JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
3, **36**, 1998

ELASTO-ELECTRIC LONGITUDINAL HARMONIC WAVES IN POROUS LONG BONES FILLED WITH PHYSIOLOGICAL FLUID

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Transmission of elasto-electric longitudinal harmonic waves in porous long bones filled with physiological fluid is investigated. The complete set of equations of the problem is obtained on the basis of the Biot theory of elastic waves in fluid-saturated porous media and the linear equations of electrokinetics, by means of quantities analogous to those in the theory of electrical transmission lines. Experimental findings from the biomechanical literature, supporting applicability of the proposed description, are presented. The electric signal associated with the propagation of longitudinal elastic waves in a wet long bone shaft can potentially be used for monitoring these waves during their application in e.g. bone porosity measurements.

Key words: cortical bone, Biot's poroelasticity, electrokinetics, elasto-electric waves

1. Introduction

Porous elastic dielectric solid filled with a viscous ionic fluid is a natural biomechanical (or bioelectromechanical) model of human cortical bone filled with physiological fluid (cf Carter and Hayes, 1978; Martin, 1984; Salzstein et al., 1987; Natali and Meroi, 1989; Uklejewski, 1994). This cortical bone tissue forms tuboidal shafts of long bones in the human and mammalian skeletons. The volume porosity of the human cortical bone ranges from 0.05 to

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0.30, the typical porosity value for normal human cortical bone is ~ 0.15 (cf Martin, 1984).

In the range of mechanical loads occuring during normal physical activity of individuals, cortical bone tissue behaves like a linear poroelastic material (cf e.g. Natali and Meroi, 1989).

The two populations of bone cells: osteoblasts and osteoclasts maintain the dynamical equilibrium between the two processes which proceed continuously in bone tissue, i.e. the new bone formation process (carried by the osteoblasts) and the bone resorption (carried by the osteoclasts). Both these two kinds of bone tissue cells are *electro-sensitive*, i.e. they can modify their cell metabolism (cf Basset, 1982; Zichner, 1984) and even migrate (in opposite directions) in response to a constant or pulsating one-directional electric field – as it was prooved in a spectacular way by Ferrier et al. (1986).

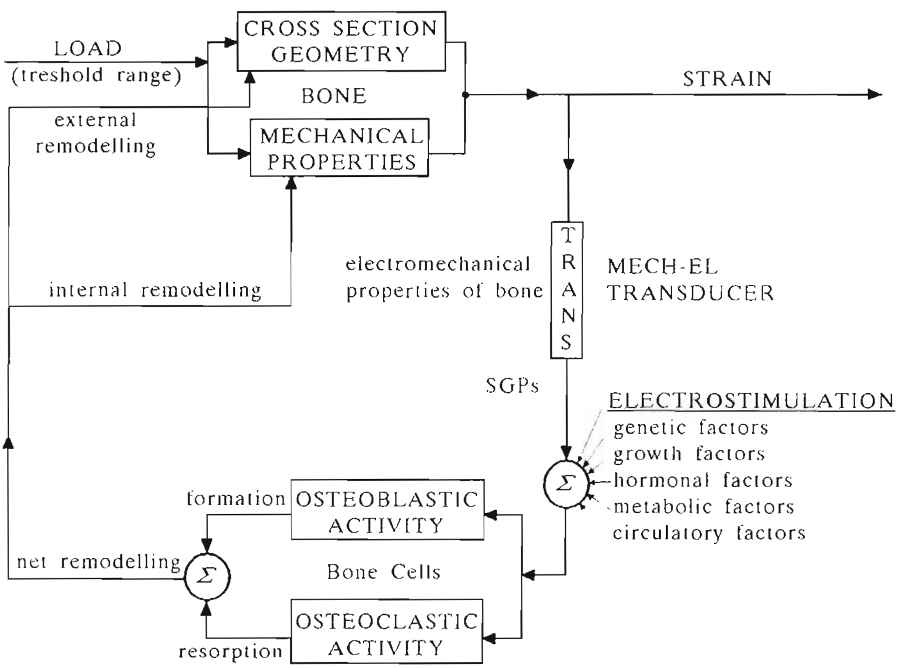

Fig. 1. Schematic diagram of the bone remodelling process (after Hart et al., 1984, modified)

Mechanical strains of porous bone filled with physiological fluid can generate electrical potentials (SGPs – Strain Generated Potentials) which in turn can stimulate the bone tissue cells to adaptive response of the tissue (Fig.1). In the mechano-adaptive response of bone (e.g. of a long bone) one can distinguish two remodelling processes: the external remodelling process, i.e. the adaptive changes of cross-section geometry of a bone (e.g. of a long bone shaft), and the internal remodelling, i.e. the adaptive changes of mechanical properties of cortical bone tissue (due to adaptive changes of porosity and internal architecture of cortical bone tissue of a long bone shaft wall).

As it was shown by the autor (cf Uklejewski, 1993, 1994) piezoelectricity is *the initial*, whereas streaming potential – *the direct* generating mechanism of the strain generated potentials (SGPs) in porous cortical bone filled with physiological fluid. The initial piezoelectric polarization of collagenous osteonic lamellae (due to *the initial stresses* induced in these lamellae during the cortical bone tissue growth (cf Ascenzi and Benvenuti, 1977; Fung, 1988) is an origin of the bound surface electric charge on pore walls of the adult cortical bone matrix and – therefore – of the electrokinetic potential Zeta of this cortical bone i.e., when filled with physiological ionic fluid.

During small linear elastic deformations of wet cortical bone the piezoelectric polarization of osteonic lamellae varies very small from its initial value (cf Uklejewski, 1993, 1994) thus, the generation of the electromechanical potentials SGPs can be effectively described by means of the linear equations of classical electrokinetics in dielectric porous media filled with ionic fluid (such a description has been used by Salzstein et al. (1987), but without explanation of its theoretical background).

These few introductory remarks underline the role of the elasto-electric potentials (SGPs) in bone biomechanics and the importance of the problem of adequate, possibly simple and effective, mathematical description of the generation of those potentials in long bones during their periodical loading.

2. Long bone shaft as a transmission line

In Fig.2 the longitudinal and transversal sections of a long bone shaft has been shown schematically. We note the interesting properties of long bone shafts, resulting from the existence of the remodelling (adaptive) processes which proceed in living bones according to its mechanical strain history.

As it was shown by Lazenby (1986): "... internal and external remodelling do not proceed independently. Net bone loss occurs where it can be most afforded without precipitously compromissing strength". – It means that the cross-section geometry of a long bone is strictly connected with the distribution of bone porosity in the cross-section region, and both these parameters, i.e.

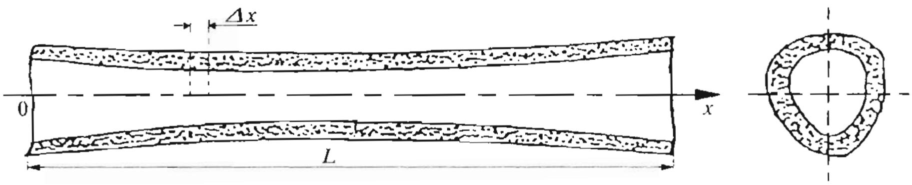

Fig. 2. Long bone shaft as a hollow (tubular) conduit; the product $S_x(1-f_x)$ (where: S_x – area of cross-section x, f_x – porosity of an element $S_x\Delta x$; $0 \leqslant x \leqslant L$) has practically the same value along the shaft (it is due to the bone adaptive remodelling process)

the cross-section geometry and the porosity distribution in the cross-section region are remodelled under the influence of bone bending loads.

Because every element $S_x\Delta x$ of a long bone shaft (where: S_x – cross-section area at the distance x; $0 \leqslant x \leqslant L$, L – long bone shaft length) transfers in the axial direction of bone the same compression loads – it can be assumed, in a healty long bone, that the compression strength of every element $S_x\Delta x$ of long bone shaft has the same value. Thus, we can assume that the value of the product: $S_x(1-f_x)$ (where f_x – the mean bone porosity in element $S_x\Delta x$) is the same along the bone shaft length

$$\forall x_1, x_2 \in < 0, L > \qquad S_{x_1}(1-f_{x_1}) = S_{x_2}(1-f_{x_2}) \tag{2.1}$$

If, moreover, for a given long bone the cross-section area of its shaft varies very small along the shaft length as it occurs usually (cf Gies and Carter, 1982; Piekarski, 1981), then also the following approximate equality holds

$$\forall x_1, x_2 \in < 0, L > \qquad S_{x_1} f_{x_1} \approx S_{x_2} f_{x_2} \tag{2.2}$$

i.e. the mean bone porosity f_x in the bone shaft element $S_x\Delta x$ is *approximately* the same along the length of long bone shaft.

The properties (2.1), (2.2) allow us to introduce in a natural way the mechanical and the electrical parameters per unit length of the long bone shaft, and to treate the long bone shaft as a mechano-electric transmission line with macroparameters distributed practically uniformly along the line length.

3. Elasto-electric longitudinal harmonic waves in a wet porous long bone shaft viewed as a transmission line

In works by Uklejewski (1993), (1994) the autor has presented a continuum theory of generation of electromechanical potentials (SGPs) in wet cortical bone, which combines the Biot theory of poroelasticity with the differential equations of classical electrokinetics. Now this theory will be adapted to a wet long bone shaft viewed as a transmission line.

Before that, however, the reasons for applicability of the Biot's theory of poroelasticity to the description of wet cortical bone (and a long bone shaft) mechanics will be presented.

The continuum approach to the problem of the electromechanical potentials (SGPs) generation in porous cortical bone filled with physiological fluid was used earlier by Salzstein et al. (1987). The theory of Salzstein et al. combines the classical electrokinetics with a theory of poroelasticity – formulated in terms of the modern mixture theory (cf Bowen, 1976) which has been applied earlier by Mow et al. (1980) to the description of poroelastic behaviour of articular cartilage. Some quantitative disagreements between the values of the electromechanical potentials SGPs of wet cortical bone obtained on the basis of the Salzstein et al. (1987) theory and the measurement resuts (cf Salzstein and Pollack, 1987; Scott and Korostoff, 1990) are mainly due to the assumption (in the Salzstein et al. theory) that the material of both the fluid and the solid bone phases is *incompressible*. Chandler (1981) showed that: "Models assuming an incompressible frame of porous medium are grossly inadequate in describing the temporal features of quasi-static flow. [...] all the physics necessary to explain transient fluid pressures (and streaming potentials) resulting from quasi-static flow are contained in Biot's wave theory". The theory of propagation of elastic waves in fluid saturated porous media with a compressible solid frame was formulated by Biot (1956) and then verified experimentally (cf Plona, 1980; Johnson et al., 1982); this theory is by now classical (cf monographs: Bourbie et al., 1987; Allard, 1993). Also on the basis of the modern mixture theory approach one can show that the linear equations of the semi-phenomenological Biot's theory are well formulated (cf Kubik, 1982; Katsube and Carrol, 1987; Ehlers and Kubik, 1994).

The Biot's consolidation theory (which can be treated as the quasi-static limit of his wave theory (cf Chandler and Johnson, 1981) has been used by Nowinski and Davies (1970, 1971) for the description of the poroelastic behaviour of bone elements and for the interpretation of some findings resulting from the Sedlin (1965) investigations on the rheological properties of cortical bone. The Biot's theory has been used also in the problems of ultrasonic wave propagation in cortical and cancellous bone (cf Williams, 1992).

3.1. Equations of the Biot theory for a long bone shaft transmission line under a longitudinal harmonic load

We consider a porous long bone shaft filled with physiological fluid and assume that at $x = 0$ on the cross-section of the shaft acts a longitudinal harmonic load $P(t)$ with the pulsation ω

$$P(t) = P_0 \sin \omega t = \operatorname{Im}\Big(P_0 e^{j\omega t}\Big) \tag{3.1}$$

The cortical bone of long bone shaft wall can be treated in these conditions as an *isotropic* material – it was experimentally proved by Huiskes (1982), who wrote: "It can be concluded from these results that when assuming cortical bone material to exhibit linear elastic, homogeneous and *transversely isotropic* behaviour an excellent agreement between theoretical and experimental results can be obtained, although some local inaccuraties due to inhomogenity should be expected. [...] However, when torsion is not considered and *only* the most significant *longitudinal* stress components are of interest, a good approximation can be obtained from assuming the cortical bone to be *isotropic*".

The deformation of the porous cortical bone of a long bone shaft wall is described by the following one-dimensional dynamical state equations of Biot's theory:

— constitutive equations

$$\begin{aligned} \sigma_{11} &= (2N + A)\frac{\partial w_1}{\partial x_1} + Q\frac{\partial W_1}{\partial x_1} \\ \sigma &= Q\frac{\partial w_1}{\partial x_1} + R\frac{\partial W_1}{\partial x_1} \end{aligned} \tag{3.2}$$

— equations of motion

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} &= (\rho_1 + \rho_a)\frac{\partial^2 w_1}{\partial t^2} - \rho_a\frac{\partial^2 W_1}{\partial t^2} + b\Big(\frac{\partial w_1}{\partial t} - \frac{\partial W_1}{\partial t}\Big) = \\ &= \rho_1\frac{\partial^2 w_1}{\partial t^2} + \Big(b + \rho_a\frac{\partial}{\partial t}\Big)\Big(\frac{\partial w_1}{\partial t} - \frac{\partial W_1}{\partial t}\Big) \\ \frac{\partial \sigma}{\partial x_1} &= (\rho_2 + \rho_a)\frac{\partial^2 W_1}{\partial t^2} - \rho_a\frac{\partial^2 w_1}{\partial t^2} + b\Big(\frac{\partial W_1}{\partial t} - \frac{\partial w_1}{\partial t}\Big) = \\ &= \rho_2\frac{\partial^2 W_1}{\partial t^2} + \Big(b + \rho_a\frac{\partial}{\partial t}\Big)\Big(\frac{\partial W_1}{\partial t} - \frac{\partial w_1}{\partial t}\Big) \end{aligned} \tag{3.3}$$

where σ_{11} is the normal component of the stress tensor of the solid frame in the x direction, σ is the fluid stress, w_1 $(\partial w_1/\partial t = w_1^*)$ is the component of the vector of the solid frame displacement (velocity of displacement) in the x direction, W_1 $(\partial W_1/\partial t = W_1^*)$ is the component of the vector of the fluid displacement (velocity of displacement) in the x direction, b is the coefficient of permeability of porous medium, N, A, Q, R are the Biot-Willis elastic coefficients, and ρ_1, ρ_2, ρ_a are the inertial coefficients.

Because the system under discussion is linear and stationary (i.e. its parameters values not change in time), and the mechanical load at the cross-section $x = 0$ is a harmonic function of time, thus *the other quantities vary also sinusoidally*, i.e.

$$\begin{aligned} \sigma_{11}(x,t) &= \mathrm{Im}\Big[\sigma_{11}(x)e^{j\omega t}\Big] & \qquad w_1^*(x,t) &= \mathrm{Im}\Big[w_1^*(x)e^{j\omega t}\Big] \\ \sigma(x,t) &= \mathrm{Im}\Big[\sigma(x)e^{j\omega t}\Big] & \qquad W_1^*(x,t) &= \mathrm{Im}\Big[W_1^*(x)e^{j\omega t}\Big] \end{aligned} \tag{3.4}$$

where the quantities $\sigma_{11}(x)$, $\sigma(x)$, $w_1^*(x)$, $W_1^*(x)$ are the complex amplitude functions of the stresses in the solid frame of porous bone, fluid stresses, velocity of displacement of solid frame and velocity of fluid displacement, respectively; ω is the pulsation and $j = \sqrt{-1}$ is the imaginary unit.

In Fig.3 we present schematically – on the basis of the electromechanical analogy given in Appendix – an element dx of a bone transmission line, i.e. of a wet long bone shaft during propagation of longitudinal harmonic elastic waves.

The transmission line 1 corresponds to the solid frame of wet long bone shaft, whereas the transmission line 2 corresponds to the fluid phase of wet long bone shaft.

The constitutive relations in complex form **for a wet long bone shaft** under longitudinal harmonic load one can obtain on the basis of the scheme given in Fig.3 applying formally the second Kirchhoff's law to the elementary circuits formed by the element 1 – 1' of the line 1 and "the earth", and by the element 2 – 2' of the line 2 and "the earth". We have

$$\begin{aligned} \frac{dw_1^*(x)}{dx} &= Z_1^m S\sigma_{11}(x) - Z_{12}^m\sigma(x) \\ \frac{dW_1^*(x)}{dx} &= -Z_{12}^m S\sigma_{11}(x) - Z_2^m\sigma(x) \end{aligned} \tag{3.5}$$

The equations (3.5) are *isomorphic* with the Biot's constitutive relations for a poroelastic material filled with fluid (cf Appendix and Eqs (3.2) in which

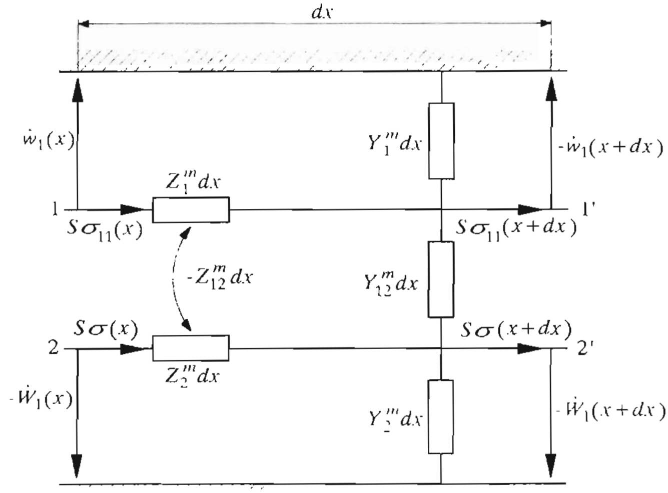

Fig. 3. The formal analog scheme of an element dx of a shaft of wet long bone during propagation of longitudinal harmonic elastic waves, on the basis of the Biot theory and the electro-mechanical analogy presented in Appendix. Z_1^m, Z_2^m, Z_{12}^m – the mechanical impedances (Y_1^m, Y_2^m, Y_{12}^m – the admitances) per unit length of a bone shaft of cross-section area S; w^*, W^* – velocities of displacements of the solid and the fluid phase; σ_{11}, σ – stresses in the solid and the fluid phase of wet porous cortical bone of the shaft

(3.4) are introduced). The mechanical impedances Z_1^m, Z_2^m, Z_{12}^m per unit length should be determined experimentally, or they may be calculated if the Biot-Willis poroelastic coefficients for cortical bone and the geometry of long bone shaft are known.

The equations of elastodynamics in complex form **for a wet long bone shaft** under longitudinal harmonic load can be obtained by applying formally the first Kirchhoff's law to the nodal points of the circuit element given in Fig.3; we have

$$
\begin{aligned}
\frac{d}{dx}[S\sigma_{11}(x)] &= Y_1^m w_1^*(x) + Y_{12}^m[w_1^*(x) - W_1^*(x)] \\
\frac{d}{dx}[S\sigma(x)] &= Y_2^m W_1^*(x) + Y_{12}^m[W_1^*(x) - w_1^*(x)]
\end{aligned}
\tag{3.6}
$$

The equations (3.6) are isomorphic with the Biot's equations of motion for a poroelastic material filled with fluid (cf Appendix and Eqs (3.3) in which (3.4) are introduced). The mechanical admitances Y_1^m, Y_2^m, Y_{12}^m per unit length of wet long bone shaft play similar role as the inertial coefficients ρ_1,

ρ_2, ρ_a in Biot's equations of motion for porous cortical bone filled with physiological fluid (long bone shaft wall material). The admitances Y_1^m, Y_2^m, Y_{12}^m should be determined experimentally, or they may be calculated if the material coefficients for cortical bone and the geometry of long bone shaft are known.

Equations (3.5) and (3.6) can be written in the matrix form, and we obtain then the following homogeneous state equatuion

$$\frac{d}{dx}\begin{bmatrix} w_1^*(x) \\ W_1^*(x) \\ S\sigma_{11}(x) \\ S\sigma(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & Z_1^m & -Z_{12}^m \\ 0 & 0 & -Z_{12}^m & Z_2^m \\ Y_1'^m & -Y_{12}^m & 0 & 0 \\ -Y_{12}^m & Y_2'^m & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1^*(x) \\ W_1^*(x) \\ S\sigma_{11}(x) \\ S\sigma(x) \end{bmatrix} \tag{3.7}$$

where

$$Y_1'^m = Y_1^m + Y_{12}^m \qquad Y_2'^m = Y_2^m + Y_{12}^m \tag{3.8}$$

We can say that the equations (3.5) and (3.6) or the state equation (3.7) are *the Biot's equations for a wet long bone shaft transmission line under longitudinal harmonic load.*

3.2. Solution of mechanic state equation for long bone shaft transmission line

Equation (3.7) can be written in the form

$$\frac{d\mathbf{S}(x)}{dx} = \mathbf{A}\mathbf{S}(x) \tag{3.9}$$

where $\mathbf{S}(x) = [w_1^*(x), W_1^*(x), S\sigma_{11}(x), S\sigma(x)]^\top$ is the state vector, and $\mathbf{A}$ is the matrix of the system.

The solution of equation (3.9) is given by

$$\mathbf{S}(x) = e^{\mathbf{A}x}\mathbf{S}(0) \tag{3.10}$$

where $e^{\mathbf{A}x}$ is the mechanic transmission matrix of a wet long bone shaft, and $\mathbf{S}(0)$ is the state vector on the input of the system, representing the boundary conditions for the cross-section of long bone shaft at $x = 0$.

The matrix $e^{\mathbf{A}x}$ can be determined by using the Cayley-Hamilton theorem (cf Ogata, 1967) and as the result one obtains (cf Uklejewski, 1994)

$$\begin{aligned} e^{\mathbf{A}x} &= \Big[\frac{\sinh(\gamma_1 x)}{\gamma_1(\gamma_1^2-\gamma_2^2)} - \frac{\sinh(\gamma_2 x)}{\gamma_2(\gamma_1^2-\gamma_2^2)}\Big]\mathbf{A}^3 + \frac{\cosh(\gamma_1 x)-\cosh(\gamma_2 x)}{\gamma_1^2-\gamma_2^2}\mathbf{A}^2 + \\ &+ \Big[\frac{\gamma_1^2\sinh(\gamma_2 x)}{\gamma_2(\gamma_1^2-\gamma_2^2)} - \frac{\gamma_2^2\sinh(\gamma_1 x)}{\gamma_1(\gamma_1^2-\gamma_2^2)}\Big]\mathbf{A} + \frac{\gamma_1^2\cosh(\gamma_2 x)-\gamma_2^2\cosh(\gamma_1 x)}{\gamma_1^2-\gamma_2^2}\mathbf{I} \end{aligned} \tag{3.11}$$

As it wiil be seen in the following the quantities γ_1, γ_2 appearing in Eq (3.11) are the propagation constants (i.e. they have a constant value for a given pulsation ω) of the two kinds of longitudinal elastic waves propagating along the shaft of wet long bone: γ_1 is the propagation constant for the Biot's slow wave, and γ_2 is the propagation constant for the longitudinal fast wave

$$\gamma_1 = \sqrt{\overline{P} + K} \qquad \gamma_2 = \sqrt{\overline{P} - K} \tag{3.12}$$

where

$$K = \sqrt{(\Delta P)^2 + Q^2} \qquad \overline{P} = \frac{P_1 + P_2}{2}$$
$$\Delta P = \frac{P_1 - P_2}{2} \qquad Q^2 = Q_1 Q_2 \tag{3.13}$$

and

$$P_1 = Z_1^m Y_1'^m + Z_{12}^m Y_{12}^m \qquad Q_1 = -Z_{12}^m Y_2'^m - Z_1^m Y_{12}^m$$
$$P_2 = Z_2^m Y_2'^m + Z_{12}^m Y_{12}^m \qquad Q_2 = -Z_{12}^m Y_1'^m - Z_2^m Y_{12}^m \tag{3.14}$$

As it is known (cf Chandler and Johnson, 1981) in the limit of zero frequency the Biot slow wave equatuion describes a quasi-static flow in fluid-saturated porous media; in those conditions the inertial forces (associated with the inertial coefficients ρ_1, ρ_2, ρ_a in equations of motion (3.3)) can be neglected in relation to the viscous drag forces. Substituting formally in the above presented equations of motion

$$\rho_1 = 0 \qquad \rho_2 = 0 \qquad \rho_a = 0 \tag{3.15}$$

we have then also (Fig.3; cf Uklejewski, 1994)

$$Y_1^m = 0 \qquad Y_2^m = 0 \qquad Y_{12}^m = G \tag{3.16}$$

where: G – a real value, then: $\gamma_2 = 0$ (because $\overline{P} = K$, Eq (3.12)) – i.e. in the limit of zero frequency the longitudinal fast wave will not propagate.

The propagation constants γ_1, γ_2 are the complex quantities

$$\gamma_k = \alpha_k + \mathrm{j}\beta_k \qquad \alpha_k > 0 \qquad k = 1,2 \tag{3.17}$$

The real parts α_k of the quantities γ_k are the measures of the attenuation of the moving waves, whereas the imaginary parts β_k are associated with the phase velocities v_k and with the wavelength λ_k by the formulas

$$v_k = \frac{\omega}{\beta_k} \qquad \lambda_k = \frac{2\pi}{\beta_k} \qquad k = 1,2 \tag{3.18}$$

which provides a physical interpretation of the propagation constants γ_k.

Interpretating on the basis of the scheme given in Fig.3 the propagation of two kinds of longitudinal elastic waves in a porous transmission line, one can say that the Biot's slow wave propagates along the two-conduit line composed of the conduits 1 and 2 (where 2 acts as a return conduit), whereas the longitudinal fast wave propagates parallely ("in phase") along the both "earth-return circuits" 1 and 2.

On the basis of Eqs (3.9) and (3.11) one obtains the following complex function describing fluid velocity $W_1^*(x) - w_1^*(x)$, i.e.

$$
\begin{aligned}
W_1^*(x) - w_1^*(x) = {} & \frac{(K-\Delta P-Q_1)W_1^*(0) - (K+\Delta P-Q_2)w_1^*(x)}{2K}\cosh(\gamma_1 x) + \\
& + \frac{(K-\Delta P-Q_1)A_2^0 - (K+\Delta P-Q_2)A_1^0}{2K\gamma_1}\sinh(\gamma_1 x) + \\
& + \frac{(K+\Delta P+Q_1)W_1^*(0) - (K-\Delta P+Q_2)w_1^*(x)}{2K}\cosh(\gamma_2 x) + \\
& + \frac{(K+\Delta P+Q_1)A_2^0 - (K-\Delta P+Q_2)A_1^0}{2K\gamma_2}\sinh(\gamma_2 x)
\end{aligned}
\tag{3.19}
$$

where

$$
Z_1^m S\sigma_{11}(0) - Z_{12}^m S\sigma(0) = A_1^0 \qquad Z_2^m S\sigma(0) - Z_{12}^m S\sigma_{11}(0) = A_2^0 \tag{3.20}
$$

If one express the hyperbolic functions in terms of exponential functions, then in Eq (3.19) the incident waves and the reflected waves can be isolated. The incident waves are associated with the term $e^{-\gamma x}$, whereas the reflected waves – with the term $e^{\gamma x}$.

3.3. Electric potentials associated with propagation of harmonic longitudinal elastic waves in wet long bone shafts

During the propagation of harmonic longitudinal elastic waves in a wet long bone shaft the oscillatory flow of ionic physiological fluid in cortical bone pores occur. This flow – described by the fluid relative velocity $W_1^*(x) - w_1^*(x)$ (Eq (3.19)) – produces the electrokinetic streaming currents and streaming potentials.

In Fig.4 the electric scheme of an element dx of wet long bone shaft under harmonic longitudinal mechanical load, placed in the air over a conducting layer, is presented.

In Fig.4 $E_{str}^0(x)$ represents the electromotive force of voltage sources of mechanical (deformation) origin, distributed along the wet long bone shaft. It

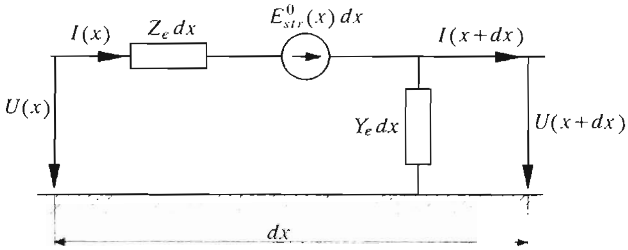

Fig. 4. The electric scheme of an element dx of wet long bone shaft under harmonic longitudinal mechanical load, $U(x)$ – electric voltage associated with propagation of longitudinal elastic waves in wet long bone shaft, E^0_{str} – electromotive force of voltage sources of mechanical origin, per unit length; $Z_e(= R + \mathrm{j}\omega L)$ – electric impedance of bone transmission line per unit length, $Y_e(= G + \mathrm{j}\omega C)$ – electric admitance of bone transmission line per unit length

will be seen in the following that the electromotive force $E^0_{str}(x)$ depends on the relative velocity $W_1^*(x) - w_1^*(x)$ of ionic fluid in bone pores.

We define E^0_{str} as

$$E^0_{str} = Z_e S J_{str} \tag{3.21}$$

where J_{str} – the streaming current density produced by the mechanical deformation of wet bone shaft porous wall, S – the cross-section area of long bone shaft, and $Z_e(= R + \mathrm{j}\omega L)$ is the electric impedance per unit length of wet long bone shaft.

The streaming current density J_{str} can be determined from the linear equations of classical electrokinetics, which are here presented in the vector form

$$\begin{aligned} \boldsymbol{W}^* - \boldsymbol{w}^* &= -A_{11}\nabla p - A_{12}\nabla V \\ \boldsymbol{J} &= -A_{21}\nabla p - A_{22}\nabla V \end{aligned} \tag{3.22}$$

where $\boldsymbol{W}^* - \boldsymbol{w}^*$ is the vector of relative velocity of ionic fluid in bone pores, p – the fluid pressure, V – the electric potential, $\boldsymbol{J}$ – the vector of the electric current density, A_{11} – the hydraulic permeability of porous bone, A_{22} – the electric conductivity, A_{12} – the electrokinetic coefficient of electroosmosis, A_{21} – the electrokinetic coefficient of streaming current.

The phenomenological linear equations of classical electrokinetics in porous media (given e.g. in Katchalsky and Curran, 1965; Grodzinsky, 1983) can be derived on the basis of the microscopic flow equations and the homogenization method (cf Gałka et al., 1994).

Equations (3.22) contain the description of four electrokinetic phenomena: the streaming potential (when $\boldsymbol{J} = \boldsymbol{0}$), the streaming current (when $\nabla V = \boldsymbol{0}$), the electroosmosis (when $\nabla p = \boldsymbol{0}$), and the electroosmotic pressure (when $\boldsymbol{W}^* - \boldsymbol{w}^* = \boldsymbol{0}$).

Substituting $\nabla V = \boldsymbol{0}$ in Eqs (3.22) we obtain the following formula for the streaming current density vector $\boldsymbol{J}_{str}$

$$\boldsymbol{J}_{str} = -A_{21}\nabla p = A_{21}A_{11}^{-1}(\boldsymbol{W}^* - \boldsymbol{w}^*) \tag{3.23}$$

The relative velocity $\boldsymbol{W}^* - \boldsymbol{w}^*$ of ionic fluid in bone pores for the considered 1D problem of wet long bone shaft under harmonic longitudinal load is given by Eq (3.19).

On the basis of the scheme from Fig.4 and the Kirchhoff's laws we can write the following non-homogeneous electric state equation

$$\frac{d}{dx}\begin{bmatrix} U(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} 0 & -Z_e \\ -Y_e & 0 \end{bmatrix}\begin{bmatrix} U(x) \\ I(x) \end{bmatrix} + \begin{bmatrix} E_{str}^0(x) \\ 0 \end{bmatrix} \tag{3.24}$$

where: $I(x)$ is the complex amplitude function of the electric current in wet long bone shaft under harmonic longitudinal load ($I = JS$, where J – the current density, S – the cross-section area) and $U(x)$ is the complex amplitude function of the electric voltage associated with the propagation of harmonic longitudinal waves along a wet long bone shaft.

The Eqs (3.7) and (3.24) together with Eqs (3.21), (3.23) and (3.19) constitute the *complete set of equations* for elasto-electric longitudinal harmonic waves in porous long bones filled with physiological fluid.

3.3.1. Solution of the electric state equation

Equation (3.24) can be written as

$$\frac{d\boldsymbol{S}(x)}{dx} = \mathbf{B}\boldsymbol{S}(x) + \boldsymbol{E}(x) \tag{3.25}$$

where: $\boldsymbol{S}(x) = [U(x), I(x)]^\top$ is the electric state vector, $\mathbf{B}$ is the matrics of the system, and $\boldsymbol{E} = [E_{str}^0(x), 0]^\top$ is the vector of the electromotive forces of mechanical origin.

The solution of the non-homogeneous state equation (3.25) have the form (cf e.g. Ogata, 1967)

$$\boldsymbol{S}(x) = \mathrm{e}^{\mathbf{B}x}\boldsymbol{S}(0) + \int\limits_0^x \mathrm{e}^{\mathbf{B}(x-\xi)}\boldsymbol{E}(\xi)\,dxi \tag{3.26}$$

where $e^{\mathbf{B}x}$ is the electric transmission matrix of a wet long bone shaft, and $\boldsymbol{S}(0)$ is the state vector on the input of the system representing the electric boundary conditions for the cross-section of long bone shaft at $x = 0$.

The matrix $e^{\mathbf{B}x}$ can be determined by using the Cayley-Hamilton theorem (cf e.g. Ogata, 1967) and as result one obtains

$$e^{\mathbf{B}x} = \frac{e^{\gamma_e x} - e^{-\gamma_e x}}{2\gamma_e}\mathbf{B} + \frac{e^{\gamma_e x} + e^{-\gamma_e x}}{2}\mathbf{I} \tag{3.27}$$

where γ_e is the propagation constant of the electric current and voltage waves propagating along wet long bone shaft, given by the formula

$$\gamma_e = \sqrt{Z_e Y_e} \qquad\qquad \mathrm{Re}(\gamma_e) > 0 \tag{3.28}$$

In the solution (3.26) of the state eqution (3.24) appear only the exponential functions $e^{\pm\gamma_e x}$ and the products of the exponential functions (such as: $e^{\pm\gamma_1 x}e^{\pm\gamma_e x}$ and $e^{\pm\gamma_2 x}e^{\pm\gamma_e x}$) with the corresponding coefficients, thus, integrating in (3.26) is easy.

4. Conclusions

The propagation of longitudinal harmonic elasto-electric waves in porous long bone shafts filled with a physiological ionic fluid has been investigated. The theoretical description of the problem is proposed in form which combines the Biot theory of elastic waves in fluid-saturated porous solids and the linear equations of classical electrokinetics, and which uses the quantities and schemes analogous to those in the theory of electrical transmission lines. Experimental findings from the biomechanical literature, supporting the proposed theoretical model, are presented. The complete set of equations of the problem is given (Eqs (3.7), (3.24) together with Eqs (3.21), (3.23) and (3.19)), and the solutions of the mechanical (homogeneous)state equation (3.7) and the electrical (non-homogeneous) state equation (3.24) are obtained in the same way: by using the Cayley-Hamilton theorem and the transmission matrix method. The electric voltage $U(x)$ associated with the propagation of longitudinal elastic waves in wet long bone shafts can – theoretically – be used to monitor these waves, and it is possible because *the porous cortical bone* (long bone shaft wall material) filled with a physiological ionic fluid *acts as a mechano-electric transducer*.

A. Appendix

In references Uklejewski and Krakowski (1982), Malecki and Uklejewski (1991) it has been shown that the system of two coupled electrical transmission lines shown in Fig.5 is the electrical analog of a one-dimensionsional dynamical problem of the Biot's theory for fluid-saturated porous media (see the *Confrontation of analogical equations*). The following notations for the electrical quantities are used: $i_1(J_1)$ – current (current density) in line 1; $i_2(J_2)$ – current (current density) in line 2; S – cross-section area of line 1 (line 2); Ψ_1, Ψ_2 – magnetic flux associated with line 1 (line 2); u_1, u_2 – line voltage of line 1 (line 2); $u = \partial\Psi/\partial t$ – relationship between voltage and magnetic flux; L_1, L_2 – selfinductance of line 1 (line 2) per unit length; M_e – mutual inductance between lines 1 and 2 per unit length; C_1, C_2 – capacitance of line 1 (line 2) per unit length; ;C_w – capacitance between lines 1 and 2 per unit length; G_w – conductance between lines 1 and 2 per unit length.

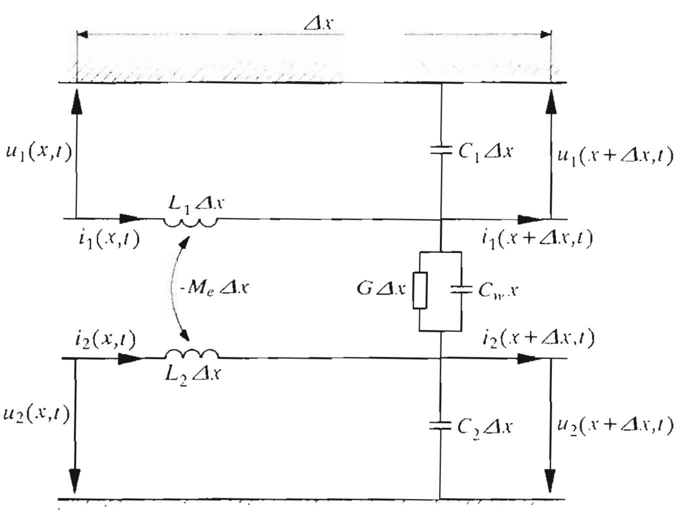

Fig. 5. The electric analogon for one-dimensional dynamic problem of Biot's mechanics of poroelastic materials filled with fluid: an element Δx of two electric coupled transmission lines

A.1. Confrontation of analogical equations

A.1.1. Biot's equations for 1D dynamical state of strain of porous medium

— Constitutive equations

$$\begin{aligned}\sigma_{11} &= (2N + A)\frac{\partial w_1}{\partial x_1} + Q\frac{\partial W_1}{\partial x_1}\\ \sigma &= Q\frac{\partial w_1}{\partial x_1} + R\frac{\partial W_1}{\partial x_1}\end{aligned} \tag{A.1}$$

$$\begin{aligned}\frac{\partial w_1}{\partial x_1} &= \frac{1}{M + 2N}\sigma_{11} - \frac{Q}{R(M + 2N)}\sigma\\ \frac{\partial W_1}{\partial x_1} &= -\frac{Q}{R(M + 2N)}\sigma_{11} + \frac{2N + A}{R(M + 2N)}\sigma\end{aligned} \tag{A.1a}$$

— Equations of motion

$$\begin{aligned}\frac{\partial \sigma_{11}}{\partial x_1} &= \rho_1\frac{\partial^2 w_1}{\partial t^2} + \Big(b + \rho_a\frac{\partial}{\partial t}\Big)\Big(\frac{\partial w_1}{\partial t} - \frac{\partial W_1}{\partial t}\Big)\\ \frac{\partial \sigma}{\partial x_1} &= \rho_2\frac{\partial^2 W_1}{\partial t^2} + \Big(b + \rho_a\frac{\partial}{\partial t}\Big)\Big(\frac{\partial W_1}{\partial t} - \frac{\partial w_1}{\partial t}\Big)\end{aligned} \tag{A.2}$$

A.2. Equations of the electric coupled transmission lines (Fig.5)

$$\begin{aligned}i_1 = J_1 S &= \frac{L_2}{L_1L_2 - M_e^2}\frac{\partial(-\Psi_1)}{\partial x} + \frac{M_e}{L_1L_2 - M_e^2}\frac{\partial(-\Psi_2)}{\partial x}\\ i_2 = J_2 S &= \frac{M_e}{L_1L_2 - M_e^2}\frac{\partial(-\Psi_1)}{\partial x} + \frac{L_1}{L_1L_2 - M_e^2}\frac{\partial(-\Psi_2)}{\partial x}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\frac{\partial(-\Psi_1)}{\partial x} &= L_1(J_1S) - M_e(J_2S)\\ \frac{\partial(-\Psi_2)}{\partial x} &= -M_e(J_1S) + L_2(J_2S)\end{aligned} \tag{A.3a}$$

$$\begin{aligned}\frac{\partial J_1}{\partial x} &= C_1\frac{\partial^2(-\Psi_1)}{\partial t^2} + \Big(G_w + C_w\frac{\partial}{\partial t}\Big)\Big(\frac{\partial(-\Psi_1)}{\partial t} - \frac{\partial(-\Psi_2)}{\partial t}\Big)\\ \frac{\partial J_2}{\partial x} &= C_2\frac{\partial^2(-\Psi_2)}{\partial t^2} + \Big(G_w + C_w\frac{\partial}{\partial t}\Big)\Big(\frac{\partial(-\Psi_2)}{\partial t} - \frac{\partial(-\Psi_1)}{\partial t}\Big)\end{aligned} \tag{A.4}$$

Acknowledgements

The autor would like to thank Prof. W. Derski – the retired V-ce Director of the IPPT PAN and Prof. I. Malecki – the retired General Director of the IPPT PAN for long-term, unfailing, scientific guidance and for inspiring me with the Mechanics and Acoustics of Porous Media, especially with the research work in this field of Maurice A. Biot and with his approach to the scientific investigations: *"Deeper physical insight combined with theoretical simplicity provides the short-cuts leading immediately to the core of extremely complex problems and to straightforward solutions. – This cannot be achieved by methods which are sophisticated and ponderous even in simple cases"* (M.A. Biot: Science and Engineer, *Appl. Mech. Rev.*, 1963, **16**, 2, 89).

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Podłużne harmoniczne fale sprężysto-elektryczne w porowatych kościach długich wypełnionych płynem fizjologicznym

Streszczenie

Przedmiotem pracy jest zagadnienie transmisji sprężysto-elektrycznych podłużnych fal harmonicznych w porowatych kościach długich wypełnionych płynem fizjologicznym. Zbiorczy układ równań zagadnienia wyprowadzono na podstawie Biotowskiej teorii propagacji fal sprężystych w ośrodkach porowatych nasyconych cieczą oraz liniowych równań elektrokinetyki, stosując opis za pomocą wielkości analogicznych do stosowanych w teorii elektrycznych linii przesyłowych. Uzasadniono przydatność zaproponowanego modelu teoretycznego przywołując z literatury wyniki badań eksperymentalnych dotyczące bioelektromechaniki kości. Sygnał elektryczny towarzyszący propagacji podłużnych fal sprężystych w trzonie kości długiej nasyconej płynem fizjologicznym mógłby, przypuszczalnie, zostać użyty celem monitorowania tych fal w zastosowaniu ich np. do pomiaru porowatości kości.

Manuscript received January 25, 1998; accepted for print May 13, 1998