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WEIGHT FUNCTIONS OF THE STRESS INTENSITY FACTORS K_1, K_2 AND K_3 FOR A SINGLE RADIAL CRACK EMANATING FROM A SEMI-CIRCULAR SIDE NOTCH

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Weight functions of stress intensity factors for a single radial crack emanating from a semi-circular side notch are derived for the loading modes I and II using the boundary element method (BEM) together with the complex stress function $Z(z)$ applied at the crack tip and satisfying the Bueckner type singularity. The mode III weight function, being identical with the known solution of two symmetrical cracks emanating from a circular hole, is also shown to complete the present analysis. All weight functions, corresponding to particular modes, are represented in a unified form convenient to create the database suitable for rapid calculations of the stress intensity factors K for any elastic stresses distributed along the potential crack path. Accuracy of the present approach is about ninete nine percent, verified by comparing the values of K obtained using the weight function method with particular solutions known from the literature and to the BEM results.

Key words: crack, stress intensity factor, weight function

1. Introduction

The stress intensity factor K is one of the most important parameters used in fracture mechanics for estimating the safety margins and durability of engineering structures, when the presence of material imperfections as cracks, flaws and inclusions is tolerable. In such cases an actual value of K, estimated for the real structure, is compared to the critical K values, obtained from experimental tests and recognised as characteristics of the material, representing quantitatively possible failure modes. In this way brittle fracture conditions, fatigue crack growth rate and some effectts of the corrosive environment may be considered for the structure under static and variable loading conditions.

Since the value of K depends on many factors; namely, shape of the body, crack geometry and its location, loading and displacement conditions, residual stresses, temperature field etc., there exist a great number of particular solutions of K available in the literature (cf Murakami (1987); Nisitani et al. (1992); Sih (1973a,b); Tada et al. (1973)), and many methods developed for a convenient approach when solving various crack problems.

The weight function method, suggested by Bueckner (1970), (1973) and Rice (1972) has the virtue of being most versatile, due to the possibility of putting together different linear elastic stress fields – as far as the principle of superposition remains valid. Since the appropriate weight functions and the resultant stress distributions $\sigma_{1j}(x)$ along the potential crack path of the uncracked body are found, the related stress intensity factors K_1, K_2 and K_3, corresponding to the opening, sliding and tearing loading modes, respectively, may be determined by the following simple integration

$$K_j = \int_0^a \sigma_{1j}(x) m^{(j)}(x,a)\, dx \tag{1.1}$$

where

a	–	crack length
$m^{(j)}(x,a)$	–	known weight functions adequate for a cracked body
$\sigma_{1j}(x)$	–	components of the stress tensor released on the crack surface in the directions corresponding to three loading modes $j = 1,2,3$.

Some approximate solutions of the weight functions method can also be found in Wang (1993) and (1994).

Some improvements of the weight function method were published by Molski (1992) and (1994a). It has been shown that any form of the classical weight function $m^{(j)}$ may be represented by two different functions: the correction one F_j and the unitary weight function $w^{(j)}$, accompanied by the scale coefficient $\sqrt{\pi a}$. Since F_j describes the effect of a uniform load applied directly to the crack surface, the $w^{(j)}$ function, having the features of the weight density function, qualifies the load along the crack surface and estimates its contribution to the stress intensity factor value.

In the present study a plane elastic problem of semi-circular side notch with a single radial crack loaded symmetrically on both sides, as shown in Fig.1, has been analysed. The normal and tangential stress components $\sigma_{1j}(x)$ that appear in uncracked body, are released on the crack faces and form multiaxial stress state, represented in the vicinity of the crack tip by three different stress intensity factors K_1, K_2 and K_3.

Thus, the aim of the present work is to determine weight functions corresponding to the modes I and II, since the mode III solution obtained by means of the weight function method is identical to that of the symmetric problem of two opposite radial cracks emanating from a circular hole, published by Molski (1996).

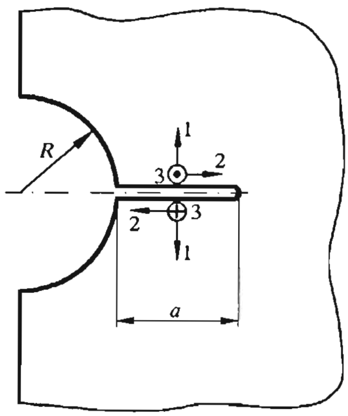

Fig. 1. Single edge crack emanating from a semi-circular notch, subjected to multiaxial load on the surface

2. Determination of mode I and mode II weight functions

2.1. General principles of the method

The method applied to evaluation of weight functions for the mode I and mode II, i.e. for $j = 1,2$, is the same as that used in solving the case of *symmetric circular hole with two opposite radial cracks, described by Molski* (1996).

The method is based on Betti's reciprocity theorem and Bueckner's procedure which demonstrate that if a known stress or displacement field, corresponding to the known value of the stress intensity factor K, is applied directly in the vicinity of the crack tip as the only loading condition, the displacements $u_j^*(r,\varphi)$ and $v_j^*(r,\varphi)$ of the remaining part of the whole body stand for the weight function components. Though the method provides the weight functions valid for the whole body, only solutions of $m^{(j)}$ related to the crack-line and corresponding to the relative opening and sliding displacements of the crack faces; i.e., $v_1^*(r,\pi) - v_1^*(r,-\pi)$ and $u_2^*(r,\pi) - u_2^*(r,-\pi)$, are of interest here.

The complex stress function $Z_j(z)$ satisfying Bueckner's singularity of the stress field near the crack tip is given by

$$Z_j(z) = \frac{B_j}{\sqrt{z^3}} \tag{2.1}$$

where

- z – complex number in the polar coordinate system (r,φ) originated at the crack tip, $z = r\exp(\mathrm{i}\varphi)$
- B_j – Bueckner's parameter, representing the strength of the singularity of the stress field for the modes I $(j=1)$ and II $(j=2)$, respectively, for a pair of self-equilibrating forces P_j applied at a small distance c from the crack tip, being the only boundary traction imposed on the body, $B_j = P_j\sqrt{c}/\pi$.

So far, the only reasonable and most accurate way in determining the crack face displacements $v_1^*(r,\pm\pi)$ and $u_2^*(r,\pm\pi)$ are numerical techniques based on the finite or boundary element methods.

2.2. Boundary element modelling and numerical results

Numerical calculations of weight functions for different ratios of the crack length a to the notch radius R have been made using the BEM software program Cracker (cf Portela and Aliabadi (1993)). Due to the symmetry of the present problem, only one half of the semi-circular side notch with one crack face in a quadrant of the plane has been modelled by 198 circular and straight boundary elements. In order to improve accuracy of the numerical results in the way suggested by Tada et al. (1973), a small semi-circle with its diameter equal to 0.5% of the crack length a has been modelled at the crack tip region by 17 nodes of 8 circular quadratic boundary elements. The boundary displacements, corresponding to the stress function $Z_j(z)$, have been applied to each of 17 nodes as the boundary conditions of the problem, causing exactly the same effect as that produced by the self-equilibrating pair of forces P_j. Numerically obtained crack face displacements i.e., $v_1^*(r,\pi)$ for the opening mode I and $u_2^*(r,\pi)$ for the sliding mode II, have been analysed separately and interpreted as displacement weight functions.

Since the weight functions in this case do not depend on the elastic material constants E and ν, their values have been conveniently chosen as $E = 1$ and $\nu = 0$ to simplify the output data analysis.

The next step consists in transforming computed crack face displacements $v_1^*(r,\pi)$ and $u_2^*(r,\pi)$ into the correction and unitary weight functions, respectively, (cf Molski (1992), (1994a,b)). To facilitate the description of weight functions for various a/R ratios in the whole domain, a new parameter $s = a/(a+R)$ has been introduced, where $s \in< 0,1 >$.

Two different correction functions: $F_1(s)$ and $F_2(s)$, shown by the solid and dashed lines, respectively, in Fig.2, have been obtained by numerical integration of the previously found and normalized displacement functions $v_1^*(r,\pi)$ and $u_2^*(r,\pi)$. They depend only on the parameter s and indicate influence of the uniform normal σ_{11} and tangential σ_{12} stresses, released or applied directly to the crack surfaces, on K_1 and K_2, respectively. Numerical values of the correction functions $F_1(s)$ and $F_2(s)$ interpolated by polynomials are given by Eqs (2.2) and (2.3). To complete the analysis, the $F_3(s)$ correction function for the mode III is also shown in Fig.2 by a dotted line, for the tearing uniform load distributed along the crack surfaces. More details related to the mode III weight function solution can be found by Molski (1996).

$$
\begin{aligned}
F_1(s) &= 1.1215 - 0.882s + 2.426s^2 - 4.414s^3 + 6.763s^4 + \\
&- 5.884s^5 + 1.991s^6 && (2.2) \\
F_2(s) &= 1.1215 - 0.202s - 0.050s^2 + 0.146s^3 + 0.251s^4 + \\
&+ 0.066s^5 - 0.211s^6 && (2.3) \\
F_3(s) &= 1.0 - 0.156s + 0.362s^2 - 0.515s^3 + 0.309s^4 && (2.4)
\end{aligned}
$$

The values of stress intensity factors K_1, K_2 and K_3 are expressed now by Eqs (2.5), valid for the uniform stresses σ_{11}, σ_{12} and σ_{13} released along the crack surfaces

$$
\begin{aligned}
K_1 &= \sqrt{\pi a}\sigma_{11}F_1(s) \\
K_2 &= \sqrt{\pi a}\sigma_{12}F_2(s) \\
K_3 &= \sqrt{\pi a}\sigma_{13}F_3(s)
\end{aligned}
\tag{2.5}
$$

In order to take into consideration the effects of non-uniform stresses $\sigma_{1j}(x)$ distributed along the crack sides, the unitary weight function, being in fact a weight density function, should also enter the integral to qualify the load applied.

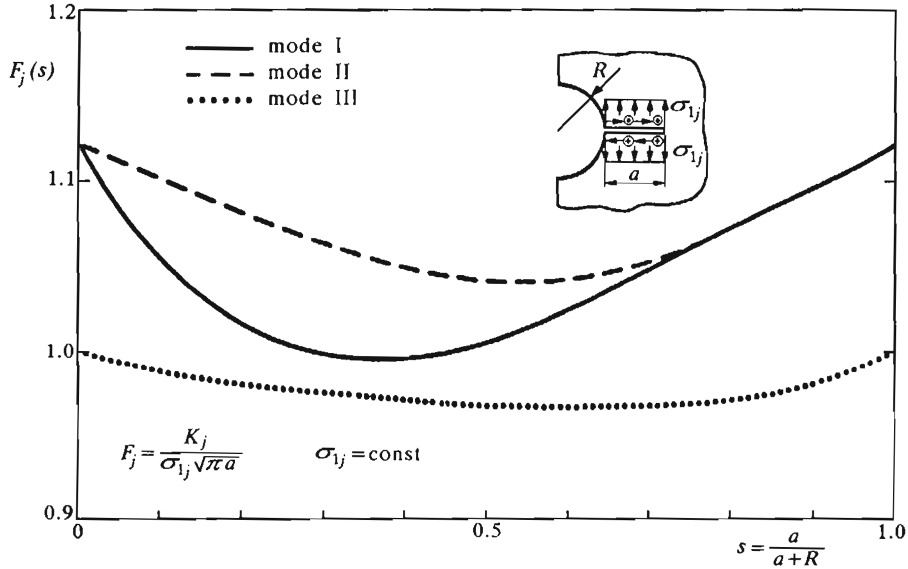

Fig. 2. Correction functions $F_j(s)$ for the modes I, II and III

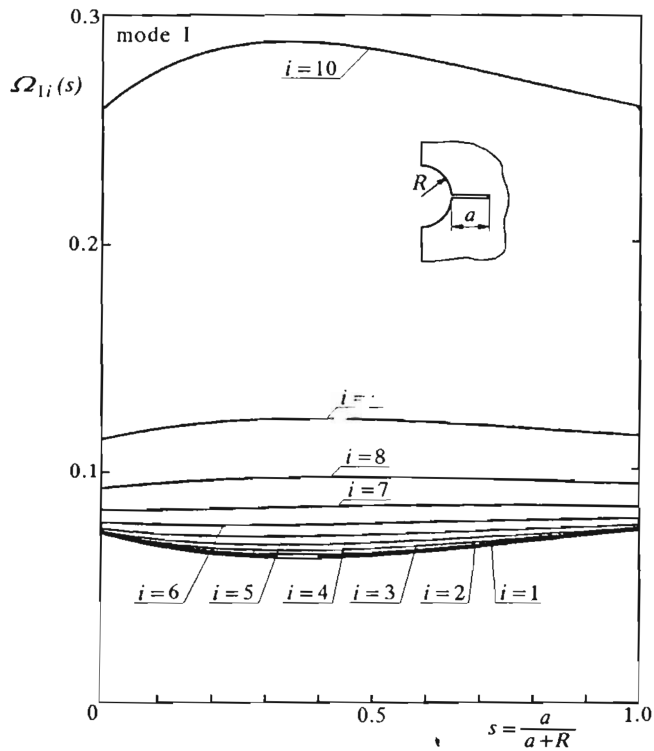

Fig. 3. Fractional values of the unitary weight function integrals (weight coefficients) for the mode I: $\Omega_{1i}(s)$ versus s

In that case the stress intensity factor K_j is represented by

$$K_j = \sqrt{\pi a} F_j(s) \int_0^1 \sigma_{1j}\Big(\frac{x}{a}\Big) w^{(j)}\Big(\frac{x}{a}, s\Big)\, d\Big(\frac{x}{a}\Big) \tag{2.6}$$

where the value of the integral being a weighted averaged quantity, may be interpreted as an equivalent uniform stress $(\sigma_{eq})_j$ applied directly to the crack faces that gives the same stress intensity factor value as the real non-uniform one.

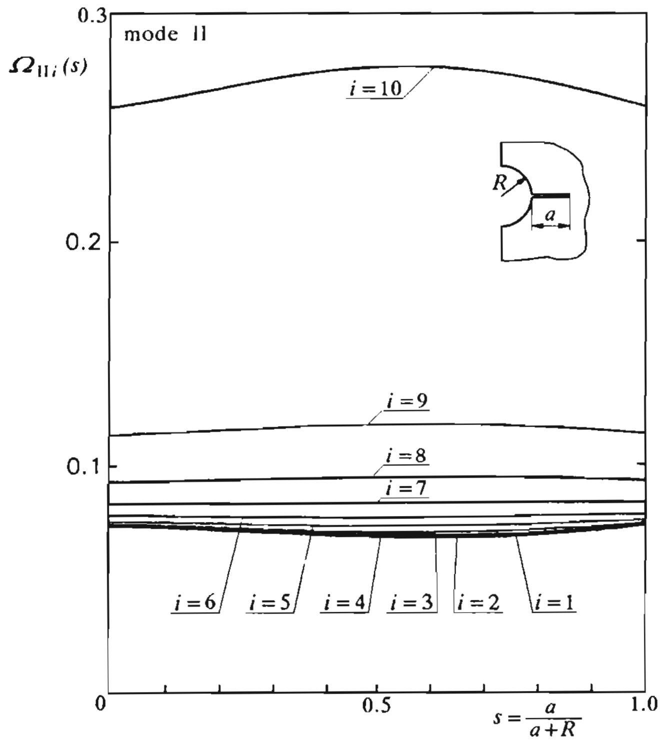

Fig. 4. Fractional values of the unitary weight function integrals (weight coefficients) for the mode II: $\Omega_{IIi}(s)$ versus s

As Molski (1992) and (1994a) showed, for numerical purposes in the case of nonuniform stress along the crack path, it is more convenient to use fractional values of the unitary weight function integrals $\Omega_i(s)$, which are obtained by dividing the whole crack length a into ten equal segments i and integrating the $w(x/a, s)$ function numerically, starting from the crack end opposite to the considered crack tip. The courses of fractional integral values $\Omega_{Ii}(s)$, $\Omega_{IIi}(s)$ and $\Omega_{IIIi}(s)$ of the unitary weight functions versus the shape parameter s are shown in Fig.3, Fig.4 and Fig.5. They are interpreted as weight coefficients, valuating the non-uniform stresses for each tenth of the whole crack length.

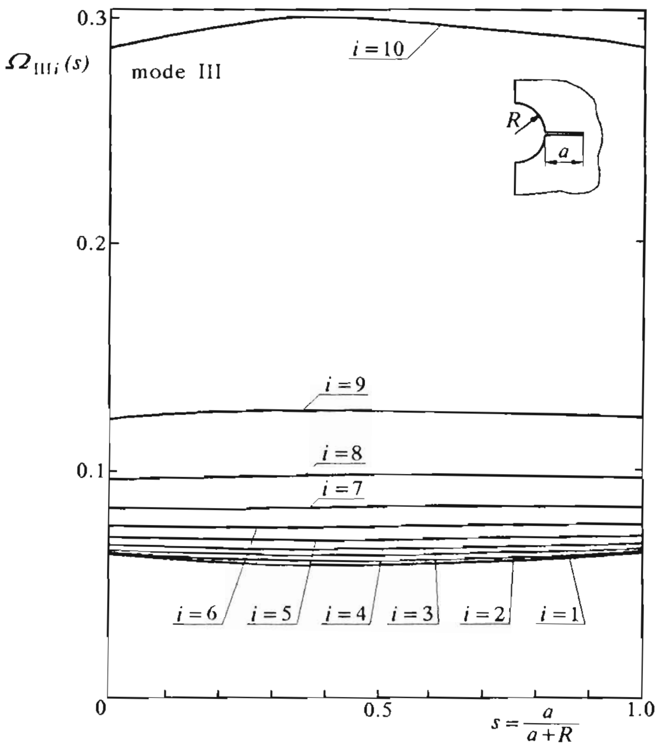

Fig. 5. Fractional values of the unitary weight function integrals (weight coefficients) for the mode III: $\Omega_{IIIi}(s)$ versus s

Once obtained correction functions $F_j(s)$ and fractional integral values $\Omega_{Ii}(s)$, $\Omega_{IIi}(s)$ and $\Omega_{IIIi}(s)$, being only functions of s, are interpolated by polynomials and incorporated into the main program (cf Molski and Truszkowski (1995)) enabling the stress intensity factors K_j (see Eq (2.6)), for any load distribution to be calculated.

All normalized K_j values obtained in this way and used below for comparative studies in the accuracy assessment, are indicated as the results obtained by means of the unitary weight function method.

3. Assessment of the accuracy

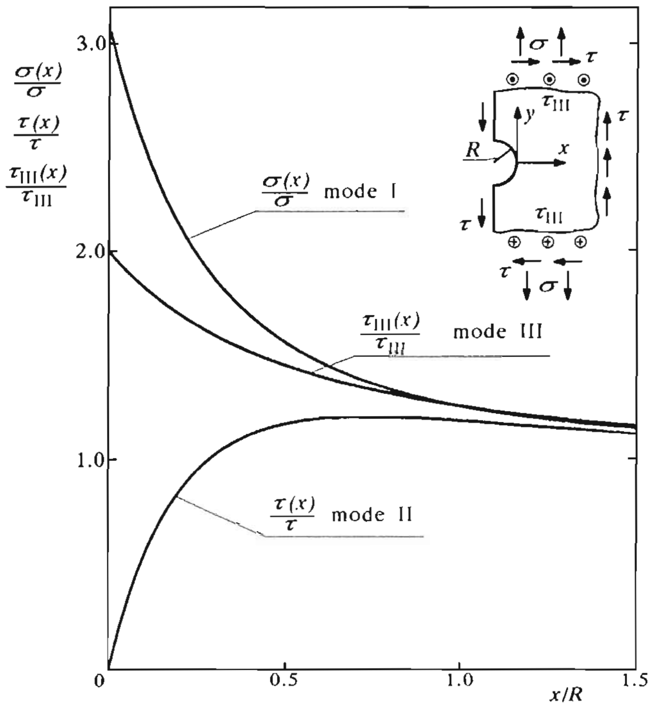

Fig. 6. Elastic stress distributions along the potential crack path for various loading conditions

To assess accuracy of the present approach, the values of stress intensity factors K_1, K_2 and K_3 calculated using the unitary weight function method have been compared to the reference solutions from the literature (cf Murakami (1987); Nisitani and Isida (1982); Sih (1973a); Tada et al. (1973)) and to the BEM results obtained by the author for modes I and II. Uniform loads: σ for the mode I, τ for the mode II and τ_{III} for the mode III have been applied to the plate sufficiently far from the cracked area. The resultant stresses $\sigma_{11}(x/R)$ and $\sigma_{12}(x/R)$ that appear along the potential crack path ahead of the notch (Fig.6) have been determined for an uncracked body using the BEM analysis, since only the mode III exact solution to this problem is known from the theory of elasticity.

These stresses, together with the weight coefficients $\Omega_i(s)$ and correction functions $F(s)$, contribute to calculation of the stress intensity factors for various a/R ratios according to Eq (2.6). The K results obtained in that way, are compared to the reference K values known from the literature for these particular loading conditions, and to the ones derived by the author using BEM for modes I and II.

The results of normalised stress intensity factors for three loading modes are shown in Tables 1, 2 and 3, where the three new correction functions $Y_{\rm I}$, $Y_{\rm II}$ and $Y_{\rm III}$, appropriate for the loading conditions shown in Fig.6, are defined as follows

$$Y_{\rm I} = \frac{K_1}{\sigma\sqrt{\pi(a+R)}} \qquad Y_{\rm II} = \frac{K_2}{\tau\sqrt{\pi(a+R)}} \qquad Y_{\rm III} = \frac{K_3}{\tau_{\rm III}\sqrt{\pi(a+R)}} \tag{3.1}$$

For all the values of K shown in the tables, the agreement is very good with the maximal difference not exceeding one percent.

Table 1. Mode I correction functions $Y_{\rm I}$

a/R	$Y_{\rm I}(a/R)$		
	BEM	[9,11]	UWFM*
0.02	0.464	0.462	0.462
0.05	0.684	0.680	0.680
0.10	0.865	0.863	0.863
0.20	1.020	1.019	1.019
0.30	1.078	1.076	1.079
0.40	1.103	1.101	1.105
0.50	1.114	1.113	1.117
0.60	1.119	1.118	1.122
0.80	1.122	1.121	1.126
1.00	1.123	1.122	1.127
1.50	1.123	–	1.127
2.00	1.123	–	1.128
2.50	1.123	–	1.128

* – Unitary Weight Function Method.

Table 2. Mode II correction functions Y_{II}

a/R	$Y_{\mathrm{II}}(a/R)$	
	BEM	UWFM*
0.10	0.119	0.118
0.25	0.329	0.326
0.40	0.490	0.486
0.50	0.570	0.566
0.75	0.710	0.706
1.00	0.794	0.792
1.50	0.888	0.886
2.00	0.937	0.936

Table 3. Mode III correction functions Y_{III}

a/R	$Y_{\mathrm{III}}(a/R)$	
	Sih (exact)	UWFM*
0.01	0.19753	0.1975
0.04	0.38105	0.3806
0.10	0.56302	0.5619
0.30	0.80615	0.8057
0.50	0.89581	0.8968
1.00	0.96825	0.9695
1.50	0.98712	0.9863
2.00	0.99381	0.9910

4. Concluding remarks

The boundary element method accompanied with the complex stress function $Z(z)$, describing the Bueckner type singularity at the vicinity of the crack tip, have appeared to be a very effective numerical tool in determination of the stress intensity factor weight functions of modes I and II, for the problem of a single radial crack emanating from a semi-circular side notch.

All three weight functions, including that for the tearing mode, are generally different, however, for relatively long cracks compared with the notch radius R, i.e. in the range $0.7 < s < 1.0$, mode I and mode II weight functions are quite similar.

The unitary weight functions, represented by the weight coefficients $\Omega_{\mathrm{I}i}(s)$, $\Omega_{\mathrm{II}i}(s)$ and $\Omega_{\mathrm{III}i}(s)$, as well as the correction functions $F_j(s)$ are scale-independent and valid for any a/R ratio.

The accuracy of the unitary weight function method used for computation of the stress intensity factors for the three loading modes appeared to be satisfactory with the maximal error much smaller than one percent compared to some particular K solutions known from the literature and to the BEM results obtained by the author. This is a very optimistic conclusion considering applications of the present solutions, since the stresses distributed ahead of the notch and applied for the unitary weight function method were derived numerically. This fact might cause additional small errors in the values of K. Thus, a similar accuracy of about 99% may be expected for any other loads, including the residual and thermoelastic stress fields, as far as their distributions along the potential crack path of unckracked body are properly identified.

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Numerical results have been obtained using the boundary element software program Cracker from Wessex Institute of Technology, UK

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Funkcje wagowe typu I, II i III dla pojedynczej szczeliny wychodzącej z dna półokrągłego karbu

Streszczenie

Rozważano plaskie zagadnienie pojedynczej szczeliny wychodzącej z dna półokrągłego karbu. Wyznaczono funkcje wagowe, służące do obliczania wartości współczynników intensywności naprężenia K_1 i K_2 dla dowolnych pól naprężeń uwalnianych na powierzchni szczeliny. Rozwiązania te uzyskano łącząc numeryczną metodę elementu brzegowego MEB z analityczną, zespoloną funkcją naprężeń $Z(z)$, opisującą osobliwość Buecknera w otoczeniu wierzchołka szczeliny. Otrzymane rozwiązania funkcji wagowych przedstawiono w zunifikowanej formie, umożliwiającej utworzenie komputerowej bazy danych. Podano również funkcję wagową dla przypadku ścinania poprzecznego – typ III, która jest identyczna z wcześniej opublikowanym rozwiązaniem dla otworu kołowego z dwiema symetrycznymi szczelinami.

We wszystkich przypadkach oszacowany błąd obliczeń współczynników K nie przekroczył jednego procenta.

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