogenisation consists of subsequent passages from one scale into another (Mei and Auriault, 1989) to arrive at the description in the natural observation scale.

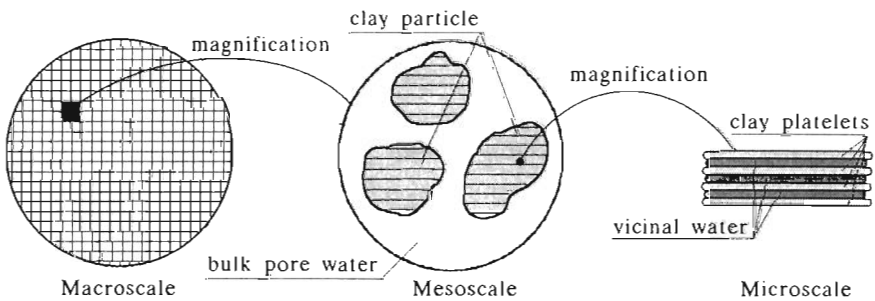

Fig. 3. Multi-scale model for clay used by Murad et al. (1995)

Sometimes the kind of the process under consideration determines the way of the passage from one scale into another, i.e. a given method of homogenisation and then the subsequent transitions between the scales are sometimes done using different homogenisation techniques or by merging the theories of mixture and homogenisation. Such a modelling was exemplified by the transfer from the micro-, through meso-, to macro-scale, applied by Murad et al. (1995) in order to obtain the mathematical model of the process of consolidation of swelling silty soils.

A simultaneous occurrence of diffusive-sorptive and filtration processes in a porous medium was taken into account by, among others, Łydżba (1996), (1997) and Łydżba and Auriault (1996). In this case there were distinguished only two scales: micro and macro. The diffusive processes in capillaries of smaller diameters were introduced by appropriate formulation of the local description of RVE, i.e. the solid together with the gas diffusing in the micropores was modelled within the framework of diffuso-elasticity (a phenomenological

approach based theory) whereas the condition of the gas in the macropores was described by means of the Navier-Stokes equations for a barotropic viscous Newtonian liquid.

To generalise, the formulation of the micromechanical model for a given porous medium must have the character of *a dedicated description*, i.e. in possibly the most detailed way take into account its discrete intrinsic structure as well as the kind of the liquid filling the pores.

2.3. Conditions at the RVE boundary – closure hypothesis

It is beyond doubt that the formulated local description valid within the RVE is not enough to determine the localisation law mentioned before. This law is supposed to render it possible to calculate the value and microscopic distribution of the physical fields when the values of macroscopic physical variables are known. Hence, it is necessary to introduce boundary conditions at the boundary RVE-the rest of the material. *These conditions must reproduce, as closely as possible, the in situ state of RVE inside the material under consideration* (Suquet, 1987).

Noticeably, the introduction of a specific form of boundary conditions into the local description might also be interpreted as imposing a limitation on the class of possible interactions between the RVE and the rest of the material. This limitation is often referred to as the closure hypothesis as it allows separating the RVE from the rest of the material being considered, hence making the material behaviour analysis confined to the analysis of RVE only.

In the case of microscopically heterogeneous solids, such as polycrystals or solids with voids or cracks, the closure hypothesis is typically the condition of uniform stresses or deformations (Hashin, 1983, 1988; Kachanov, 1992). The introduction of such a hypothesis form is justifiable only when void or crack distribution in the solid is dilute and the size of a single heterogeneity is much smaller than that of RVE.

When analysing the process of liquid flow through a porous medium, the only closure hypothesis known to the author, regardless of the homogenisation (averaging) technique used, is the condition of local periodicity (see e.g. Sanchez-Palencia, 1980; Barrere et al., 1992). However, it should be noted that the application of this condition is physically justified only for porous media which exhibit a periodic structure, i.e. generated from a unit cell by the condition of periodicity. Of course such a unit cell constitutes the RVE.

Only in the case of flow through random media, instead of the local periodicity hypothesis, the condition of the local stationariness of the fluid velocity field is used (Rubinstain and Torquato, 1989).

3. Different homogenisation techniques

In general, among homogenisation (averaging) methods one can distinguish, depending on the kind of heterogeneous medium the analyses are applicable to, two main groups of techniques: methods applied solely to single-phase micro-heterogeneous solids and those applied to multiphase media. Since, in the case of porous media, we typically deal with many phases occurring in the medium, the present discussion is focused on methods dedicated to multiphase media. For instance, no homogenisation methods applied to stratified composite media (see e.g. Woźniak (1995), as one in this group) have been discussed here.

3.1. Self-consistent scheme

This method was introduced by Hershey and Kröner (Zaoui, 1987) as an approximation scheme to determine elasticity effective parameters for polycrystalline media. The basic idea of this method is to replace complicated interactions between an element of the aggregate and any other element by the interaction between an element and a substitute homogeneous continuous medium displaying the sought effective parameters of elasticity (Fig.4).

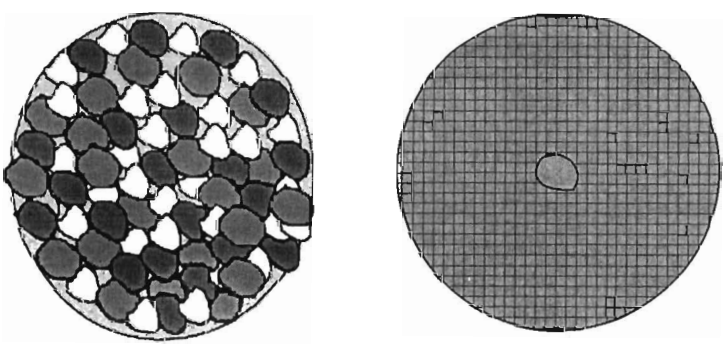

Fig. 4. Interactions between grains modelled according to the self-consistent scheme

For a macro-homogeneous medium (natural observation scale), the substitute continuous medium surrounding a given aggregate element is treated as infinite with uniform boundary conditions (at infinity).

Below, we will show the framework of the self-consistent calculation scheme for a polycrystalline-structure elastic medium (for details see Buisson et al., 1990).

The assumption that RVE is composed of N-grains with different elastic stiffness values is accepted. If, for each grain belonging to the RVE, we define the following characteristic function

$$h^{\alpha}(r) = \begin{cases} 1 & \text{for} \quad r \in V_{\alpha} \\ 0 & \text{for} \quad r \notin V_{\alpha} \end{cases} \tag{3.1}$$

the values of particular components of the elastic stiffness tensor within the RVE are expressed as follows

$$C_{ijkl}(r) = \sum_{\alpha=1}^{N} C_{ijkl}^{\alpha} h^{\alpha}(r) \tag{3.2}$$

where

V^{α} – volume occupied by a grain α

r – spatial co-ordinate

C_{ijkh}^{α} – values of the elastic stiffness tensor components for a grain α.

The local distribution of the issue being considered is made up by the equilibrium equations and the constitutive equations for a linear-elastic material

$$\sigma_{ij,j} = 0 \qquad \sigma_{ij}(r) = C_{ijkh}(r) e_{kh}(\boldsymbol{u}) \tag{3.3}$$

where

$e_{kh}(\boldsymbol{u})$ – deformation tensor

$\boldsymbol{u}$ – displacement vector.

By substituting

$$\overline{C}_{ijkh}(r) = C_{ijkh}(r) - C_{ijkh}^{eff} \qquad \overline{u}_{i,j}(r) = u_{i,j}(r) - U_{i,j} \tag{3.4}$$

with an additional condition that at the boundary of the entire space considered, i.e. at infinity, $\overline{u}_i(r) = 0$ (an assumption of the self-consistent scheme that, at infinity, there occurs a uniform deformation state $U_{i,j}$) Eqs (3.3) are transformed for the following form

$$\left[C_{ijkl}^{eff} \overline{u}_{k,l}(r)\right]_{,i} + f_j(r) = 0 \qquad f_j(r) = \left[\overline{C}_{ijkl}(r) u_{k,l}(r)\right]_{,i} \tag{3.5}$$

The tensor C_{ijkh}^{eff} is the elastic stiffness tensor for the homogenised medium the values of which we are looking for.

The solution of Eqs (3.5), using the Green function, can be presented in the following way

$$\overline{u}_n(r) = \int_{-\infty}^{+\infty} G_{nj}(r - r') f_j(r)\, dr' \tag{3.6}$$

where the function $G_{nj}(r)$ is the solution of the system of equations given below

$$C^{eff}_{ijkl} G_{km,lj}(r - r') + \delta_{im}\delta(r - r') = 0 \tag{3.7}$$

where $\delta(r)$ and δ_{im} stand for the Dirac function and Kronecker delta, respectively.

Using Eqs (3.4), (3.5) and (3.6), we obtain

$$e_{mn}(r) = E_{mn} + \int_{-\infty}^{+\infty} \Gamma_{mnij}(r - r')\overline{C}_{ijkl}(r')e_{kl}(r')\, dr' \tag{3.8}$$

where

$$\begin{aligned} e_{mn} &= \frac{1}{2}(u_{m,n} + u_{n,m}) \qquad & E_{mn} &= \frac{1}{2}(U_{m,n} + U_{n,m}) \\ \Gamma_{mnij} &= \frac{1}{2}(G_{mj,in} + G_{nj,im}) \end{aligned} \tag{3.9}$$

If we assume that, within a single grain, the elasticity tensor does not depend on the spatial variable, Eq (3.8) can be approximated by the following expression

$$\begin{aligned} \langle e_{mn}\rangle^{\alpha} &= E_{mn} + \langle \overline{C}_{ijkl}\rangle^{\alpha}\langle e_{kl}\rangle^{\alpha} A^{\alpha}_{mnij} \\ A^{\alpha}_{mnij} &= \frac{1}{\|V_{\alpha}\|}\iint_{V_{\alpha}} \Gamma_{mnij}(r - r')\, dr' dr \end{aligned} \tag{3.10}$$

where

$$\langle * \rangle = \frac{1}{\|V_{\alpha}\|}\int_{V_{\alpha}} * \, dV_{\alpha}$$

In consequence, Eq (3.10) leads to the localisation law (in terms of mean value), i.e. it determines mean deformation in the grain α when the macroscopic deformation tensor is known, i.e.

$$\langle e_{kl}\rangle^{\alpha} = \left[l^{\alpha}_{mnkl} - A^{\alpha}_{mnij}\langle \overline{C}_{ijkl}\rangle^{\alpha}\right]^{-1} E_{mn} \tag{3.11}$$

where

$$l^{\alpha}_{mnkl} = \frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})$$

Using the self-consistency condition, i.e. macroscopic deformation equals RVE-averaged deformation (over N-grains), Eq (3.11) leads to the tensor equation

in which the unknowns are the values of the components of the sought elastic stiffness effective tensor for the homogenised medium, i.e.

$$E_{kl} = \sum_{\alpha=1}^{N} \left[l^{\alpha}_{mnkl} - A^{\alpha}_{mnij} \langle \overline{C}_{ijkl} \rangle^{\alpha} \right]^{-1} f^{\alpha} E_{mn} \qquad f^{\alpha} = \frac{1}{\|V_{RVE}\|} \int_{V_{\alpha}} dV \tag{3.12}$$

The scheme presented above is known as the one-site self-consistent scheme and is applied to perfectly disordered media. In Buisson et al. (1990) one can also find the so-called N-site self-consistent scheme, where N-grains are simultaneously immersed in an infinite matrix exhibiting effective parameters of elasticity. The extension of the scheme to cover media that feature plasticity was discussed by Zaoui (1987).

The self-consistent scheme is also applied to determination of the effective parameters of elastic media with cracks or voids (Budiansky and O'Connell, 1976), which make up a certain class of porous media. This time, the analysis resolves itself to the issue of a single void or a crack placed in an infinite continuous medium which has the effective parameters to be determined. In the case of a two-dimensional issue, for a macroscopically isotropic medium, this leads to the following formulas for the effective parameters (Kachanov, 1992)

$$E^{eff} = E(1 - \pi\zeta) \qquad \nu^{eff} = \nu(1 - \pi\zeta) \tag{3.13}$$

where

ζ – so-called crack density parameter (Budiansky and O'Connell, 1976)

E, ν – Young modulus and material skeleton Poisson ratio.

A modification of the above scheme, called a differential scheme, was proposed by Hashin (1988). Again a single isolated crack in an infinite matrix is being considered yet, contrary to the self-consistent scheme, this time the analysis is performed in the incremental way. Cracking density is increased by small increments, $d\zeta$, and in each calculation step the values of effective parameter of the medium are arrived at.

For an macroscopically isotropic medium, should the issue be two-dimensional, this leads to the differential equations (which follow Eqs (3.13))

$$\begin{aligned} E^{eff} + dE^{eff} &= E^{eff}(1 - \pi d\zeta) \\ \nu^{eff} + d\nu^{eff} &= \nu^{eff}(1 - \pi d\zeta) \end{aligned} \tag{3.14}$$

which, after determining the initial condition, i.e. $E^{eff} = E$, $\nu^{eff} = \nu$ for $\zeta = 0$, yield

$$E^{eff} = E\mathrm{e}^{-\pi\zeta} \qquad \nu^{eff} = \nu\mathrm{e}^{-\pi\zeta} \tag{3.15}$$

In the case of porous media analysis, a certain drawback of the methods discussed above is a lack of possibility of extending these schemes to cover multiphase media except when the voids of the medium are filled with the Pascal ideal liquid (Kachanov, 1992).

3.2. *The method of volume and weighted averaging*

The methods most commonly used in multiphase media mechanics include those of volume averaging and averaging by weight (see e.g. Ene and Polisevski, 1987; Slattery, 1969; Whitaker, 1986). The main feature of the two methods is the use of the transition from the microscopic scale (spatial variable y) to the natural observation scale (spatial variable x) of spatial averaging (with or without a certain weighting function).

We shall start our discussion of these techniques with presenting the method of averaging by weight as the other method is a specific example of the former.

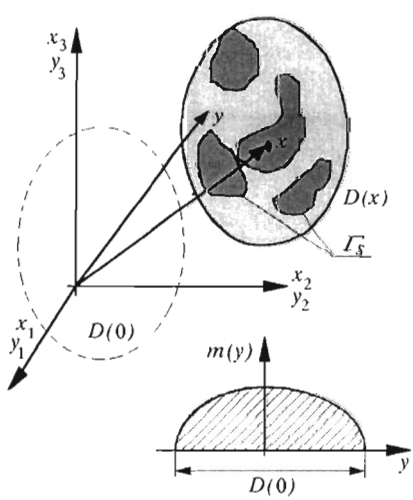

Fig. 5. Change of the observation scale by the spatial convolution with a weight function $m(y)$

Let $m(y)$ be a positive even function with the compact support in $D(0)$ (Fig.5), such that

$$\int_{D(0)} m(y)\, dy = 1 \tag{3.16}$$

By definition (Gilbert, 1990; Ene and Polisevski, 1990), the macroscopic variable associated with a given microscopic one $g^\alpha(y,t)$ is the convolution product (Fig.5)

$$\begin{aligned} m^*(g^\alpha h^\alpha) = \langle g^\alpha \rangle(x) = \int_{D(x)} g^\alpha(y) h^\alpha(y) m(x-y)\, dy = \\ = \int_{D(x)} g^\alpha(x-y) h^\alpha(x-y) m(y)\, dy \end{aligned} \tag{3.17}$$

where h^α – function characteristic for the α-phase, determined as in Eq (3.1).

In reality, the choice of $m(y)$ must be made in accordance with physical considerations.

Let us consider the microscopic variable Ψ, which – for the different phases which occur in the medium – has the following values

$$\Psi = \begin{cases} \Psi^\beta & \text{in the } \beta\text{-phase} \\ \Psi^\alpha & \text{in the } \beta\text{-phase} \end{cases} \tag{3.18}$$

The law of spatial averaging by weight has the form (see e.g. Ene and Polisevski, 1987)

$$\begin{aligned} m^*\Big(\frac{\partial \Psi}{\partial y_i}\Big) &= \frac{\partial}{\partial x_i}(m^* \Psi) - m^*\Big[(\Psi^\alpha - \Psi^\beta) \boldsymbol{n}_{\beta\alpha} \boldsymbol{e}_i \delta_{\beta\alpha}\Big] \\ m^*\Big(\frac{\partial \Psi}{\partial t}\Big) &= \frac{\partial}{\partial t}(m^* \Psi) + m^*\Big[(\Psi^\alpha - \Psi^\beta) \boldsymbol{n}_{\beta\alpha} \boldsymbol{v}^{\beta\alpha} \delta_{\beta\alpha}\Big] \end{aligned} \tag{3.19}$$

where

- $\boldsymbol{n}_{\beta\alpha}$ – unit vector normal to the phase separation surface $\beta - \alpha$ pointed from the phase β towards α
- $\delta_{\beta\alpha}$ – Dirac distribution in the three-dimensional space with respect to the $\beta - \alpha$ phase separation surface
- $\boldsymbol{v}_{\beta\alpha}$ – $\beta - \alpha$ phase separation surface velocity vector
- $\boldsymbol{e}_i$ – unit vector directed along x_i.

The first of the above relationships represents the law of spatial averaging for a certain function of weight whereas the second – a version of the general transport law with the function of weight.

If the assumption is that the weight function $m(y)$ equals $1/V$ in a sphere of V-volume, with the centre at the origin of the system of co-ordinates and is by identity equal to zero outside the sphere, Eqs (3.19) are transformed to the laws of volume averaging (Slattery, 1969; Whitaker, 1986)

$$\left\langle \frac{\partial \Psi}{\partial y_i} \right\rangle = \frac{\partial}{\partial x_i}\langle \Psi \rangle - \frac{1}{\|V\|}\int_V \left[(\Psi^\alpha - \Psi^\beta) \boldsymbol{n}_{\beta\alpha} \boldsymbol{e}_i \delta_{\beta\alpha} \right] dV$$

$$\left\langle \frac{\partial \Psi}{\partial t} \right\rangle = \frac{\partial}{\partial t}\langle \Psi \rangle + \frac{1}{\|V\|}\int_V \left[(\Psi^\alpha - \Psi^\beta) \boldsymbol{n}_{\beta\alpha} \boldsymbol{v}^{\beta\alpha} \delta_{\beta\alpha} \right] dV \tag{3.20}$$

The symbol $\langle cdot \rangle$ denotes the mean value calculated for the RVE volume (see Eq (2.1)).

Eqs (3.19) are the basis for mathematical description transition from the micro-scale to the macro-scale according to the method of spatial averaging by weight whilst Eqs (3.20) are the basic laws of the volume averaging method.

Below, we present an exemplary application of the spatial averaging method to arrive at the macroscopic description once the local description is given. The analysis focuses on the flow of a Newtonian incompressible liquid through a non-deformable porous medium (for simplicity we have also assumed no mass exchange between the phases of the system).

The use of Eq (3.20a) for the local law of liquid incompressibility (microscopic description) leads to the macroscopic law of incompressibility for the phase α

$$\operatorname{div}_x \langle \boldsymbol{v}^\alpha \rangle = 0 \tag{3.21}$$

Certainly, Eq (3.21) is a classical condition of incompressibility for a continuous medium.

The averaging of the momentum conservation equation for a Newtonian liquid leads to (we assumed that the process is quasi-static, hence the inertial terms are negligible)

$$0 = -\frac{\partial \langle p \rangle}{\partial x_j} + \mu \nabla^2 \langle v_j \rangle + \frac{1}{\|V_{VER}\|}\int_{\Gamma_s} \left[-p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i} \right) \right] n_i \, d\Gamma \tag{3.22}$$

where Γ_s – phase separation surface (Fig.5).

Unfortunately, Eq (3.22) fails to be expressed solely by macroscopic variables. Unless additional simplifying solutions (phenomenological-nature postulates (Slattery, 1969) or a value order comparative analysis (Whitaker, 1986)) are introduced, this is the ultimate form of the averaged local equation of momentum conservation, which – due to the occurrence of both microscopic and macroscopic variables in it – cannot be treated as a macroscopic description.

The reason for the above-mentioned *halting* of the process of transition from one observation scale to the other is that the methods of spatial averaging are lacking the closure hypothesis. This is the primary shortcoming of the two method discussed. Therefore, these methods are typically used to obtain macroscopic equations of equilibrium, with the expressions for interactions between multiphase medium components not being defined univocally (the last term in Eq (3.22) represents the interaction between the skeleton and the flowing liquid). The constitutive equations that define these interactions are then obtained by analysing the second law of thermodynamics (see e.g. Hassanizadeh and Gray, 1990). Such a way of modelling is called the hybrid theory of mixtures (Murad et al., 1995).

In order to effect a full transition by the volumetric averaging technique, some researchers apply the periodicity condition as the closure hypothesis (Gilbert, 1990; Barrere et al., 1992).

3.3. Asymptotic homogenisation method for periodic structures

The central assumption of this method of homogenisation is the condition that the medium exhibits a periodic structure, i.e. it is periodicity-generated from a single element, the so-called unit cell. The necessary condition for homogenisation to be possible is the so-called separation of the scales, i.e. if l denotes the unit cell size and L one of volumetric dimensions of the medium being considered, the following must be satisfied: $\varepsilon = l/L \ll 1$ (Bensoussan et al., 1978; Sanchez-Palencia, 1980).

The solution of the micromechanical issue (local description) is sought in the form of asymptotic expansions depending on the ε-parameter while the closure hypothesis is made up by the condition of the so-called local periodicity of the physical fields under consideration, which will be detailed hereafter in this paper.

Mathematically, the above issue can be presented as follows: in the domain Ω, the boundary value problem is given

$$L_\varepsilon(u_\varepsilon) = f \qquad \text{in } \Omega \tag{3.23}$$

and appropriate boundary conditions.

The differential operator L_ε displays rapidly oscillating coefficients. Assuming that the sequence u_ε converges to a limit $u(0)$, the objective of the asymptotic homogenisation method is to find the expansion in the form

$$u_\varepsilon = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots \tag{3.24}$$

which will be asymptotic in general, or at least to determine the first part of the expansion convergent to the solution of Eq (3.23) as $\varepsilon \to 0$. We are also looking for the so-called homogenised operator L, such that $u^{(0)}$ is the solution of the following

$$L\left(u^{(0)}\right) = f \qquad \text{in } \Omega \tag{3.25}$$

Eq (3.25) is the equivalent macroscopic description of the process under consideration which has been looked for.

Classically, the passage from the description (3.23) to (3.25) is done by applying the so-called *method of two-scale asymptotic expansions* (Bensoussan et al., 1978; Sanchez-Palencia, 1980), i.e. Eq (3.24) is postulated in the following form

$$u_\varepsilon(x) = u^{(0)}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^{(1)}(x, \frac{x}{\varepsilon}) + \varepsilon^2 u^{(2)}(x, \frac{x}{\varepsilon}) + \dots \tag{3.26}$$

while it is assumed that each term $u^{(i)}(x,y)$ is periodic in y, i.e. for an established x, the $u^{(i)}$ – values at the boundaries of a unit cell are the same. Then by substituting Eq (3.26) into Eq (3.23) and identifying the terms at the respective powers of ε-parameter, we arrive at a sequence of equations for each term of the expansion (3.26) .

When imposed on the particular terms of $u^{(i)}(x,y)$, the periodicity condition makes us look for the solutions of thus-obtained equations within a unit cell; i.e. the so-called cell boundary-value problem. In consequence, after averaging for the y-variable (within the unit cell), this process results in the sought macroscopic description of the issue being considered, i.e. in Eq (3.25). Importantly, when substituting Eq (3.26) into Eq (3.23), at the same time the spatial derivative operators should be changed according to the following dependence

$$\nabla = \nabla_x + \varepsilon^{-1}\nabla_y \tag{3.27}$$

where: ∇_x and ∇_y – gradient operators calculated with respect to the spatial variables x and y, respectively.

Noticeably, the process we described is relatively simple, being at the same time very effective, which is proved by many important results obtained in this way. Let us mention only some of them, obtained for fluid-saturated porous media: equations of poro-elasticity for a two-phase medium (Auriault

and Sanchez-Palencia, 1977), mathematical description of partially-saturated media consolidation (Auriault, 1987), dynamics of a multiphase medium (Auriault, 1991; Boutin and Auriault, 1990), Darcy's law of filtration (e.g. Keller, 1980), consolidation theory of porous media with fissures (Auriault and Boutin, 1992, 1993), generalised law of liquid filtration through a porous medium taking into account weak inertia effects (Mei and Auriault, 1991), theory of thermo-diffuso-elasticity for composite media (Gałka et al., 1994), consolidation equations for media with sorption phenomena (Murad et al., 1995; Łydżba, 1996, 1997).

The method of asymptotic homogenisation also allows for an analysis of the effect of the intrinsic structure of a porous medium on the values of macroscopic description constants. We shall exemplify this with the classical constitutive equation of Biot's poro-elasticity theory

$$\langle \boldsymbol{\sigma}_{ij} \rangle^{\top} = C^{eff}_{ijkh} e_{kh}(\boldsymbol{u}) - \alpha_{ij} p \tag{3.28}$$

where $\langle \boldsymbol{\sigma}_{ij} \rangle^{\top}$ is the total stress tensor, the tensors $\mathbf{C}^{eff}$ and $\boldsymbol{\alpha}$ are the effective parameters of the medium in question.

Using the apparatus of the asymptotic theory of homogenisation, in (Łydżba, 1996a) it was shown how porosity and crack development affect the value of the $\boldsymbol{\alpha}$-tensor, and a modified definition of this tensor was proposed

$$\alpha_{ij} = \frac{1}{3} D_{ijkh} \delta_{kh} \tag{3.29}$$

where the tensor $\mathbf{D}$, defined as

$$D_{ijkh} = \frac{C_{ijkh} - C^{eff}_{ijkh}}{K_s} \qquad K_s = \lambda + \frac{2}{3}\mu \tag{3.30}$$

can be interpreted as a measure of the material elastic stiffness loss in relation to its initial stiffness (material without pores or cracks). In the above equation, $\mathbf{C}$ denotes the elastic stiffness tensor of the material making up the skeleton and K_s its volumetric deformation modulus. Hence, the principal axes of the $\boldsymbol{\alpha}$-tensor are the principal directions of porous medium elastic stiffness loss.

An objection which rises as to the applicability of the above-cited method of homogenisation to any porous media is the kind of the closure hypothesis, i.e. the periodicity condition. It seems, however, that if we confine ourselves to looking for a form of an equivalent macroscopic description and an analysis of the role of the intrinsic structure of porous medium, then the applicability of this methods might also be extended to cover not only periodic media (see e.g.. Rubinstain and Torquato, 1989; Kröner, 1980).

A certain variant of the two-scale asymptotic expansion method discussed above is the so-called *two-scale convergence method* (Allaire, 1992). The method is based on the following theorem proved by Nguetseng (1989):

Theorem: Let u_ε be a bounded sequence in $L^2(\Omega)$ (Ω being an open set of R^N, $L^2(\Omega)$ denotes classically the Sobolev space). There exists a subsequence, still denoted by u_ε, and a function $u_0(x,y) \in L^2(\Omega \times Y)$ ($Y = (0;1)^N$ is the unit cube) such that

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \int_\Omega \int_Y u_0(x,y) \varphi(x,y)\, dy dx \tag{3.31}$$

for any smooth function $\varphi(x,y)$, which is Y-periodic in y. Such a sequence u_ε is said to two-scale converge to $u_0(x,y)$.

The idea of the method is to multiply Eq (3.23) by a test function of the type $\varphi(x, x/\varepsilon)$ (function $\varphi(x,y)$ is Y-periodic in y) and then, after integration by parts, the two-scale limit as $\varepsilon \to 0$ is sought with the help of the above theorem. This yields a variational formulation for the first term of the expansion of Eq (3.26). After averaging the corresponding partial differential equation with respect to y-variable the macroscopic description (3.25) is recovered. The method is detailed in Allaire (1992).

4. Conclusions

In this paper we have shown various methods of homogenisation applied to porous media mechanics. The advantage of these methods is that the information available in the level of heterogeneity (micro-scale) is then, using the appropriate process of transition, transferred to the natural observation scale. By the same token, these methods very often make it also possible to determine the effect of the intrinsic structure of a porous medium on the values of macroscopic description effective parameters.

The following methods were presented: self-consistent, spatial averaging by volume and by weight and asymptotic homogenisation. It was found out that the self-consistent method might be useful for determination of the effective parameters of single-phase porous media. In particular, it renders it possible to identify the possibility of determining the effect of micro-cracking development or, for instance, that of void growth, on the macroscopic mechanical behaviour

of a solid. On the other hand, it is not possible to extend this scheme to cover the analysis of multiphase media.

In the case of the method of averaging by volume or by weight, we have pointed out the lack of the closure hypothesis in them. This results in *halting* the process of transition from one scale into the other. We have indicated a possibility of supplementing this method by assuming the closure hypothesis in the form of the periodicity condition or by *transferring the undefined expression* and analysing it following the second principle of thermodynamics.

The presentation ends with a discussion of the asymptotic homogenisation method. This technique can be applied equally successfully to either single-phase porous media or multiphase ones. The author believes it is the most effective of all the homogenisation methods discussed.

References

1. ALLAIRE G., 1992, Homogenization and Two-Scale Convergence, *SIAM J. Math. Anal.*, **23**, 6, 1482-1518

2. AURIAULT J.L., 1991, Dynamic Behaviour of Porous Media, in *Transport Processes in Porous Media*, J. Bear and M.Y. Corapcioglu (edit.), Kluwer Academic Publishers, 471-519

3. AURIAULT J.L., BOUTIN C., 1992, Deformable Porous Media with Double Porosity. Quasistatics. I: Coupling Effects, *Transport in Porous Media*, **7**, 63-82

4. AURIAULT J.L., BOUTIN C., 1993, Deformable Porous Media with Double Porosity. Quasistatics. II: Memory Effects, *Transport in Porous Media*, **10**, 153-169

5. AURIAULT J.L., SANCHEZ-PALENCIA E., 1977, Étude de comportement macroscopique d'un milieu poreux saturé déformable, *Journal de Mecanique*, **16**, 4, 575-603

6. AURIAULT J.L, 1987, Nonsaturated Deformable Porous Media: Quasistatics, *Transport in Porous Media*, **2**, 45-64

7. BARRERE J., GIPOULOUX O., WHITAKER S., 1992, On the Closure Problem for Darcy's Law, *Transport in Porous Media*, **7**, 209-222

8. BEDFORD A., DRUMHELLER D.S., 1983, Theories of Immiscible and Structured Mixtures, *Int. J. Engng. Sci.*, **21**, 8, 863-960

9. BENSOUSSAN A., LIONS J.L., PAPANICOLAU G., 1978, Asymptotic Analysis for Periodic Structures, North-Holland Publishing Company, Amsterdam

10. BOUTIN C., 1996, Microstructulal Effects in Elastic Composities, *Int. J. Solids Structures*, **33**, 7, 1023-1051

11. BOUTIN C., AURIAULT J.L., 1990, Dynamic Behaviour of Porous Media Saturated by a Viscous Fluid. Application to Bituminous Concretes, *Int. J. Engng. Sci.*, **28**, 11, 1157-1181

12. BUDIANSKY B., O'CONNELL R.J., 1976, Elastic Moduli of a Cracked Solid, *Int. J. Solids and Struct.*, **12**, 81-96

13. DE BUHAN P., TALIERCIO A., 1991, A Homogenization Approach to the Yield Strength of Composite Material, *Eur. J. Mech., A/Solids*, **10**, 2, 129-154

14. BUISSON M., MOLINARI A., BERVEILLER M., 1990, On a Constitutive Relation for Heterogeneous Thermoelastic Media, *Arch. Mech.*, **42**, 2, 197-221

15. EMERIAULT F., CAMBOU B., MAHBOUBI A., 1996, Homogenization for Granular Materials: Non-Reversible Behaviour, *Mechanics of Cohesive-Frictional Materials*, *1*, 199-218

16. ENE H.I., POLISEWSKI D., 1987, *Thermal Flow in Porous Media*, D. Reidel Publishing Company, Boston

17. GAŁKA A., TELEGA J.J., WOJNAR R., 1994, Thermodiffusion in Heterogeneous Elastic Solids and Homogenization, *Arch. Mech.*, **46**, 3, 267-314

18. GEE L.M. ET AL., 1990, Liquid to Solidlike Transitions of Molecularly Thin Films under Shear, *J. Chem. Phys.*, **93**, 3, 1895-1906

19. GILBERT F., 1990, Change of Scale in Multiphase Media: the Case of Saturated Soils, in: *Geomaterials: Constitutive Equations and Modelling*, F. Darve (edit.), 329-347

20. HASHIN Z., 1983, Analysis of Composite Materials: a Survey, *J. Appl. Mech.*, **50**, 481-505

21. HASHIN Z., 1988, The Diffrential Scheme and its Application to Cracked Materials, *J. Mech. Phys. Solids*, **36**, 719-734

22. HASSANIZADEH S.M., GRAY W.G., 1990, Mechanics and Thermodynamics of Multiphase Flow in Porous Media Including Interphase Boundaries, *Adw. Water Res.*, 169-186

23. HILL R., 1965, A Self Consistent Mechanics of Composite Materials, *J. Mech. Phys. Solids*, **13**, 213-222

24. KACHANOV M., 1992, Effective Properties of Cracked Solids: Critical Review of Some Basic Concepts, *Appl. Mech. Rev.*, **45**, 8, 304-335

25. KAVIANY M., 1991, *Principles of Heat Transfer in Porous Media*, Springer-Verlag, New York

26. KISIEL I. (EDIT.), 1982, *Mechanika skał i gruntów*, seria Mechanika Techniczna, VII, PWN, Warszawa

27. LASOŃ M., 1988, Specific Surface of Porous Materials, *Arch. Min. Sci.*, **33**, 475-497

28. KELLER J.B., 1980, Darcy's Law for Flow in Porous Media and the Two-Space Method, in: *Nonlinear Partial Differential Equations in Engineering and Applied Science*, edit. Sternberg R., Kalinowski A., Papadakis J., Lecture Notes in Pure and Applied Mathematics, 54, 429-443

29. KRÖNER E., 1972, *Statistical Continuum Mechanics*, Springer-Verlag, Wien

30. KRÖNER E., 1980, Effective Elastic Moduli of Periodic and Random Media: a Unification, *Mechanics Research Comunications*, **7**, 5, 323-327

31. ŁYDŻBA D., 1996a, Introduction to the Homogenization Method in Geomechanics, Report de EUDIL de Universite des Sciences et Technologies de Lille

32. ŁYDŻBA D., AURIAULT J.L., 1996, Gas Filtration Through Porous Coal Medium. Effect of the Gas Constrained in Micropores, *Arch. Mech.*, **48**, 3, 447-473

33. ŁYDŻBA D., 1996, Consolidation of Porous Medium Saturated with Gas and with Sorption Phenomenon. Part I – Asymptotic Analysis, *Studia Geot. Mech.*, 1-2, 45-58

34. ŁYDŻBA D., 1997, Consolidation of Porous Medium Saturated with Gas and with Sorption Phenomenon. Part II – Parameters' Identification, *Studia Geot. Mech.*, 1-2, 63-77

35. MEI C.C., AURIAULT J.L., 1989, Mechanics of Heterogeneous Porous Media with Several Spatial Scales, *Proc. R. Soc. Lond.*, A 426, 391-423

36. MEI C.C., AURIAULT J.L., 1991, The Effect of Weak Inertia on Flow Through a Porous Medium, *J. Fluid Mech.*, **222**, 647-663

37. MURAD M.A., BENNETHUM L.S., CUSHMAN J.H., 1995, A Multi-Scale Theory of Swelling Porous Media: I. Application to One-Dimensional Consolidation, *Transport in Porous Media*, **19**, 93-122

38. NGUETSENG G., 1989, A General Convergence Result for a Functional Related to the Theory of Homogenization, *SIAM J. Math. Anal.*, **20**, 608-623

39. NIGMATULIN R.I., 1981, Three-Dimensional Averaging in the Mechanics of Heteregoneous Media, *Fluids Mechanics-Soviet Research*, **10**, 4, 72-105.

40. RUBINSTAIN J., TORQUATO S., 1989, Flow in Random Porous Media: Variational Principles and Rigorous Bounds, *J. Fluid Mech.*, **206**, 25-46

41. SANCHEZ-PALENCIA E., 1987, Boundary Layers and Edge Effects in Composities, in: *Homogenization Techniques for Composities Media*, Lecture Notes in Physics, 272, Springer-Verlag, Berlin

42. SANCHEZ-PALENCIA E., 1980, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, 127, Springer-Verlag, Berlin

43. SANCHEZ-PALENCIA E., 1974, Comportement local et macroscopique d'un type de milieux Physiques Hétérogenes, *Int. J. Engng. Sci.*, **12**, 331-351

44. SLATTERY J.C., 1969, Single Phase Flow Through Porous Media, *AIChE J.*, **15**, 866-872

45. SOBCZYK K., 1982, *Fale stochastyczne*, PWN, Warszawa

46. SUQUET P., 1987, Elements of Homogenization for Inelastic Solid Mechanics, in: *Homogenization Techniques for Composities Media*, Lecture Notes in Physics, 272, Springer-Verlag, Berlin

47. WHITAKER S., 1986, Flow in Porous Media I: A Theoretical Derivation of Darcy's Law, *Transport in Porous Media*, **1**, 3-25

48. WOŹNIAK C., 1995, Microdynamics: Continuum Modelling the Simple Composite Materials, *J. Theor. Appl. Mech.*, **33**, 267-289

49. ZAOUI A., 1987, Approximate Statistical Modelling and Applications, in: *Homogenization Techniques for Composite Media*, Lecture Notes in Physics, 272, Springer-Verlag, Berlin

Metody homogenizacji stosowane w mechanice ośrodków porowatych

Streszczenie

W artykule dokonano przeglądu i porównania metod homogenizacji stosowanych w mechanice ośrodków porowatych. W szczególności przedstawiono metody: objętościowego uśredniania, uśredniania wagowego, metody samouzgodnień, asymptotycznej metody homogenizacji dla struktur periodycznych oraz tzw. metody zbieżności wg podwójnej skali. Omówiono zagadnienie hipotezy zamykającej dla reprezentatywnej elementarnej objętości.

Manuscript received December 11, 1997; accepted for print February 25, 1998