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STOCHASTIC HOMOGENIZATION OF THE FIRST GRADIENT-STRAIN MODELLING OF ELASTICITY

BARBARA GAMBIN

Institute of Fundamental Technological Research, Polish Academy of Sciences

e-mail: bgambin@ippt.gov.pl

Stochastic Γ-convergence concept of the mathematical theory of homogenization in a version is applied to calculation of effective energy of non-simple elastic body with a microinhomogeneous random structure. The problem of homogenization in the case considered is solved by the application and generalization of the idea introduced by Dal Maso and Modica (1986). The authors proved the theorem of convergence for a special class of stochastic integral functionals. The form of homogenized, non-random integral functional was given. The first gradient-strain modelling of elasticity leads to integral functionals of internal energy depending on the second gradients of displacements. The main theorem formulated in the paper is a generalization of the result of Dal Maso and Modica (1986) for integrands of functionals depending on the second gradients of displacement limited, however the, to linear constitutive relations. The form of effective non-random, integral functional is given. It is interpreted as a internal energy of homogenized, effective non-simple material body with well defined effective, constant properties. An example of the Kirchhoff plate with thickness randomly changing in one direction is considered and effective stiffness coefficients are explicitly calculated. They depend on a volume fraction of stiffeners in a matrix material.

Key words: stochastic homogenization, effective properties, gradient-strain modelling

1. Introduction

To introduce the functional formulation of non-simple body let us start from equations of the model.

The equations of statical equilibrum of a body occupying the domain A

in R^3 are (cf Hlavacek and Hlavacek (1969))

$$\tau_{ij,i}(\boldsymbol{x}) - \mu_{ijk,ik}(\boldsymbol{x}) + X_j(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in A \qquad i,j,k = 1,2,3 \tag{1.1}$$

here $\tau_{ij} = \tau_{ji}$ and μ_{ijk} denote the stress tensor and the couple stress tensor, respectively. $\boldsymbol{X}$ is the body force vector per unit volume.

We suppose that the internal energy per unit volume has the form

$$\mathcal{A}\Big(\varepsilon_{ij}(\boldsymbol{x}), \kappa_{ijk}(\boldsymbol{x})\Big) = \frac{1}{2}K_{ijkl}(\boldsymbol{x})\varepsilon_{ij}(\boldsymbol{x})\varepsilon_{kl}(\boldsymbol{x}) + \\ + \frac{1}{2}M_{ijklmn}(\boldsymbol{x})\kappa_{ijk}(\boldsymbol{x})\kappa_{lmn}(\boldsymbol{x}) + N_{ijklm}(\boldsymbol{x})\varepsilon_{ij}(\boldsymbol{x})\kappa_{klm}(\boldsymbol{x}) \tag{1.2}$$

where

$$\varepsilon_{ij}(\boldsymbol{x}) = \frac{1}{2}\Big(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\Big) \qquad \kappa_{ijk}(\boldsymbol{x}) = \varepsilon_{jk,i}(\boldsymbol{x}) \tag{1.3}$$

$$K_{ijkl} = K_{klij} = K_{jikl} \qquad M_{ijklmn} = M_{lmnijk} \\ N_{ijklm} = N_{jiklm} \tag{1.4}$$

and **K**, **M**, **N** are bounded and measurable functions in $\overline{A}$ ($\overline{A}$ is the closure of the region A). Displacement boundary conditions are homogeneous in the displacement field, i.e. $\boldsymbol{u} = \boldsymbol{0}$ on ∂A.

Then the constitutive relations become

$$\tau_{ij} = \frac{\partial \mathcal{A}}{\partial \varepsilon_{ij}} = K_{ijpq}\varepsilon_{pq} + N_{ijpqr}\kappa_{pqr} \\ \mu_{ijk} = \frac{\partial \mathcal{A}}{\partial \kappa_{ijk}} = N_{pqijk}\varepsilon_{pq} + M_{ijkpqr}\kappa_{pqr} \tag{1.5}$$

Moreover, we assume that the form $\mathcal{A}(\cdot,\cdot)$ is coercive, i.e.

$$\mathcal{A}(\varepsilon_{ij}, \kappa_{ijk}) \geq c \sum_{i,j,k=1}^{3} (\varepsilon_{ij}^2 + \kappa_{ijk}^2) \tag{1.6}$$

where $c > 0$ is a constant.

The global internal energy of a body has the form

$$\int_A \Big[\frac{1}{2}K_{ijkl}(\boldsymbol{x})\varepsilon_{ij}(\boldsymbol{x})\varepsilon_{kl}(\boldsymbol{x}) + \frac{1}{2}M_{ijklmn}(\boldsymbol{x})\kappa_{ijk}(\boldsymbol{x})\kappa_{lmn}(\boldsymbol{x}) + \\ + N_{ijklm}(\boldsymbol{x})\varepsilon_{ij}(\boldsymbol{x})\kappa_{klm}(\boldsymbol{x})\Big]\, dx \tag{1.7}$$

We rewrite it in the abbreviate notation

$$F(u,A) \equiv \int_A f\Big(x, \nabla u(x), \nabla\nabla u(x)\Big)\, dx \tag{1.8}$$

where the integrand $f(x,p,q)$ is given by the following relations

$$p = \nabla u \qquad\qquad q = \nabla\nabla u \tag{1.9}$$

$$f(x,\nabla u,\nabla\nabla u) = \frac{1}{2}\mathbf{K}(x)\nabla u\nabla u + \frac{1}{2}\mathbf{M}(x)\nabla\nabla u\nabla\nabla u + \\ +\frac{1}{2}\mathbf{N}(x)\nabla u\nabla\nabla u \tag{1.10}$$

The minimum of the energy functional

$$\min_{v\in W_0^{2,2}(A)^3}\Big\{F(v,A) + \int_A Xv\, dx\Big\} \tag{1.11}$$

is reached for a displacement field $u \in W_0^{2,2}(A)^3$, which is a unique solution of Eqs (1.1) ÷ (1.5) and $W_0^{2,2}(A)^3 = W_0^{2,2}(A) \times W_0^{2,2}(A) \times W_0^{2,2}(A)$, $W_0^{2,2}(A)$ is the Sobolev space.

If the material is homogeneous the material coefficients **K**, **M**, **N** are constant tensors and the solution u can be found using standard numerical procedures. But if the material is microinhomogeneous, i.e. the material coefficients are not longer constant tensors, but they are tensor fields rapidly changing in the domain, the problem becomes much more complicated. In such a situation the numerical procedures are ill-posed problems. Then an overall, macroscopic description of microinhomogeneities is needed to establish the global behaviour of the body. If the size of inhomogeneity approaches zero then different methods of homogenization are applied. If the microstructure is assumed to be periodic, the classical, deterministic homogenization gives good prediction of the global behaviour. In the case of periodic inhomogeneous non-simple body, the homogenization of the first gradient-strain modelling of elasticity was presented by Bytner and Gambin (1988). If the microstructure of the medium is not strictly periodic but only *stochastically periodic* one needs the concept of stochastic homogenization. Such inhomogeneities correspond to a large number of real physical microstructures in material bodies.

In the considered model a stochastic microstructure causes the material coefficients **K**, **M**, **N** to be random fields in the domain A. Then Eqs (1.1) ÷ (1.5) become stochastic differential equations, and Eq (1.8) becomes a stochastic integral functional. We will use the method of stochastic homogenization to pass from the point of view of stochastic differential equations to be solved (Eqs (1.1) ÷ (1.5)) to the viewpoint of random functional to be minimized. The stochastic homogenization process will be defined under general assumptions about the probabilistic properties of random functionals.

Below the main theorem is formulated and one gets the form of effective, macroscopic internal energy of homogeneous, homogenized medium. The theorem forms a base of numerical calculations of effective material parameters. They depend on probabilistic characteristics of random inhomogeneities. The example of analytical calculations is given.

2. The class $\mathcal{F}$ of integral functionals and its structure

Let $\mathcal{A}_0$ be the family of all open bounded subsets of R^3.

We introduce the class of all functionals $\mathcal{F}$, $F \in \mathcal{F}$

$$F: \quad L^2_{loc}(R^3)^3 \times \mathcal{A}_0 \to \overline{R} \qquad (\overline{R} = R \cup +\infty) \tag{2.1}$$

such that

$$F(\boldsymbol{u}, A) = \begin{cases} \int_A f\Big(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}), \nabla\nabla \boldsymbol{u}(\boldsymbol{x})\Big)\, d\boldsymbol{x} & \text{if } \boldsymbol{u}\Big|_A \in W^{2,2}(A)^3 \\ +\infty & \text{otherwise} \end{cases} \tag{2.2}$$

where

$$f: \quad R^3 \times (R^3 \times R^3) \times (R^3 \times R^3 \times R^3) \to R^+ \tag{2.3}$$

is any function $f(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q})$ Lebesque measurable in $\boldsymbol{x}$, convex in $\boldsymbol{q}$ and satisfying the inequalities

$$\begin{aligned} \lambda|q|^2 \le f(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q}) \le \Lambda(1 + |p|^2 + |q|^2) \\ \left|\sqrt{f(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q})} - \sqrt{f(\boldsymbol{x}, \boldsymbol{p}', \boldsymbol{q}')}\right| \le S(|\boldsymbol{p} - \boldsymbol{p}'|) \end{aligned} \tag{2.4}$$

where λ, Λ, S are positive constants.

As we want to study the random integral functionals, that are measurable maps of a probabilistic space (Ω, ξ, P) into $\mathcal{F}$, and their convergence, we need a topological structure on $\mathcal{F}$.

To create it, we define a distance d between two functionals F, G

$$d(F, G) = \sum_{i,j,k=1}^{\infty} \frac{1}{2^{i+j+k}} \Big| \arctan T_{1/i} F(w_j, B_k) - \arctan T_{1/i} G(w_j, B_k) \Big| \tag{2.5}$$

where

$\mathcal{W} = \{w_i, i \in N\}$ – countable dense subset of $W^{2,2}(R^3)^3$

$\mathcal{B} = \{B_k, k \in N\}$ – countable dense subfamily of $\mathcal{A}_0$

and T_ε is a Yosida transform of F defined in appendix.

Following idea of the proof given by Dal Maso and Modica (1986) we claim that the metric space $(\mathcal{F}, d)$ is compact, complete and separable. Besides, the convergence in metric is equivalent to the $\Gamma(L^2)$ convergence

$$\lim_{n\to\infty} d(F_n, F_\infty) = 0 \Leftrightarrow \Gamma(L^2)^- \lim_{n\to\infty} F_n = F_\infty \tag{2.6}$$

the definition of $\Gamma(L^2)^-$ convergence is given in appendix.

Having the distance d, we have the Borel σ-field $\sum_d$ generated in $\mathcal{F}$. Now we call F the random integral functional iff it is measurable function

$$F: \ \Omega \to \mathcal{F} \qquad\qquad F(w) \in \mathcal{F} \tag{2.7}$$

between (Ω, ξ) and $(\mathcal{F}, \sum_d)$.

The real microinhomogeneous, random structure can not be defined arbitrary. Having in mind the aim of homogenization, i.e. to obtain a homogenized, non-random structure, special assumption about random integral functionals should be assumed. To introduce the intuitive properties of random fields such as; translational invariance and independence of probabilistic model of the scale (homothety) the following mappings of functions, sets in R^3 and functionals are defined. For every $c > 0$ and $\varepsilon > 0$, we define the operators τ_c and ρ_ε, respectively, of translation and homothety.

$$\begin{aligned} &\tau_c \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{c}) &&\tau_c A = \{\boldsymbol{x} \in R^3 : \ \boldsymbol{x} - \boldsymbol{c} \in A\} \\ &\rho_\varepsilon \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) &&\rho_\varepsilon A = \{\boldsymbol{x} \in R^3 : \ \varepsilon \boldsymbol{x} \in A\} \end{aligned} \tag{2.8}$$

Finally, if $F \in \mathcal{F}$ then the functionals $\tau_c F$ and $\rho_\varepsilon F$ are defined by

$$\begin{aligned} (\tau_c F)(\boldsymbol{u}, A) &= F(\tau_c \boldsymbol{u}, \tau_c A) \\ (\rho_\varepsilon F)(\boldsymbol{u}, A) &= \varepsilon^3 F(\rho_\varepsilon \boldsymbol{u}, \rho_\varepsilon A) \end{aligned} \tag{2.9}$$

for every $\boldsymbol{u} \in L^2(R^3)^3$, $A \in \mathcal{A}_0$.

If f denotes the inegrand of F then

$$\begin{aligned} \tau_c F(\boldsymbol{u}, A) &= \int_A f\Big(\boldsymbol{x} + \boldsymbol{c}, \nabla \boldsymbol{u}(\boldsymbol{x}), \nabla\nabla \boldsymbol{u}(\boldsymbol{x})\Big)\, d\boldsymbol{x} \\ \rho_\varepsilon F(\boldsymbol{u}, A) &= \int_A f\Big(\frac{\boldsymbol{x}}{\varepsilon}, \nabla \boldsymbol{u}(\boldsymbol{x}), \nabla\nabla \boldsymbol{u}(\boldsymbol{x})\Big)\, d\boldsymbol{x} \end{aligned} \tag{2.10}$$

for every $\boldsymbol{u} \in W^{2,2}(A)^3$, $A \in \mathcal{A}_0$.

Let us introduce the small parameter ε being the size of inhomogeneity. The random fields $\mathbf{K}(\boldsymbol{x}/\varepsilon, \omega)$, $\mathbf{L}(\boldsymbol{x}/\varepsilon, \omega)$, $\mathbf{M}(\boldsymbol{x}/\varepsilon, \omega)$ on R^3 are assumed to be the random fields on (Ω, ξ, P).

Then the problem of minimization of the random functional

$$F_\varepsilon(\omega)(\boldsymbol{u}, A) = \begin{cases} \int_A f\Big(\frac{\boldsymbol{x}}{\varepsilon}, \nabla \boldsymbol{u}, \nabla\nabla \boldsymbol{u}\Big)(\omega)\, dx & \text{if } \boldsymbol{u}\Big|_A \in W^{2,2}(A)^3 \\ +\infty & \text{otherwise} \end{cases} \tag{2.11}$$

is given by

$$\min_{\boldsymbol{u} \in W_0^{2,2}(A)^3} \Big\{ F_\varepsilon(\omega)(\boldsymbol{u}, A) + \int_A X\boldsymbol{u}\, dx \Big\}$$

For every fixed $\varepsilon > 0$ and $\omega \in \Omega$ the above variational formulation describes the behaviour of random microinhomogeneous non-simple material with displacements caused by external body forces $\boldsymbol{X}$.

Let $(F_\varepsilon)_{\varepsilon > 0}$ be a family of random integral functionals over the same probability space (Ω, ξ, P). We say that F_ε is a stochastic homogenization process modelled on a fixed random inegral functional F over Ω if $F_\varepsilon \sim \rho_\varepsilon F$ that is to F_ε and $\rho_\varepsilon F$ the same distribution laws apply.

Let F be a random functional. We say that F is stochastically 1-periodic (with period 1), if

$$F \sim \tau_z F \qquad \forall z \in Z^3$$

where Z is a set of integer numbers.

3. Main theorem of stochastic homogenization

Let F_ε be a stochastic homogenization process modelled on a stochastically 1-periodic random functional F. Suppose that there exists $M > 0$, that two families of random variables

$$\Big(F(\cdot)(\boldsymbol{u}, A)\Big)_{\boldsymbol{u} \in L^2_{loc}(R^3)^3} \qquad \Big(F(\cdot)(\boldsymbol{u}, B)\Big)_{\boldsymbol{u} \in L^2_{loc}(R^3)^3} \tag{3.1}$$

are independent whenever $A, B \in \mathcal{A}_0$ and distance $(A, B) \geq M$.

Then F_ε converges in probability as $\varepsilon \to 0^+$ to the single functional $F_0 \in \mathcal{F}$ independent of ω (i.e. to the constant random integral functional

F_0) given by

$$F_0(\boldsymbol{u}, A) = \begin{cases} \int\limits_A f_0\Big(\nabla \boldsymbol{u}(\boldsymbol{x}), \nabla\nabla \boldsymbol{u}(\boldsymbol{x})\Big)\, dx & \text{if } \boldsymbol{u}\Big|_A \in W^{2,2}(A)^3 \\ +\infty & \text{otherwise} \end{cases} \tag{3.2}$$

where

$$f_0(\boldsymbol{p}, \boldsymbol{q}) = \lim_{\varepsilon\to 0} \int\limits_{\Omega} \min_{\boldsymbol{u}} \Big\{ \frac{1}{|Q_{1/\varepsilon}|} F(\omega)(\boldsymbol{u}, Q_{1/\varepsilon}) : \\ u_i(\boldsymbol{x}) - p_{ij}x_j - \frac{1}{2} q_{ijk}x_jx_k \in W_0^{2,2}(Q_{1/\varepsilon}) \Big\}\, dP(\omega) \qquad i = 1,2,3 \tag{3.3}$$

$$Q_{1/\varepsilon} = \Big\{ \boldsymbol{x} \in R^3 : \ |x_i| < \frac{1}{\varepsilon} \ \ i = 1,2,3 \Big\} \qquad |Q_{1/\varepsilon}| = \Big(\frac{2}{\varepsilon}\Big)^3$$

The convergence in probability is defined by

$$P\Big\{ \omega : \ d(F^\varepsilon, F_0) > \eta \Big\} \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \qquad \forall \eta > 0$$

In the periodic case the homogenized, effective functional was obtained by the solution of variational problems defined on the single cell of periodicity of the structure. In the stochastically periodic case we shall solve the sequence of periodic problems defined on $Q_{1/\varepsilon}$ for every ω

$$\min_{\boldsymbol{u}} \Big\{ \frac{1}{|Q_{1/\varepsilon}|} F(\omega)(\boldsymbol{u}, Q_{1/\varepsilon}) : \ \ u_i - p_{ij}x_j - \frac{1}{2} q_{ijk}x_jx_k \in W_0^{2,2}(Q_{1/\varepsilon}) \Big\}$$

In every cube $Q_{1/\varepsilon}$ we must solve the problem for fixed $\omega \in \Omega$. Then, the mean value is calculated and the limit of the above sequence as ε tends to 0 gives the proper answer. The results jutify numerical procedure which could be applied to calculation of special cases of the random structure. The very simple example of layered structures can be calculated analytically in order to illustrate the result.

3.1. Example

For every ε let us consider a plate formed by parallel stiffeners randomly distributed in a matrix material. The elastic energy of a plate is

$$F^\varepsilon(\omega)(u, A) = \int\limits_A M^\varepsilon_{\alpha\beta\gamma\delta}(\omega, \boldsymbol{x}) u_{,\alpha\beta}\, u_{,\gamma\delta}\ dx_1 dx_2 \\ A \subset R^2 \qquad u \in W^{2,2}(A) \tag{3.4}$$

where

$$M^{\varepsilon}_{\alpha\beta\gamma\delta}(\omega,x_1)=\frac{2}{3}\Big(h^{\varepsilon}(\omega,x_1)\Big)^3\widetilde{B}_{\alpha\beta\gamma\delta}$$

$$\widetilde{B}_{1111}=\widetilde{B}_{2222}=\frac{E}{1-\nu^2}$$

$$\widetilde{B}_{1122}=\widetilde{B}_{2211}=\frac{E\nu}{1-\nu^2}$$

$$\widetilde{B}_{1212}=\widetilde{B}_{1221}=\widetilde{B}_{2112}=\widetilde{B}_{2121}=\frac{E}{2(1+\nu)}$$

and E denotes the Young modulus, ν the Poisson ratio, $h^{\varepsilon}(\omega,x_1)$ the width of the plate changing in x_1-direction.

Let us suppose that $h^{\varepsilon}(\omega,x_1)$ is a random field (here, a stochastic process because the problem is 1-dimensional) defined in the following way

$$h^{\varepsilon}(\omega,x_1)=\sum_{k=1}^{\infty}X^k(\omega)I^k_{\varepsilon}(x_1) \tag{3.5}$$

where

$$I^k_{\varepsilon}(x_1)=\begin{cases}1 & \text{if } x_1\in<k\varepsilon,(k+1)\varepsilon> \qquad k\in Z\\ 0 & \text{otherwise}\end{cases}$$

and $X^k(\omega)$ is the family of random variables defined over the same probability space (Ω,ξ,P) with $\Omega=(h_1,h_2)^z$, Fig.1.

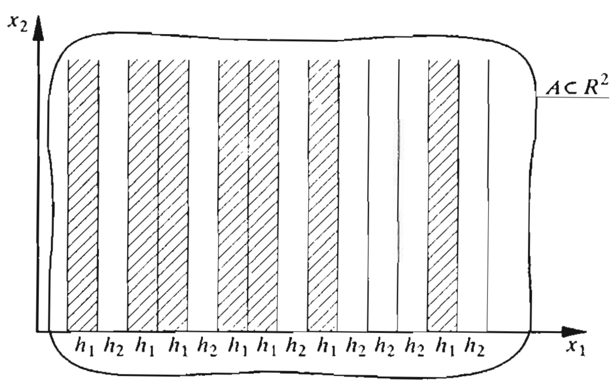

Fig. 1.

The above formulation corresponds to the plate ε-layers structure of which is filled with two kinds of materials: matrix (width h_2) and stiffeners (width h_1), chosen indepedently by Bernoulli's law. The number h_1

appears with probability V and the number h_2 with probability $(1-V)$. Under these assumptions we can apply the homogenization theorem. By solving the Euler equation of minimization problem in quadratures and using the theorem of large numbers (cf Dal Maso and Modica (1986)), we get the energy of homogenized plate

$$F_0(u, A) = \widetilde{M}_{\alpha\beta\gamma\delta} \int_A u_{,\alpha\beta}\, u_{,\gamma\delta}\; dx_1 dx_2 \tag{3.6}$$

where

$$\begin{aligned}
\widetilde{M}_{1111} &= \frac{2}{3}\frac{E}{1-\nu^2}\frac{h_1^3 h_2^3}{V(h_2^3-h_1^3)+h_1^3} \\
\widetilde{M}_{2222} &= \nu^2 \widetilde{M}_{1111} + \frac{2}{3}E[V(h_1^3-h_2^3)+h_2^3] \\
\widetilde{M}_{1122} &= \widetilde{M}_{2211} = \nu \widetilde{M}_{1111} \\
\widetilde{M}_{1212} &= \widetilde{M}_{2121} = \widetilde{M}_{1221} = \widetilde{M}_{2112} = \frac{E}{3(1+\nu)}[V(h_1^3-h_2^3)+h_2^3]
\end{aligned}$$

The obtained expression depends on two widths h_1 and h_2 and the volume fraction V of the stiffeners (matrix volume fraction is $1-V$).

4. Conclusions

The main theorem applied to the simplest, and in a sense unphysical, example (for details see Caillerie (1984)) gives an illustration how to deal with a random structure. It is clear that explicit calculations are possible to perform similarly as in the periodic case only in one-dimensional mode of inhomogeneity. In general only some approximations (bounds) or numerical calculations should be performed. Due to the increasing capacities of computers it has become possible to compute the homogenized properties of a random medium by the simulation of a representative volume element. The basic idea how to do it correctly in agreement with stochastic homogenization is presented by Sab (1992). A generalization to a non-simple material is straightforward. From the practical point of view the main theorem is important for applications in different plate theories.

References

1. BYTNER S., GAMBIN B., 1988, Homogenization of First Strain-Gradient Body, *J. Theor. Appl. Mech.*, **26**, 3

2. CAILLERIE D., 1984, Thin Elastic and Periodic Plates, *Math. Meth. Appl. Sci.*, **6**, 159-191

3. GAMBIN B., 1994, Stochastic Homogenization, *Control and Cybernetics*, **23**, 4

4. HLAVACEK I., HLAVACEK M., 1969, On the Existance and Uniqueness of Solution and Some Variational Principles in Linear Theories of Elasticity with Couple Stresses, *Appl. Mat.*, **14**, 5

5. DAL MASO G., MODICA L., 1986, Nonlinear Stochastic Homogenization, *Ann. di Mat. Pur. ed Applic.*, 4, CXLIV, 374-389

6. SAB K., 1992, On the Homogenization and the Simulation of Random Materials, *Eur. J. Mech., A/Solids*, **11**, 5, 585-607

A. Appendix

Let (X,τ) be a topological space and F_h, $h \in N$, $F_h : x \to \overline{R}$ be a sequence of functions.

We define

$$\Gamma^{-}(\tau)\limsup_{\substack{h\to\infty \\ y\to x}} F_h(y) = \sup_{U\in\tau(x)}\limsup_{h\to\infty}\inf F_h(y)$$

$$\Gamma^{-}(\tau)\liminf_{\substack{h\to\infty \\ y\to x}} F_h(y) = \sup_{U\in\tau(x)}\liminf_{h\to\infty}\inf F_h(y) \qquad y \in U \tag{A.1}$$

where $\tau(x)$ is the family of open sets, for the topology τ, containig X.

When

$$\Gamma^{-}(\tau)\limsup_{\substack{h\to\infty \\ y\to x}} F_h(y) = \Gamma^{-}(\tau)\liminf_{\substack{h\to\infty \\ y\to x}} F_h(y) \tag{A.2}$$

we shall denote their common value by

$$\Gamma^{-}(\tau)\lim_{\substack{h\to\infty \\ y\to x}} F_h(y) \tag{A.3}$$

or briefly by

$$\Gamma^{-}(\tau)\lim_{h\to\infty} F_h(x) \tag{A.4}$$

We shall say that $F = \Gamma^-(\tau)\lim\limits_{h\to\infty} F_h$, iff

$$\forall x \in X \qquad F(x) = \Gamma^-(\tau)\lim_{h\to\infty} F_h(x) \tag{A.5}$$

The ε-Yosida transform of functional $F \in \mathcal{F}$ is the functional $T_\varepsilon F \in \mathcal{F}$ defined by

$$T_\varepsilon F(\boldsymbol{u}, A) = \inf\Big\{F(\boldsymbol{v}, A) + \frac{1}{\varepsilon}\int\limits_A |\boldsymbol{v}-\boldsymbol{u}|^2\,dx\ ,\ \ \boldsymbol{v}\in L^2_{loc}(R^3)^3\Big\} \tag{A.6}$$

or

$$T_\varepsilon F(\boldsymbol{u}, A) = \min\Big\{F_A(\boldsymbol{v}) + \frac{1}{\varepsilon}\int\limits_A |\boldsymbol{u}-\boldsymbol{v}|^2\,dx\ ,\ \ \boldsymbol{v}\in W^{2,2}(A)^3\Big\} \tag{A.7}$$

where

$$F_A(\boldsymbol{v}) = F(\boldsymbol{v}, A) \qquad \forall \boldsymbol{v} \in L^2(A)^3$$

Stochastyczna homogenizacja gradientowej teorii sprężystości

Streszczenie

W ramach metody matematycznej teorii homogenizacji wykorzystano koncepcję Γ-zbieżności funkcjonałów całkowych do wyznaczania efektywnych własności ośrodków gradientowych z losową mikrostrukturą. Głównym rezultatem pracy jest rozszerzenie i uogólnienie twierdzenia o nieliniowej homogenizacji stochastycznej podanego przez Dal Maso and Modica (1986). Mikrostruktura ma charakter losowy, konieczne założenia matematyczne ograniczają losowość do tzw. geometrii stochastycznie periodycznej. Funkcjonał całkowy będący Γ granicą losowych funkcjonałów stochastycznych zależy od drugich gradientów pola przemieszczeń. Pokazano, że zbieżność minimów ciągu homogenizacyjnego jest zbieżnością w prawdopodobieństwie. Przykład płyty Kirchoffa z losowym użebrowaniem w 1 kierunku ilustruje zastosowanie sformułowanego w pracy twierdzenia do wyznaczenia zastępczych modułów sztywności płyty. Zależą one od udziałów objętościowych matrycy i żeber oraz od własności sprężystych obu składników konstrukcji.

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