JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
3, **36**, 1998

# FLOW OF CONDUCTIVE FLUIDS THROUGH POROELASTIC MEDIA WITH PIEZOELECTRIC PROPERTIES

JÓZEF JOACHIM TELEGA

RYSZARD WOJNAR

*Institute of Fundamental Technological Research Polish Academy of Sciences*
*e-mail: jtelega@ippt.gov.pl, rwojnar@ippt.gov.pl*

The aim of this contribution is to elaborate a general framework for modelling flows of electrolytes through porous piezoelectric media. Organic materials like animal and human bones provide an example of materials to which our results apply, though in wet bones the piezoelectric effect is smaller than the electrokinetic one. Those materials may be treated as piezoelectric porous materials through which a conductive fluid flows. The present work is confined to a regular distribution of pores. On the interfaces between the piezoelectric skeleton and conductive fluid natural jump conditions are imposed.
By using the method of two-scale asymptotic expansions, the macroscopic phenomenological equations describing electrokinetics of such a two-phase structure are derived and the formulae for the effective mechanical and nonmechanical coefficients are given.

*Key words:* homogenization, porous medium, elektrokinetics

## 1. Introduction

Porous materials are either natural or man-made, cf Sahimi (1995). Porous rocks as well as animal and human cartilage and bone provide important classes of natural porous media (cf Mow et al., 1984; Sahimi, 1995; Bourgeat et al., 1995; Bielski and Telega, 1997).

From the mechanical point of view one distinguishes between rigid and deformable skeletons (Sahimi, 1995; Cieszko and Kubik, 1996a,b). The present contribution is devoted to the study of flow of electrolytes through a piezoelectric skeleton. By using the method of two-scale asymptotic expansions, equations of electrokinetics involving the piezoelectric effect have been derived.

The present contribution is a continuation of our previous papers on homogenization of piezoelectric composites (Telega, 1991; Gałka et al., 1992) and homogenization of the flow of electrolytes through porous media (Gałka et al., 1994; Wojnar and Telega, 1997). In the papers Gałka et al. (1994), Wojnar and Telega (1997), Auriault and Strzelecki (1981) it was assumed that the skeleton is rigid. The essential novelty of our paper consists in assuming that the deformable skeleton reveals piezoelectric properties.

For earlier developments related to electromechanical properties of connective tissues the reader is referred to Demiray and Güzelsu (1977), Güzelsu and Demiray (1979), Grodzinsky (1983), Uklejewski (1993).

We observe that a different approach to modelling of bones as porous piezoelectric materials was proposed by Avdeev and Regirer (1979). These authors did not precise the equations governing the fluid flow. Their approach resembled that typical for mixture theory.

Our approach seems to be lucid from both the physical and mathematical points of view. In fact, from the equations of the fluid filling the pores and the equations of the piezoelectric skeleton, after homogenization we arrive at the equations of electrokinetics.

The general macroscopic model offers further possibilities of simplifications when some effects may be neglected. For instance, in wet bones the piezoelectric effect is significantly smaller then the electrokinetic effect (cf Johnson and Katz, 1987; Salzstein and Pollack, 1987). However, our model enables to obtain in a natural manner the effective model for dry bones.

Throughout the paper the following notation has been assumed

$$\frac{\partial}{\partial x_i} = \partial_{x_i} \qquad \text{and} \qquad \frac{\partial}{\partial y_i} = \partial_{y_i}$$

## 2. Notations and basic relations

Let $\Omega \subset \mathbb{R}^3$ be a bounded sufficiently regular domain. This domain consists of two parts

$$\Omega = \Omega_S \cup \Omega_L \qquad\qquad \overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_L$$

We assume that $\Omega_S$ is made of piezoelectric material (the skeleton) while $\Omega_L$ is filled with a conductive fluid.

The interface solid-liquid is denoted by $\Gamma$; $\partial\Omega_S$ ($\partial\Omega_L$) is the boundary of $\Omega_S$ ($\Omega_L$). We have $\partial\Omega_S \cup \partial\Omega_L = \partial\Omega$.

## 2.1. Skeleton

The material of the skeleton is described by the linear relations between stress $S^S_{ij}$, strain $e_{ij}$, electric induction $D^S_i$ and electric field $E_i$ as follows

$$S^S_{ij} = a_{ijmn}e_{mn} - \pi_{kij}E_k \qquad\qquad D^S_i = \pi_{imn}e_{mn} + \epsilon^S_{ik}E_k \tag{2.1}$$

where $a_{ijmn}$ is the elasticity tensor, $\pi_{kij}$ denotes the piezoelectric tensor and $\epsilon^S_{ik}$ is the tensor of dielectric coefficients (cf Landau and Lifshitz, 1957); moreover

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\Big(\partial_{x_j}u_i + \partial_{x_i}u_j\Big) \tag{2.2}$$

The electric field $E_i$ is given by gradient of electric potential $\Phi$

$$E_i = -\partial_{x_i}\Phi \tag{2.3}$$

The processes considered are slow, near the equilibrium and are governed by the following equations

$$\partial_{x_j}S^S_{ij} + f^S_i = 0 \qquad\qquad \partial_{x_i}D^S_i = 0 \tag{2.4}$$

where $f^S_i$ denotes the mass force of nonelectric origin.

## 2.2. Liquid

The electrolyte is treated as a conductive viscous fluid with the viscosity tensor $\eta_{ijmn}$. The linear relation between the stress $S^L_{ij}$, pressure $p$ and liquid strain rate $e_{ij}(\boldsymbol{v})$ is assumed

$$S^L_{ij} = -p\delta_{ij} + \eta_{ijmn}e_{mn}(\boldsymbol{v}) \tag{2.5}$$

The relation between the electric current $J_i$, electric field $E_i$, liquid velocity $v_i$ and electric charge $q$ has the following form

$$J_i = \sigma_{ij}E_j + qv_i - d_{ij}\partial_{x_j}q \tag{2.6}$$

where $\sigma_{ij} = b_{ij}q$ is the electrical conductivity, $b_{ij} = eB_{ij}$, $B_{ij}$ is the mobility of free charges while $d_{ij}$ is the coefficient of diffusion. If the dielectric tensor of the liquid is denoted by $\epsilon^L_{ik}$, then

$$D^L_i = \epsilon^L_{ik}E_k \tag{2.7}$$

For an isotropic liquid these tensors are of the form

$$\eta_{ijmn} = \eta\Big(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} - \frac{2}{3}\delta_{ij}\delta_{mn}\Big) + \zeta\delta_{ij}\delta_{mn}$$

$$\sigma_{ij} = \sigma\delta_{ij} \qquad b_{ij} = b\delta_{ij}$$

$$d_{ij} = d\delta_{ij} \qquad \epsilon^L_{ik} = \epsilon^L\delta_{ij}$$

where $\eta$, $\zeta$, $\sigma$, $b$, $d$ and $\epsilon^L$ are constants.

The tensor $e_{ij}(\boldsymbol{v})$ has the usual form

$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}\Big(\partial_{x_j}v_i + \partial_{x_i}v_j\Big) \tag{2.8}$$

The electric field $E_i$, similarly as in the case of skeleton, is given by

$$E_i = -\partial_{x_i}\Phi \tag{2.9}$$

The flow of liquid is stationary

$$\partial_{x_j}S^L_{ij} + f^L_i + qE_i - k\partial_{x_i}q = 0 \tag{2.10}$$

and incompressible

$$\partial_{x_i}v_i = 0 \tag{2.11}$$

In Eq (2.10) the coefficient $k = \kappa_B T/e$, where $\kappa_B$ is the Boltzmann constant and $e$ stands for the value of the elementary charge; the term $f^L_i$ denotes the mass force of nonelectric origin (cf Wojnar and Telega, 1997).

The last term on the left-hand side of Eq (2.10) is given by the Einstein-type relation between the coefficients $d_{ij}$ and $b_{ij}$

$$d_{ij} = k_B T b_{ij}$$

Moreover, we have $v_i = b_{ij}f^{(1)}_j$ where $f^{(1)}_i$ stands for the force acting on the particle (Lifshits and Pitayevskiy, 1979).

The vector $D_i$ satisfies the Gauss equation

$$\partial_{x_i}D_i = q \tag{2.12}$$

and the stationarity implies

$$\partial_{x_i}J_i = 0 \tag{2.13}$$

In our paper (Wojnar and Telega, 1997) we have studied the flow of two ion species electrolyte through a nondeformable dielectric porous medium, cf also Teso et al. (1997).

### 2.3. Solid-liquid interface

The conditions at the solid-fluid interface $\Gamma$ are assumed to be given by

$$\begin{array}{lll} [\![S_{ij}]\!]n_j = 0 & [\![\Phi]\!] = 0 & [\![D_i]\!]n_i = \gamma \\ v_i = 0 & J_i n_i = 0 & \end{array} \tag{2.14}$$

where $\gamma$ is the density of electric charges on the surface of skeleton (cf Teso et al., 1997; Hunter, 1981). Here $[\![\cdot]\!]$ stands for the jump on $\Gamma$, e.g.

$$[\![S_{ij}]\!]n_j = S^L_{ij}n_j - S^S_{ij}n_j \qquad [\![\Phi]\!] = \Phi_L - \Phi_S$$

with $\Phi_S$ and $\Phi_L$ denoting the values of potential $\Phi$ on both sides of $\Gamma$.

The coefficients $a_{ijmn}$, $\epsilon_{ij}$, $\eta_{ijmn}$, $b_{ij}$ and $d_{ij}$ satisfy the usual symmetry and positivity conditions, for instance

$$\begin{array}{lll} \exists\, c > 0 \quad & a_{ijmn}\xi_{ij}\xi_{ij} \geqslant c\|\boldsymbol{\xi}\|^2 & \forall\, \xi_{ij} \in \mathbb{E}^3_s \\ & \epsilon_{ij}\eta_i\eta_j \geqslant c\|\boldsymbol{\eta}\|^2 & \forall\, \boldsymbol{\eta} \in \mathbb{R}^3 \\ & d_{ij}\eta_i\eta_j \geqslant c\|\boldsymbol{\eta}\|^2 & \forall\, \boldsymbol{\eta} \in \mathbb{R}^3 \end{array} \tag{2.15}$$

## 3. Equations of microperiodic porous media

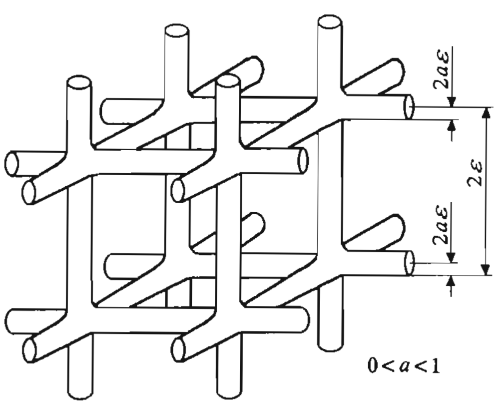


Fig. 1. Example of a skeleton, after Allaire (1989)

Consider now a porous medium with an $\varepsilon Y$-periodic microstructure. Here $\varepsilon$ is a small positive parameter, $\varepsilon = l/L$, and $l$, $L$ are characteristic lengths of the micro- and macro-scale, cf Fig.1. The basic cell $Y$ consists of two parts: $Y_S$ and $Y_L$ with $\overline{Y} = \overline{Y}_S \cup \overline{Y}_L$.

For a fixed $\varepsilon > 0$ all the relevant quantities have now the superscript $\varepsilon$. From Eqs (2.1) $\div$ (2.13) we obtain the set of equations for the fields $u_i$, $v_i$, $\Phi$, $p$ and $q$.

— in $\Omega_S^\varepsilon$

$$\begin{aligned} &\partial_{x_j}\Big(a^\varepsilon_{ijmn}\partial_{x_n}u^\varepsilon_m + \pi^\varepsilon_{kij}\partial_{x_k}\Phi^\varepsilon\Big) = 0 \\ &\partial_{x_i}\Big(\pi^\varepsilon_{imn}\partial_{x_n}u^\varepsilon_m - \epsilon^{S\varepsilon}_{ik}\partial_{x_k}\Phi^\varepsilon\Big) = 0 \end{aligned} \tag{3.1}$$

— in $\Omega_L^\varepsilon$

$$\begin{aligned} &\partial_{x_j}\Big(-p^\varepsilon\delta_{ij} + \varepsilon^2\eta^\varepsilon_{ijmn}\partial_{x_n}v^\varepsilon_m\Big) + f^g_i - q^\varepsilon\partial_{x_i}\Phi^\varepsilon - k\partial_{x_i}q^\varepsilon = 0 \\ &\partial_{x_i}v^\varepsilon_i = 0 \qquad\qquad \partial_{x_i}\Big(\epsilon^{L\varepsilon}_{ik}\partial_{x_k}\Phi^\varepsilon\Big) = -q^\varepsilon \\ &\partial_{x_i}\Big(-b^\varepsilon_{ij}q^\varepsilon\partial_{x_j}\Phi^\varepsilon + q^\varepsilon v^\varepsilon_i - d^\varepsilon_{ij}\partial_{x_j}q^\varepsilon\Big) = 0 \end{aligned} \tag{3.2}$$

The conditions imposed on the solid-liquid interface $\Gamma^\varepsilon$ read

$$\begin{aligned} &[\![S^\varepsilon_{ij}]\!]n^\varepsilon_j = 0 \qquad [\![\Phi^\varepsilon]\!] = 0 \qquad [\![D^\varepsilon_i]\!]n_i = \gamma^\varepsilon \\ &v^\varepsilon_i = 0 \qquad\qquad J^\varepsilon_i n_i = 0 \end{aligned} \tag{3.3}$$

where $S^\varepsilon_{ij}$, $D^\varepsilon_i$ and $J^\varepsilon_i$ are given by

$$S^\varepsilon_{ij} = \begin{cases} a^\varepsilon_{ijmn}\partial_{x_n}u^\varepsilon_m + \pi^\varepsilon_{kij}\partial_{x_k}\Phi^\varepsilon & \text{in } \Omega^\varepsilon_S \\ -p^\varepsilon\delta_{ij} + \varepsilon^2\eta^\varepsilon_{ijmn}\partial_{x_n}v^\varepsilon_m & \text{in } \Omega^\varepsilon_L \end{cases} \tag{3.4}$$

$$D^\varepsilon_i = \begin{cases} \pi^\varepsilon_{imn}\partial_{x_n}u^\varepsilon_m - \epsilon^{S\varepsilon}_{ik}\partial_{x_k}\Phi^\varepsilon & \text{in } \Omega^\varepsilon_S \\ -\epsilon^{L\varepsilon}_{ik}\partial_{x_k}\Phi^\varepsilon & \text{in } \Omega^\varepsilon_L \end{cases} \tag{3.5}$$

$$J^\varepsilon_i = -b^\varepsilon_{ij}q^\varepsilon\partial_{x_j}\Phi^\varepsilon + q^\varepsilon v^\varepsilon_i - d^\varepsilon_{ij}\partial_{x_j}q^\varepsilon \qquad \text{in } \Omega^\varepsilon_L \tag{3.6}$$

Note that in Eqs $(3.2)_1$ and $(3.4)_2$ the following rescaling is introduced

$$\eta_{ijmn} \to \varepsilon^2\eta_{ijmn} \tag{3.7}$$

According to the method of two scale asymptotic expansions we assume the following substitution for the scalar field $\Phi^\varepsilon$ and vector field $u^\varepsilon_i$

$$\begin{aligned} &\Phi^\varepsilon = \Phi^{(0)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon\Phi^{(1)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2\Phi^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \ldots \qquad \boldsymbol{y} = \boldsymbol{x}/\varepsilon \\ &u^\varepsilon_i = u^{(0)}_i(\boldsymbol{x},\boldsymbol{y}) + \varepsilon u^{(1)}_i(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 u^{(2)}_i(\boldsymbol{x},\boldsymbol{y}) + \ldots \qquad \boldsymbol{y} = \boldsymbol{x}/\varepsilon \end{aligned} \tag{3.8}$$

as well as analogous expansions for $p^\varepsilon$, $q^\varepsilon$, $v_i^\varepsilon$ and $J_i^\varepsilon$.

Assume also that in Eq $(3.3)_3$ we have

$$\gamma^\varepsilon = \gamma(\boldsymbol{x},\boldsymbol{y}) = \gamma_k^{(0)}(\boldsymbol{y})\partial_{x_k}\Phi^{(0)}(\boldsymbol{x},\boldsymbol{y}) \qquad \boldsymbol{y}\in\Gamma_Y \tag{3.9}$$

where $\gamma_k^{(0)}(\boldsymbol{y})$ is a given function; particularly it may be constant.

Next, taking into account the relation

$$\partial_{x_i} f(\boldsymbol{x},\boldsymbol{y}) = \Big(\partial_{x_i} + \frac{1}{\varepsilon}\partial_{y_i}\Big) f(\boldsymbol{x},\boldsymbol{y}) \qquad \boldsymbol{y} = \boldsymbol{x}/\varepsilon$$

and comparing terms with the same power of $\varepsilon$, we arrive at the homogenized set of equations.

According to the above given division of cell $Y$ we define three types of averages

$$\langle(\cdot)\rangle = \frac{1}{|Y|}\int\limits_Y (\cdot)\, d\boldsymbol{y} \qquad \langle(\cdot)\rangle_\alpha = \frac{1}{|Y|}\int\limits_{Y_\alpha} (\cdot)\, d\boldsymbol{y} \qquad \alpha = S, L$$

Note that $\partial Y_S = \Gamma_Y \cup P_S$ while the surface $\partial Y_L = \Gamma_Y \cup P_L$; $\Gamma_Y$ is the contact surface solid-liquid; and $P_S$ and $P_L$ are parts of the surfaces of the solid and liquid, respectively, coinciding with the boundary of $Y$.

## 4. Results of homogenization

The results of homogenization depend strongly on the interface conditions (3.3). The conditions $(3.3)_1$ and $(3.3)_3$ imposed on $\Gamma^\varepsilon$ can be rewritten also as

$$\Big[a_{ijmn}^\varepsilon \partial_{x_n} u_m^\varepsilon + \pi_{kij}^\varepsilon \partial_{x_k}\Phi^\varepsilon\Big] n_j = p^\varepsilon n_i$$

$$\Big[\pi_{imn}^\varepsilon \partial_{x_n} u_m^\varepsilon - \epsilon_{ik}^{S\varepsilon}\partial_{x_k}\Phi^\varepsilon + \epsilon_{ik}^{L\varepsilon}\partial_{x_k}\Phi^\varepsilon\Big] n_i = \gamma^\varepsilon$$

and after substitution of expansions (3.8) we get

$$\begin{aligned}
&\Big[a_{ijmn}\Big(\partial_{x_n} + \frac{1}{\varepsilon}\partial_{y_n}\Big)(u_m^{(0)} + \varepsilon u_m^{(1)} + \varepsilon^2 u_m^{(2)} + \ldots) + \\
&+\pi_{kij}\Big(\partial_{x_k} + \frac{1}{\varepsilon}\partial_{y_k}\Big)(\Phi^{(0)} + \varepsilon\Phi^{(1)} + \varepsilon^2\Phi^{(2)} + \ldots)\Big] n_j\Big|_S = \\
&= (p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \ldots)\delta_{ij} n_j\Big|_L
\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\Big[\pi_{imn}\Big(\partial_{x_n}+\frac{1}{\varepsilon}\partial_{y_n}\Big)(u_m^{(0)}+\varepsilon u_m^{(1)}+\varepsilon^2 u_m^{(2)}+...)+\\&-\epsilon_{ik}^S\Big(\partial_{x_k}+\frac{1}{\varepsilon}\partial_{y_k}\Big)(\varPhi^{(0)}+\varepsilon\varPhi^{(1)}+\varepsilon^2\varPhi^{(2)}+...)\Big]n_i|S=\\&=\gamma^\varepsilon-\epsilon_{ik}^L\Big(\partial_{x_k}+\frac{1}{\varepsilon}\partial_{y_k}\Big)(\varPhi^{(0)}+\varepsilon\varPhi^{(1)}+\varepsilon^2\varPhi^{(2)}+...)n_i\Big|_L\end{aligned}\tag{4.2}$$

Then on $\varGamma_Y$ Eqs (4.1) and (4.2) at $\varepsilon^{-1}$ yield

$$\begin{aligned}&\Big[a_{ijmn}\partial_{y_n}u_m^{(0)}+\pi_{kij}\partial_{y_k}\varPhi^{(0)}\Big]n_j=0\\&\Big[\pi_{imn}\partial_{y_n}u_m^{(0)}-\epsilon_{ik}^S\partial_{y_k}\varPhi^{(0)}+\epsilon_{ik}^L\partial_{y_k}\varPhi^{(0)}\Big]n_i=0\end{aligned}\tag{4.3}$$

The analysis of terms of the order $\varepsilon^{-2}$ appearing in Eqs (3.1) is carried out in Appendix A. Combined with the interface conditions (4.3) it yields

$$u_i^{(0)}=u_i^{(0)}(\boldsymbol{x})\qquad\qquad \varPhi^{(0)}=\varPhi^{(0)}(\boldsymbol{x})\tag{4.4}$$

Eqs (A.12) are satisfied provided that

$$\begin{aligned}u_m^{(1)}&=A_m^{(pq)}\partial_{x_q}u_p^{(0)}+B_{mq}\partial_{x_q}\varPhi^{(0)}+P_m p^{(0)}\\\varPhi^{(1)}&=R_{pq}\partial_{x_q}u_p^{(0)}+F_q\partial_{x_q}\varPhi^{(0)}+Sp^{(0)}\end{aligned}\tag{4.5}$$

and coefficients $A_m^{(pq)}$, $B_{mq}$ etc. are $Y$-periodic solutions to the following local equations on $Y_S$

$$\begin{aligned}&\partial_{y_j}\Big(a_{ijpq}+a_{ijmn}\partial_{y_n}A_m^{(pq)}+\pi_{kij}\partial_{y_k}R_{pq}\Big)=0\\&\partial_{y_i}\Big(\pi_{ipq}+\pi_{imn}\partial_{y_n}A_m^{(pq)}-\epsilon_{ik}^S\partial_{y_k}R_{pq}\Big)=0\\&\partial_{y_j}\Big(a_{ijmn}\partial_{y_n}B_{mq}+\pi_{qij}+\pi_{kij}\partial_{y_k}F_q\Big)=0\\&\partial_{y_i}\Big(\pi_{imn}\partial_{y_n}B_{mq}-\epsilon_{iq}^S-\epsilon_{ik}^S\partial_{y_k}F_q\Big)=0\\&\partial_{y_j}\Big(a_{ijmn}\partial_{y_n}P_m+\pi_{kij}\partial_{y_k}S\Big)=0\\&\partial_{y_i}\Big(\pi_{imn}\partial_{y_n}P_m-\epsilon_{ik}^S\partial_{y_k}\Big)=0\end{aligned}\tag{4.6}$$

The terms linked with $\varepsilon^0$ in Eqs (4.1) and (4.2) yield

$$\begin{gathered}\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(0)}+\partial_{y_n}u_m^{(1)}\Big)+\pi_{kij}\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\Big]n_j\Big|_S=p^{(0)}\delta_{ij}n_j\Big|_L\\ \Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(0)}+\partial_{y_n}u_m^{(1)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\Big]n_i\Big|_S=\\ =\gamma_k^{(0)}\partial_{x_k}\Phi^{(0)}-\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)n_i\Big|_L\end{gathered}\tag{4.7}$$

where assumption (3.9) was used. Hence, by using Eqs (4.5), we get

$$\begin{gathered}\Big(a_{ijpq}+a_{ijmn}\partial_{y_n}A_m^{(pq)}+\pi_{kij}\partial_{y_k}R_{pq}\Big)n_j\Big|_S=0\\ \Big(\pi_{ipq}+\pi_{imn}\partial_{y_n}A_m^{(pq)}-\epsilon_{ik}^S\partial_{y_k}R_{pq}\Big)n_i\Big|_S=0\\ \Big(a_{ijmn}\partial_{y_n}B_{mq}+\pi_{qij}+\pi_{kij}\partial_{y_k}F_q\Big)n_j\Big|_S=0\\ \Big(\pi_{imn}\partial_{y_n}B_{mq}-\epsilon_{iq}^S-\epsilon_{ik}^S\partial_{y_k}F_q\Big)n_i\Big|_S=\gamma_q^{(0)}-\Big(\epsilon_{iq}^L+\epsilon_{ik}^L\partial_{y_k}F_q\Big)n_i\Big|_L\\ \Big(a_{ijmn}\partial_{y_n}P_m+\pi_{kij}\partial_{y_k}S\Big)n_j\Big|_S=\delta_{ij}n_j\Big|_L\\ \Big(\pi_{imn}\partial_{y_n}P_m-\epsilon_{ik}^S\partial_{y_k}S\Big)n_i\Big|_S=0\end{gathered}\tag{4.8}$$

The interface conditions $(3.3)_2$ and $(3.3)_4$ are written as follows

$$\begin{gathered}\Big(\Phi^{(0)}+\varepsilon\Phi^{(1)}+\varepsilon^2\Phi^{(2)}+...\Big)\Big|_S=\Big(\Phi^{(0)}+\varepsilon\Phi^{(1)}+\varepsilon^2\Phi^{(2)}+...\Big)\Big|_L)\\ \Big(v_i^{(0)}+\varepsilon v_i^{(1)}+\varepsilon^2v_i^{(2)}+...\Big)\Big|_L=0\end{gathered}\tag{4.9}$$

Hence we obtain

$$\begin{array}{llll}\Phi^{(0)}\Big|_S=\Phi^{(0)}\Big|_L & \Phi^{(1)}\Big|_S=\Phi^{(1)}\Big|_L & \Phi^{(2)}\Big|_S=\Phi^{(2)}\Big|_L & \dots\\ v_i^{(0)}\Big|_L=0 & v_i^{(1)}\Big|_L=0 & v_i^{(2)}\Big|_L=0 & \text{etc.}\end{array}\tag{4.10}$$

The interface condition $(3.3)_5$ can be rewritten as

$$\begin{gathered}\Big[-b_{ij}(q^{(0)}+\varepsilon q^{(1)}+\varepsilon^2q^{(2)}+...)\Big(\partial_{x_j}+\frac{1}{\varepsilon}\partial_{y_j}\Big)(\Phi^{(0)}+\varepsilon\Phi^{(1)}+\varepsilon^2\Phi^{(2)}+...)+\\ +(q^{(0)}+\varepsilon q^{(1)}+\varepsilon^2q^{(2)}+...)(v_i^{(0)}+\varepsilon v_i^{(1)}+\varepsilon^2v_i^{(2)}+...)+\\ -d_{ij}\Big(\partial_{x_j}+\frac{1}{\varepsilon}\partial_{y_j}\Big)(q^{(0)}+\varepsilon q^{(1)}+\varepsilon^2q^{(2)}+...)\Big]n_i\Big|_L=0\end{gathered}\tag{4.11}$$

By virtue of Eq $(4.4)_2$, comparing in the last relation the terms linked with $\varepsilon^{-1}$ we conclude that

$$-d_{ij}\partial_{y_j}q^{(0)}n_i\Big|_L = 0 \tag{4.12}$$

and consequently

$$q^{(0)} = q^{(0)}(\boldsymbol{x}) \tag{4.13}$$

Eq (A.17) is satisfied provided that

$$\Phi^{(1)} = F_q\partial_{x_q}\Phi^{(0)} \tag{4.14}$$

where $F_q$ satisfies the *local* equation in $Y_L$

$$\partial_{y_i}\Big(\epsilon^L_{iq} + \epsilon^L_{ik}\partial_{y_k}F_q\Big) = 0 \tag{4.15}$$

The local problems (4.6) and (4.15) are supplemented by the interface conditions (4.8).

Eq (A.17) is satisfied provided that

$$q^{(1)} = Q_k\partial_{x_k}q^{(0)} + W_kq^{(0)}\partial_{x_k}\Phi^{(0)} \tag{4.16}$$

where $Q_k$ and $W_k$ are $Y$-periodic solutions to the local equations in $Y_L$

$$\begin{aligned}&\partial_{y_i}\Big(d_{ik} + d_{ij}\partial_{y_j}Q_k\Big) = 0\\ &\partial_{y_i}\Big(b_{ik} + b_{ij}\partial_{y_i}F_k + d_{ij}\partial_{y_j}W_k)\Big) = 0\end{aligned} \tag{4.17}$$

We recall that $\Phi^{(1)}$ has the form of Eq (4.14).

The functions $Q_k$ and $W_k$ are additionally subject to the boundary conditions

$$\begin{aligned}&\Big(b_{ik} + b_{ij}\partial_{y_j}F_k + d_{ij}\partial_{y_j}W_k\Big)n_i\Big|_L = 0\\ &\Big(d_{ik} + d_{ij}\partial_{y_j}Q_k\Big)n_i\Big|_L = 0\end{aligned} \tag{4.18}$$

Really, Eq (4.11) at $\varepsilon^0$ gives

$$\Big[-b_{ij}q^{(0)}\Big(\partial_{x_j}\Phi^{(0)} + \partial_{y_j}\Phi^{(1)}\Big) - d_{ij}\Big(\partial_{x_j}q^{(0)} + \partial_{y_j}q^{(1)}\Big]n_i\Big|_L = 0 \tag{4.19}$$

Substituting Eqs (4.14) and (4.16) we get

$$\begin{aligned}&\Big\{-b_{ij}q^{(0)}\Big(\delta_{jk} + \partial_{y_j}F_k\Big)\partial_{x_k}\Phi^{(0)} +\\ &-d_{ij}\Big[\Big(\delta_{jk} + \partial_{y_j}Q_k\Big)\partial_{x_k}q^{(0)} + \partial_{y_j}W_kq^{(0)}\partial_{x_k}\Phi^{(0)}\Big]\Big\}n_i\Big|_L = 0\end{aligned} \tag{4.20}$$

Hence we arrive at Eqs (4.18).

## 5. Macroscopic constitutive relations

Applying asymptotic expansions to the constitutive relations $(3.4)_1$, $(3.5)_2$ and comparing the terms linked with $\varepsilon^0$ we get

$$S_{ij}^{(0)} = \begin{cases} a_{ijmn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right) + \pi_{kij}\left(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\right) & \text{in } Y_S \\ -p^{(0)}\delta_{ij} & \text{in } Y_L \end{cases} \tag{5.1}$$

$$D_i^{(0)} = \begin{cases} \pi_{imn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right) - \epsilon_{ik}^S\left(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\right) & \text{in } Y_S \\ -\epsilon_{ik}^L\left(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\right) & \text{in } Y_L \end{cases} \tag{5.2}$$

We observe that

$$\langle S_{ij}^{(0)}\rangle = \langle S_{ij}^{(0)}\rangle_S + \langle S_{ij}^{(0)}\rangle_L \qquad \langle D_i^{(0)}\rangle = \langle D_i^{(0)}\rangle_S + \langle D_i^{(0)}\rangle_L \tag{5.3}$$

Using Eqs (4.5) and (4.14) we get

$$\begin{aligned} \langle S_{ij}^{(0)}\rangle &= a_{ijpq}^h \partial_{x_q} u_p^{(0)} + \pi_{qij}^h \partial_{x_q}\Phi^{(0)} + [c_{ij} - (1-f)\delta_{ij}]p^{(0)} \\ \langle D_i^{(0)}\rangle &= \pi_{ipq}^h \partial_{x_q} u_p^{(0)} - \epsilon_{iq}^h \partial_{x_q}\Phi^{(0)} + d_i p^{(0)} \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} a_{ijpq}^h &= \langle a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)} + \pi_{kij}\partial_{y_k} R_{pq}\rangle_S \\ \pi_{qij}^h &= \langle a_{ijmn}\partial_{y_n} B_{mq} + \pi_{qij} + \pi_{kij}\partial_{y_k} F_q\rangle_S \\ c_{ij} &= \langle a_{ijmn}\partial_{y_n} P_m + \pi_{kij}\partial_{y_k} S\rangle_S \\ \pi_{ipq}^h &= \langle \pi_{ipq} + \pi_{imn}\partial_{y_n} A_m^{(pq)} - \epsilon_{ik}^S\partial_{y_k} R_{pq}\rangle_S \\ \epsilon_{iq}^h &= \langle -\pi_{imn}\partial_{y_n} B_{mq} + \epsilon_{iq}^S + \epsilon_{ik}^S\partial_{y_k} F_q\rangle_S + \langle \epsilon_{iq}^L + \epsilon_{ik}^L\partial_{y_k} F_q\rangle_L \\ d_i &= \langle \pi_{imn}\partial_{y_n} P_m - \epsilon_{ik}^S\partial_{y_k} S\rangle_S \end{aligned} \tag{5.5}$$

and $f$ denotes the volume fraction of the solid (skeleton)

$$f = \frac{|Y_S|}{|Y|} \qquad 1 - f = \frac{|Y_L|}{|Y|} \tag{5.6}$$

The coefficients $A_m^{(pq)}$, $B_{mq}$, etc. have to be determined from the local equations $(4.6)_{1,2}$ and (4.15) jointly with Eqs (4.8).

## 6. Mechanics of porous medium with liquid

From Eqs (3.1) and $(3.2)_1$, by comparing the terms linked with $\varepsilon^0$, we get

$$\begin{aligned}
&\partial_{x_j}\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\Big)\Big] + \\
&+\partial_{y_j}\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(1)} + \partial_{y_n}u_m^{(2)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(1)} + \partial_{y_k}\Phi^{(2)}\Big)\Big] = 0 \qquad \text{in } Y_S \\
&-\Big(\partial_{x_i}p^{(0)} + \partial_{y_i}p^{(1)}\Big) + f_i^g + \partial_{y_j}\Big(\eta_{ijmn}\partial_{y_n}v_m^{(0)}\Big) + \\
&-q^{(0)}\Big(\partial_{x_i}\Phi^{(0)} + \partial_{y_i}\Phi^{(1)}\Big) - \kappa\Big(\partial_{x_i}q^{(0)} + \partial_{y_i}q^{(1)}\Big) = 0 \qquad \text{in } Y_L
\end{aligned} \tag{6.1}$$

On the other hand, the terms linked with $\varepsilon^1$ in the interface condition (4.1) lead to the relation

$$\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(1)} + \partial_{y_n}u_m^{(2)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(1)} + \partial_{y_k}\Phi^{(2)}\Big)\Big]n_j\Big|_S = p^{(1)}\delta_{ij}n_j\Big|_L \tag{6.2}$$

Integration of Eq $(6.1)_1$ over $Y_S$ and $(6.1)_2$ over $Y_L$ gives

$$\begin{aligned}
&\partial_{x_j}\langle a_{ijmn}\Big(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\partial_{y_k}\Big)\rangle_S + \\
&+\int\limits_{\partial Y_S}\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(1)} + \partial_{y_n}u_m^{(2)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(1)} + \partial_{y_k}\Phi^{(2)}\Big)\Big]n_j\,dA = 0 \qquad \text{in } Y_S \\
&-(1-f)\partial_{x_i}p^{(0)} + \langle f_i^g\rangle_L - (1-f)\Big(q^{(0)}\partial_{x_i}\Phi^{(0)} + \kappa\partial_{x_i}q^{(0)}\Big) + \\
&+\int\limits_{\partial Y_L}\Big[-\Big(p^{(1)} + q^{(0)}\Phi^{(1)} + \kappa q^{(1)}\Big)\delta_{ij} + \eta_{ijmn}\partial_{y_n}v_m^{(0)}\Big]n_j\,dA = 0 \qquad \text{in } Y_L
\end{aligned} \tag{6.3}$$

Adding Eqs (6.3), using the interface relation (6.2) and taking into account Eqs (5.1) and $(5.4)_1$ we obtain

$$\partial_{x_j}\langle S_{ij}^{(0)}\rangle + \langle f_i^g\rangle_L - (1-f)\Big(q^{(0)}\partial_{x_i}\Phi^{(0)} - \kappa\partial_{x_i}q^{(0)}\Big) = 0 \tag{6.4}$$

This is the macroscopic equilibrium equation for the porous medium filled with liquid.

## 7. Electrostatics of porous medium with liquid

From Eqs $(3.1)_2$ and $(3.2)_3$, by comparing the terms linked with $\varepsilon^0$, we get

$$\begin{aligned}
&\partial_{x_i}\Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(0)}+\partial_{y_n}u_m^{(1)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\Big]+ \\
&+\partial_{y_i}\Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(1)})+\partial_{y_n}u_m^{(2)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)\Big]=0 \qquad \text{in } Y_S \\
&-\partial_{x_i}\Big[\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\Big]-\partial_{y_i}\Big[\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)\Big]=q^{(0)} \qquad \text{in } Y_L
\end{aligned} \tag{7.1}$$

Further, the terms linked with $\varepsilon^1$ in the interface condition (4.1) lead to the relation

$$\begin{aligned}
&\Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(1)}+\partial_{y_n}u_m^{(2)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)\Big]n_i\Big|_S= \\
&=-\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)n_i\Big|_L
\end{aligned} \tag{7.2}$$

Integration of Eq $(7.1)_1$ over $Y_S$ and Eq $(7.1)_2$ over $Y_L$ yields

$$\begin{aligned}
&\partial_{x_i}\langle\pi_{imn}\Big(\partial_{x_n}u_m^{(0)}+\partial_{y_n}u_m^{(1)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\rangle_S+ \\
&+\int\limits_{\partial Y_S}\Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(1)}+\partial_{y_n}u_m^{(2)}\Big)-\epsilon_{ik}^S\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)\Big]n_i\,dA=0 \qquad \text{in } Y_S \\
&-\partial_{x_i}\langle\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(0)}+\partial_{y_k}\Phi^{(1)}\Big)\rangle_L-\int\limits_{\partial Y_L}\Big[\epsilon_{ik}^L\Big(\partial_{x_k}\Phi^{(1)}+\partial_{y_k}\Phi^{(2)}\Big)\Big]n_i\,dA=q^{(0)} \qquad \text{in } Y_L
\end{aligned} \tag{7.3}$$

Next, we add by sides Eqs (7.3), using the interface relation (7.2), and the results (5.2). Finally, we arrive at

$$\partial_{x_i}\langle D_i^{(0)}\rangle=q^{(0)} \tag{7.4}$$

This is the Gauss equation for the porous dielectric medium filled with dielectric liquid.

## 8. Flow of liquid in porous medium

As we have seen in Section 6, Eq $(3.2)_1$ yields the following relation, $\boldsymbol{x} \in \Omega$, $\boldsymbol{y} \in Y_L$

$$
\begin{gathered}
f_i^g - \partial_{x_i} p^{(0)} - \partial_{y_i} p^{(1)} + \partial_{y_j}\Big(\eta_{ijmn}\partial_{y_n} v_m^{(0)}\Big) + \\
-q^{(0)}\Big(\partial_{x_i}\Phi^{(0)} + \partial_{y_i}\Phi^{(1)}\Big) - \kappa\Big(\partial_{x_i} q^{(0)} + \partial_{y_i} q^{(1)}\Big) = 0
\end{gathered}
\tag{8.1}
$$

Let $\boldsymbol{u} = (u_i)$ be a $Y$-periodic and divergence-free function, which vanishes on $\Gamma$

$$
\partial_{y_i} u_i = 0 \quad \text{in} \quad Y_L \qquad\qquad u_i = 0 \quad \text{on} \quad \Gamma \tag{8.2}
$$

We look for a weak solution to Eq (8.1). To this end we multiply Eq (8.1) by $u_i$ and integrate over $Y_L$. We get

$$
\begin{aligned}
&\int\limits_{Y_L} \eta_{ijmn}\partial_{y_n} v_m^{(0)} \partial_{y_j} u_i \; d\boldsymbol{y} = \\
&= \int\limits_{Y_L} \Big(f_i^g - \partial_{x_i} p^{(0)} - q^{(0)}\partial_{x_i}\Phi^{(0)} - \kappa \partial_{x_i} q^{(0)}\Big) u_i \; d\boldsymbol{y}
\end{aligned}
$$

This equation is satisfied provided that

$$
v_m^{(0)} = \chi_m^{(s)}(\boldsymbol{y})\Big(f_s^g - \partial_{x_s} p^{(0)} - q^{(0)}\partial_{x_s}\Phi^{(0)} - \kappa\partial_{x_s} q^{(0)}\Big) \tag{8.3}
$$

The functions $\chi_m^{(s)}$ are $Y$-periodic solutions to the following local problem

$$
\begin{gathered}
\int\limits_{Y_L} \eta_{ijmn}\partial_{y_n}\chi_m^{(s)}(\boldsymbol{y})\partial_{y_j} u_i(\boldsymbol{y}) \; d\boldsymbol{y} = \int\limits_{Y_L} u_s(\boldsymbol{y}) \; d\boldsymbol{y} \qquad \forall \boldsymbol{u} \in V_Y \\
V_Y = \Big\{\boldsymbol{u} \in H^1(Y_F)^3 | \boldsymbol{u}|_G = 0, \quad \mathrm{div}_y \boldsymbol{u} = 0 \text{ in } Y_L, \quad \boldsymbol{u} \text{ is } Y-\text{periodic}\Big\}
\end{gathered}
\tag{8.4}
$$

The strong form of Eq (8.4) is given by

$$
\partial_{y_j}\Big(\eta_{ijmn}\partial_{y_n}\chi_m^{(k)}(\boldsymbol{y})\Big) = -\delta_{ik} \tag{8.5}
$$

After averaging we get

$$
\langle v_m^{(0)}\rangle_L = \langle \chi_{ms}(\boldsymbol{y})\rangle_L\Big(f_s^g - \partial_{x_s} p^{(0)} - q^{(0)}\partial_{x_s}\Phi^{(0)} - \kappa\partial_{x_s} q^{(0)}\Big) \tag{8.6}
$$

This is the Wiedemann-Darcy equation describing electrokinetics of our system.

## 9. Flow of electric current

Applying asymptotic expansions to the current law (3.6) and comparing the terms linked with $\varepsilon^0$, in $Y_L$ we get

$$J_i^{(0)} = -b_{ik}q^{(0)}\left(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\right) + q^{(0)}v_i^{(0)} - d_{ik}\left(\partial_{x_k}q^{(0)} + \partial_{y_k}q^{(1)}\right) \tag{9.1}$$

Next, using Exqs (4.14) and $(4.17)_2$ we obtain

$$\langle J_i^{(0)}\rangle_L = -b_{ij}^h q^{(0)}\partial_{x_j}\Phi^{(0)} + q^{(0)}\langle v_i^{(0)}\rangle_L - d_{ij}^h\partial_{x_j}q^{(0)} \tag{9.2}$$

where

$$\begin{aligned} b_{ij}^h &= \langle b_{ik} + b_{ij}\partial_{y_j}F_k + d_{ij}\partial_{y_j}W_k\rangle_L \\ d_{ij}^h &= \langle d_{ik} + d_{ij}\partial_{y_j}Q_k\rangle_L \end{aligned} \tag{9.3}$$

Comparing the terms linked with $\varepsilon^0$ in Eq $(3.2)_4$ we get the relation in $Y_L$

$$\begin{aligned} &-\partial_{x_i}\left[-b_{ij}q^{(0)}\left(\partial_{x_j}\Phi^{(0)} + \partial_{y_j}\Phi^{(1)}\right) + q^{(0)}v_i^{(0)} - d_{ij}\left(\partial_{x_j}q^{(0)} + \partial_{y_j}q^{(1)}\right)\right] + \\ &-\partial_{y_i}\left[-b_{ij}q^{(0)}\left(\partial_{x_j}\Phi^{(1)} + \partial_{y_j}\Phi^{(2)}\right) - b_{ij}q^{(1)}\left(\partial_{x_j}\Phi^{(0)} + \partial_{y_j}\Phi^{(1)}\right) + \right. \\ &\left.+q^{(1)}v_i^{(0)} + q^{(0)}v_i^{(1)} - d_{ij}\left(\partial_{x_j}q^{(1)} + \partial_{y_j}q^{(2)}\right)\right] = 0 \end{aligned} \tag{9.4}$$

Similarly, from the interface condition $(3.3)_5$, by comparing the terms linked with $\varepsilon^1$ we arrive at

$$\begin{aligned} &\left[-b_{ij}q^{(0)}\left(\partial_{x_j}\Phi^{(1)} + \partial_{y_j}\Phi^{(2)}\right) - b_{ij}q^{(1)}\left(\partial_{x_j}\Phi^{(0)} + \partial_{y_j}\Phi^{(1)}\right) + \right. \\ &\left.-d_{ij}\left(\partial_{x_j}q^{(1)} + \partial_{y_j}q^{(2)}\right)\right]n_i\Big|_L = 0 \end{aligned} \tag{9.5}$$

where Eqs $(4.4)_2$ and $(4.10)_2$ have been taken into account.

We now integrate Eq (9.4) over $Y_L$ and use Eqs (9.5) and $(4.10)_2$. We readily obtain

$$\partial_{x_i}\langle J_i^{(0)}\rangle_L = 0 \tag{9.6}$$

Thus the property of electric current stationarity conserves under the homogenization.

**Remark 9.1.** Eqs (5.4) could be used to a generalization of equations due to Biot (1955), provided that an evolution problem would be considered.

*Acknowledgement*

This work was supported by the State Committee for Scientific Research (KBN) – grant No. 7 T07 A 016 12. The first author was also supported through grant No. 8 T11 F 018 12.

We are also indebted to the Referee for drawing our attention to references Johnson and Katz (1987), Salzstein and Pollack (1987).

# A. Appendix

## A.1. Analysis of terms of $\varepsilon^{-2}$ order

Eqs (3.1) yield in $Y_S$

$$\partial_{y_j}\Big[a_{ijmn}\partial_{y_n}u_m^{(0)} + \pi_{kij}\partial_{y_k}\Phi^{(0)}\Big] = 0 \qquad \partial_{y_i}\Big[\pi_{imn}\partial_{y_n}u_m^{(0)} - \epsilon_{ik}^{S}\partial_{y_k}\Phi^{(0)}\Big] = 0 \tag{A.1}$$

Multiplying $(A.1)_1$ by $u_i^{(0)}$ and $(A.1)_2$ by $\Phi^{(0)}$, integrating by parts and substracting the relations obtained we get

$$\begin{aligned}
&\int\limits_{\partial Y_S} \Big[\Big(a_{ijmn}\partial_{y_n}u_m^{(0)} + \pi_{kij}\partial_{y_k}\Phi^{(0)}\Big]u_i^{(0)} + \\
&-\Big(\pi_{jmn}\partial_{y_n}u_m^{(0)} - \epsilon_{jk}^{S}\partial_{y_k}\Phi^{(0)}\Big)\Phi^{(0)}\Big]n_j\,dA + \\
&-\int\limits_{Y_S}\Big[a_{ijmn}\partial_{y_j}u_i^{(0)}\partial_{y_n}u_m^{(0)} + \epsilon_{ik}^{S}\partial_{y_i}\Phi^{(0)}\partial_{y_k}\Phi^{(0)}\Big)\,d\boldsymbol{y} = 0
\end{aligned} \tag{A.2}$$

On the other hand Eq $(3.2)_3$ yields in $Y_L$

$$\partial_{y_i}\Big(\epsilon_{ik}^{L}\partial_{y_k}\Phi^{(0)}\Big) = 0 \tag{A.3}$$

from which in a similar way as above we obtain

$$\int\limits_{\partial Y_L}\Big(\epsilon_{jk}^{L}\partial_{y_k}\Phi^{(0)}\Phi^{(0)}\Big)n_j\,dA - \int\limits_{Y_L}\Big(\epsilon_{ik}^{L}\partial_{y_i}\Phi^{(0)}\partial_{y_k}\Phi^{(0)}\Big)\,dY = 0 \tag{A.4}$$

Adding Eq (A.2) we obtain

$$
\begin{aligned}
&\int\limits_{\partial Y_S} \Big[\Big(a_{ijmn}\partial_{y_n}u_m^{(0)} + \pi_{kij}\partial_{y_k}\Phi^{(0)}\Big)u_i^{(0)} + \\
&-\Big(\pi_{jmn}\partial_{y_n}u_m^{(0)} - \epsilon_{jk}^S\partial_{y_k}\Phi^{(0)} + \epsilon_{jk}^L\partial_{y_k}\Phi^{(0)}\Big)\Phi^{(0)}\Big]n_j\ dA + \\
&-\Big[\int\limits_{Y_S}\Big(a_{ijmn}\partial_{y_j}u_i^{(0)}\partial_{y_n}u_m^{(0)} + \epsilon_{ik}^S\partial_{y_i}\Phi^{(0)}\partial_{y_k}\Phi^{(0)}\Big)\ d\boldsymbol{y} + \\
&+\int\limits_{Y_L}\Big(\epsilon_{ik}^L\partial_{y_i}\Phi^{(0)}\partial_{y_k}\Phi^{(0)}\Big)\ d\boldsymbol{y}\Big] = 0
\end{aligned}
\tag{A.5}
$$

By virtue of the interface conditions (4.3), the surface integral in the last equation vanishes and the rest implies, by Eq $(2.15)_{1,2}$ that

$$
u_i^{(0)} = u_i^{(0)}(\boldsymbol{x}) \qquad\qquad \Phi^{(0)} = \Phi^{(0)}(\boldsymbol{x}) \tag{A.6}
$$

Finally, by comparing the terms linked with $\varepsilon^{-2}$ in Eq $(3.2)_4$ we get in $Y_L$

$$
\partial_{y_i}\Big(-b_{ij}q^{(0)}\partial_{y_j}\Phi^{(0)} - d_{ij}\partial_{y_j}q^{(0)}\Big) = 0 \tag{A.7}
$$

or by Eq $(\mathrm{A.6})_2$

$$
\partial_{y_i}\Big(d_{ij}\partial_{y_j}q^{(0)}\Big) = 0 \qquad \text{in } Y_L \tag{A.8}
$$

Multiplication by $q^{(0)}$ and integration by parts leads to

$$
\int\limits_{\partial Y_L} q^{(0)}d_{ij}\partial_{y_j}q^{(0)}n_i\ dA - \int\limits_{Y_L} d_{ij}\partial_{y_i}q^{(0)}\partial_{y_j}q^{(0)}\ d\boldsymbol{y} = 0 \tag{A.9}
$$

On account of Eq (4.12) and due to periodic boundary conditions the surface integral vanishes and we get

$$
\int\limits_{Y_L} d_{ij}\partial_{y_i}q^{(0)}\partial_{y_j}q^{(0)}\ d\boldsymbol{y} = 0 \tag{A.10}
$$

Hence, Eq $(2.15)_3$ we conclude that

$$
q^{(0)} = q^{(0)}(\boldsymbol{x}) \tag{A.11}
$$

### A.2. Analysis of terms of $\varepsilon^{-1}$ order

Eqs (3.1) yield now the relations for $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y_S$

$$\begin{aligned} &\partial_{y_j}\Big[a_{ijmn}\Big(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\Big) + \pi_{kij}\Big(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\Big)\Big] = 0 \\ &\partial_{y_i}\Big[\pi_{imn}\Big(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\Big) - \epsilon_{ik}^{S}\Big(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\Big)\Big] = 0 \end{aligned} \tag{A.12}$$

Similarly, Eq $(3.2)_1$ gives for $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y_L$

$$-\partial_{y_i}p^{(0)} - q^{(0)}\partial_{y_i}\Phi^{(0)} - k\partial_{y_i}q^{(0)} = 0$$

By $(A.6)_2$ and (A.11) we have

$$\partial_{y_i}p^{(0)} = 0 \tag{A.13}$$

what means that $p^{(0)}$ does not depend on $\boldsymbol{y}$, $\boldsymbol{x} \in \Omega$

$$p^{(0)} = p^{(0)}(\boldsymbol{x}) \tag{A.14}$$

The incompressibility equation $(3.2)_2$ yields

$$\partial_{y_i}v_i^{(0)} = 0 \tag{A.15}$$

what means that the field $\boldsymbol{v}^{(0)}$ is divergence free with respect to $\boldsymbol{y} \in Y_L$.

Next, the Gauss equation $(3.2)_3$ yields for $\boldsymbol{x} \in \Omega$, $\boldsymbol{y} \in Y_L$

$$\partial_{y_i}\Big[\epsilon_{ik}^{L}\Big(\partial_{x_k}\Phi^{(0)} + \partial_{y_k}\Phi^{(1)}\Big)\Big] = 0 \tag{A.16}$$

where Eq $(A.6)_2$ has been exploited.

The current equation $(3.2)_4$ gives for $\boldsymbol{x} \in \Omega$, $\boldsymbol{y} \in Y_L$

$$\partial_{y_i}\Big[-b_{ij}q^{(0)}\Big(\partial_{x_j}\Phi^{(0)} + \partial_{y_j}\Phi^{(1)}\Big) - d_{ij}\Big(\partial_{x_j}q^{(0)} + \partial_{y_j}q^{(1)}\Big)\Big] = 0 \tag{A.17}$$

## References


1. ALLAIRE G., 1989, Homogenization of the Stokes Flow in a Connected Porous Medium, *Asymptotic Analysis*, **2**, 203-222

2. ALLAIRE G., 1991, Continuity of the Darcy's Law in the Low-Volume Fraction Limit, *Annali della Scuola Norm. Sup. di Pisa, Sci. Fis. e Mat.*, Ser. IV, fsac. 4

3. AURIAULT J.L., SANCHEZ-PALENCIA E., 1977, Étude du comportement macroscopique d'un milieu poreux saturé déformable, *J. de Mécanique*, **16**, 575-603

4. AURIAULT J.L., STRZELECKI T., 1981, On the Electro-Osmotic Flow in a Saturated Porous Medium, *Int. J. Engng Sci.*, **19**, 915-928

5. AVDEEV YU.A., REGIRER S.A., 1979, Mathematical Model of Bone Tissue as a Poroelastic Piezoelectric Material, *Mekh. Kompozitnykh Materyalov*, **5**, 851-855, in Russian

6. BIELSKI W., TELEGA J.J., 1977, Effective Properties of Geomaterials: Rocks and Porous Media, *Publ. Institute of Geophysics Polish Academy of Sciences*, A-26 (285), Warsaw

7. BIOT M.A., Theory of Elasticity and Consolidation for a Porous Anisotropic Solid, *J. Appl. Phys.*, **26**, 182-185

8. BOURGEAT A.P., CARASSO C., LUCKHAUS S., MIKELIĆ A. (edit.), 1995, *Mathematical Modelling of Flow through Porous Media*, World Scientific, Singapore-New Jersey-London-Hong Kong

9. CIESZKO M., KUBIK J., 1996, Constitutive Relations and Internal Equilibrium Condition for Fluid-Saturated Porous Solids. Nonlinear Description, *Arch. Mech.*, **48**, 893-910

10. *CIESZKO M., KUBIK J., 1996, Constitutive Relations and Internal Equilibrium* Condition for Fluid-Saturated Porous solids. Linear Description, *Arch. Mech.*, **48**, 911-925

11. DEMIRAY H., GÜZELSU N., 1977, A Mixture Model for Wet Bones – I Theory, *Int. J. Engng Sci.*, **15**, 707-718

12. GAŁKA A., TELEGA J.J., WOJNAR R., 1992, Homogenization and Thermo-Piezoelectricity, *Mech. Res. Comm.*, **19**, 315-324

13. GAŁKA A., TELEGA J.J., WOJNAR R., 1994, Equations of Electrokinetics and Flow of Electrolytes in Porous Media, *J. Techn. Physics*, **35**, 49-59

14. GRODZINSKY A.J., 1983, Electrochemical and Physicochemical Properties of Connective Tissue, *CRC Critical Reviews in Biomedical Engineering*, **9**, 133-199

15. GÜZELSU N., DEMIRAY H., 1979, Recent Advances: Electromechanical Properties and Related Models of Bone Tissues, *Int. J. Engng Sci.*, **17**, 813-851

16. HUNTER R.J., 1981, *Zeta Potential in Colloid Science*, Academic Press, London

17. JOHNSON M.W., KATZ J.L., 1987, Electromechanical Effects in Bone, in: *Handbook of Bioengineering*, Edit. R.Skalak and S.Chien, Mc Graw Hill, New York, 3.1-3.11

18. LANDAU L.D., LIFSHITZ E.M., 1957, *Electrodynamics of Continuous Media* (in Russian), Gosud. Izd. Tekhn. Teoret. Literat., Moskva

19. LIFSHITS E.M., PITAYEVSKIY L.D., 1979, *Physical Kinetics*, Nauka, Moskva, in Russian
20. MOW V.C., HOLMES M.H., LAI W.M., 1984, Fluid Transport and Mechanical Properties of Articular Cartilage: a Review, *J. Biomech.*, **17**, 377-394
21. SAHIMI M., 1995, *Flow and Transport in Porous Media and Fractured Rock: from classical to Modern Approaches*, VCH Verlagsgesellschaft, Weinheim-New York-Basel
22. SALZSTEIN R.A., POLLACK S.R., 1987, Electromechanical Potential in Cortical Bone – II. Experimental Analysis, *J. Biomechanics*, **20**, 271-280
23. SANCHEZ-PALENCIA E., 1980, *Non-Homogeneous Media and Vibration Theory*, Springer, Berlin
24. TELEGA J.J., 1991, Piezoelectricity and Homogenization. Application to Biomechanics, in: *Continuum Models and Discrete Systems*, **2**, edit. G.A.Maugin, 220-229, Longman, Essex
25. TESO A., FILHO A.D., NETO A.A., 1997, Solution of the Poisson-Boltzmann Equation for a System with Four Ionic Species, *J. Math. Biol.*, **35**, 814-824
26. UKLEJEWSKI R., 1993, Electromechanical Potentials in a Fluid-Filled Cortical Bone: Initial Stress State in Osteonic Lamellae, Piezoelectricity and Streaming Potential Roles – a Theory, *Biocybernetics and Biomedical Engng*, **13**, 97-112
27. WOJNAR R., TELEGA J.J., 1997, Electrokinetics in Dielectric Porous Media, in: *Problems of Environmental and Damage Mechanics*, edit. W.Kosiński, R.de Boer, D.Gross, Wydawnictwa IPPT PAN, Warszawa, 97-136

## Przepływ cieczy przewodzącej przez ośrodek porowaty o własnościach piezoelektrycznych

Streszczenie

W pracy podajemy ogólny opis przepływu elektrolitu przez porowaty ośrodek piezoelektryczny. Otrzymane wyniki mogą być wykorzystane do materiałów organicznych; kości zwierząt i ludzi stanowią przykład takich materiałów, choć w kościach żywych efekt piezoelektryczny jest mniejszy od elektrokinetycznego.

W niniejszej pracy ograniczamy się do regularnego (okresowego) rozkładu porów. Na powierzchniach między fazami zakładamy naturalne warunki styku.

Korzystamy z metody dwuskalowych rozwinięć asymptotycznych i wyprowadzamy makroskopowe równanie fenomenologiczne dla elektrokinetyki takiego układu dwufazowego. Podajemy też wzory matematyczne na współczynniki skuteczne (zhomogenizowane), zarówno mechaniczne jak i niemechaniczne.

*Manuscript received January 6, 1998; accepted for print January 26, 1998*