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# NONSMOOTH MECHANICS AND DYNAMIC CONTACT PROBLEMS [1]

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The aim of this contribution is to present some applications of the Nonsmooth Mechanics to the formulation of dynamic unilateral contact problems without and within friction. The study is concentrated on mechanical systems with finite degrees of freedom. The initial value problems are formulated and shocks are taken into account.

## 1. Introduction

The literature on dynamic contact-impact problem is already quite impressive (cf Gryboś (1969); Jaeger (1992); Zhong and Mackerle (1994)). Both rigid and deformable systems are the subject of investigations. Detailed studies, however, are rare. Moreau (1983) and (1986) initiated mathematically elegant studies for systems of rigid bodies. In the presence of friction even quasi-static motions may be nonsmooth in time. Nonsmooth constraints may lead to discontinuous changes in time of the velocity of a system. It appears that nonsmooth motions are properly described by the velocity functions which belong to a space of functions with locally bounded variations.

Our aim here is first to present some ideas due to Moreau (1983), (1986), (1988) and next to generalize them so as to include nonsmooth constraints and friction (Problems 1 and 2, example). Sequences of approximation of the velocity and the corresponding sequences of motions are also proposed.

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## 2. Functions of bounded and locally bounded variations

In the case of nonsmooth motions, particularly in the presence of shocks (collisions) the left-side $\boldsymbol{u}^-(t)$ and the right-side $\boldsymbol{u}^+(t)$ limits of the velocity function $\boldsymbol{u}(t)$ are in general different. Functions of bounded and, more generally, of locally bounded variations describe quite naturally such possible jumps (cf Moreau (1988)).

Let $I$ be a real interval and $(\mathcal{E}, \rho)$ a metric space. By $J$ we denote a subinterval of $I$ and $f : I \to \mathcal{E}$ is a function. The variation of $f$ on $J$ is defined by

$$\operatorname{var}(f, J) := \sup \sum_{i=1}^{n} \rho\Big(f(\tau_{i-1}), f(\tau_i)\Big) \tag{2.1}$$

where the supremum is taken over all finite sequences $\tau_0 < \tau_1 < \ldots < \tau_n$ of points of $J$ (finite sequences which are only nondecreasing are likewise acceptable). From Eq (2.1) we readily infer:

(i) $\operatorname{var}(f, J) = 0 \Longleftrightarrow f$ equals a constant on $J$.

(ii) If $a \leq b \leq c$, then

$$\operatorname{var}(f; a, c) = \operatorname{var}(f; a, b) + \operatorname{var}(f; b, c) \tag{2.2}$$

**Definition 2.1.** The function $f : I \to \mathcal{E}$ is said to be of *bounded variation* on $I$ iff $\operatorname{var}(f, I) < +\infty$, where $f \in \operatorname{bv}(I, \mathcal{E})$.

It is called the function of *locally bounded variation* on $I$ iff $\operatorname{var}(f; a, b) < +\infty$ for every compact subinterval $[a, b] \subset I$, where $f \in \operatorname{lbv}(I, \mathcal{E})$.

**Definition 2.2.** The function $f : I \to \mathcal{E}$ is said to be *absolutely continuous* on $I$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite collection of non-overlapping open subintervals $(a_i, b_i)$ of $I$, the following implication is true

$$\sum_i (b_i - a_i) < \delta \Rightarrow \sum_i \rho\Big(f(a_i), f(b_i)\Big) < \varepsilon \tag{2.3}$$

The function is said to be *locally absolutely continuous* on $I$ if it is absolutely continuous on every compact subinterval of $I$.

**Remark 2.3.** *If the space $(\mathcal{E}, \rho)$ is complete and $f \in \operatorname{lbv}(I, \mathcal{E})$, then $f$ has a right-limit $f^+(t)$ for every $t \in I$ different from the possible right end of $I$; the symmetric statement can be formulated for $f^-(t)$.* □

Essential for the study of nonsmooth motions is the notion of a *differential measure*. Let now $(X, \|\cdot\|)$ be a real Banach space and $\mathcal{K}(I)$ the space of continuous functions with compact supports in $I$. By $\mathcal{S}$ we denote the totality of the finite subsets of $I$; then for $S \in \mathcal{S}$ we may write

$$S: \tau_0 < \tau_1 < \ldots < \tau_n; \quad \tau_0, \ldots, \tau_n \in I$$

Suppose that $\theta_S^i \in [\tau_{i-1}, \tau_i]$ and let $f: I \to X$. For every $S, \varphi$ and $\theta = (\theta_S^i)$ we construct an element of $X$

$$M(S, \theta, \varphi) = \sum_{i=1}^{n} \theta_S^i \Big( f(\tau_i) - f(\tau_{i-1}) \Big) \tag{2.4}$$

Out of $\mathcal{S}$ one can make a directed set $(\mathcal{S}, \supset)$: then the concept of convergence of $(\mathcal{S}, \supset)$ to the topological space $X$ is meaningful. Thus we may formulate:

**Proposition 2.4.** *Let $f \in \mathrm{lbv}(I, X)$: for every $\varphi \in \mathcal{K}(I)$ and every $\theta$, the mapping $S \to M(S, \theta, \varphi)$ of the directed set (net) $(\mathcal{S}, \supset)$ to $X$ converges to a limit independent of $\theta$. The convergence is uniform with respect to the choice of $\theta$. Let us denote this limit by $\int \varphi\, df$. Then, for every compact subinterval $[a, b] \subset I$, supp $\varphi \subset [a, b]$, one has*

$$\| \int \varphi\, df \| \leq \max |\varphi| \mathrm{var}(f; a, b) \tag{2.5}$$

*Thus the linear mapping $\varphi \to \int \varphi\, df$ of $\mathcal{K}(I)$ to $X$ constitutes a vector measure on $I$.*

**Definition 2.5.** The $X$-valued measure constructed above is called the *differential measure* (or the Stieltjes measure) of $f \in \mathrm{lbv}(I, X)$ and denoted by $df$.

**Basic properties.**

**(i)** If $f \in \mathrm{lbv}(I, X)$ and $[a, b] \subset I$, then

$$\int_{[a,b]} df = f^+(b) - f^-(a)$$

**(ii)** $\forall a \in I \qquad \int_{\{a\}} df = f^+(a) - f^-(a)$

For every $t \in I$, different from the possible right end of this interval the right-side limit $f^+(t)$ is defined by

$$f^+(t) = \lim_{s \to t} f(s) \qquad s > t \tag{2.6}$$

## 3. Generalization of Lagrange's equations

In the sequel we shall investigate dynamic motions of mechanical systems with finite degrees of freedom. To account for the presence of nonsmooth unilateral constraints and/or friction one has to properly generalize classical Lagrange's equations. To this end we shall exploit the ideas primarily due to Moreau (1983), (1986) and next developed by Jean and Moreau (1992), Monteiro Marques (1987).

### 3.1. Preliminaries and notations

A mechanical system is supposed to have a finite number $m$ of degrees of freedom. Let $\boldsymbol{q} = (q^1,\ldots,q^m) \in \Omega \subset \mathbb{R}^m$; $q^1,\ldots,q^m$ are local coordinates in the manifold of the system possible positions. $I$ is a time interval, for instance $I = [0,T]$, $T > 0$. The function $\boldsymbol{q} : I \to \mathbb{R}^m$ is not necessarily derivable everywhere. By $\boldsymbol{u} : I \to \mathbb{R}^m$ we denote a velocity function. For smooth motions (see below), we have

$$\boldsymbol{q}(t) = \boldsymbol{q}(t_0) + \int\limits_{t_0}^{t} \boldsymbol{u}(\tau)\,d\tau \tag{3.1}$$

Here $t_0$ stands for the initial instant. To study nonsmooth motions we make the following

**Assumption.** $\boldsymbol{u} \in \mathrm{lbv}(I,\mathbb{R}^m)$.

Consequently $\boldsymbol{u}^+(\tau)$ (the right-limit) and $\boldsymbol{u}^-(\tau)$ (the left-limit) exist for each $\tau \in intI$. The initial condition is given by

$$\boldsymbol{u}(t_0) = \boldsymbol{u}^-(t_0) = \boldsymbol{u}_0 \tag{3.2}$$

Obviously we have $\dot{\boldsymbol{q}}^-(\tau) = \boldsymbol{u}^-(\tau)$, $\dot{\boldsymbol{q}}^+(\tau) = \boldsymbol{u}^+(\tau)$. According to Moreau (1988) one can justify Eqs (3.2) by imagining that the systems under investigation is already in motion before $t_0$.

Unilateral constraints are specified by

$$f_\alpha(\boldsymbol{q}) \le 0\,, \qquad \alpha \in \{1,2,\ldots,\gamma\} \tag{3.3}$$

where $f_\alpha$ are of class $C^1$ and $\nabla f_\alpha = (\partial f_\alpha/\partial q^1,\ldots,\partial f_\alpha/\partial q^m) \neq \boldsymbol{0}$ at least in a neighborhood of the corresponding surface $f_\alpha = 0$.

$\Phi \subset \Omega$ is a feasible region defined by

$$\Phi = \Big\{ q | f_\alpha(q) \leq 0 \,, \quad \alpha \in \{1, 2, \ldots, \gamma\} \Big\} \tag{3.4}$$

It is convenient to introduce the following notation: for each $q \in \Omega$

$$J(q) := \Big\{ \alpha \in \{1, 2, \ldots, \gamma\} | f_\alpha(q) \geq 0 \Big\} \tag{3.5}$$

$$\mathbb{R}^m \supset V(q) = \Big\{ v \in \mathbb{R}^m | \forall \alpha \in J(q),\ v \cdot \nabla f_\alpha(q) \leq 0 \Big\} \tag{3.6}$$

and $V(q) = \mathbb{R}^m$ if $J(q) = \emptyset$.

*Interpretation*: $V(q)$ is a set of admissible velocities at a point $q$. We have

**Theorem 3.1.** *If a motion* $t - q(t)$ *conforms to the set of constraints* ($q(t) \in \Phi$ *for each* $t$), *then*

$$u^+(t) \in V(q(t)) \qquad \wedge \qquad u^-(t) \in -V(q(t)) \tag{3.7}$$

□

We observe that Eq (3.6) must be satisfied by a frictionless motion as well as by motions with frictions. $V(q)$ is a closed convex cone of $\mathbb{R}^m$ known as the tangent cone provided that $q \in \Phi$ (cf Rockafellar (1970)).

### 3.2. Lagrange's equations: smooth motions

A motion of the system is said to be *smooth* if the velocity function $u$ is locally absolutely continuous, see Section 2. Then $\dot{u}(t) = \ddot{q}(t)$ exists for the Lebesgue measure almost every $t$.

Let $E(q, \dot{q})$ stand for the kinetic energy of the system, thus

$$E(q, \dot{q}) = \frac{1}{2} A_{ij}(q) \dot{q}^i \dot{q}^j \tag{3.8}$$

is a positive definite quadratic form in $\dot{q}$. For the sake of simplicity only scleronomic systems will be investigated.

Lagrange's equations may be written as follows

$$\frac{d}{dt}\Big(\frac{\partial E}{\partial \dot{q}^i}\Big) - \frac{\partial E}{\partial q^i} = F_i + r_i \tag{3.9}$$

where

$F_i$ – known functions of time, position and velocity (given forces)

$r_i$ – unknown reactions or constraint forces.

Obviously, if $f_\alpha(\boldsymbol{q}) < 0$ then the associated reaction is $\boldsymbol{r}^\alpha = \boldsymbol{0}$.

Eqs (3.8) and (3.9) yield

$$\ddot{\boldsymbol{q}} = \mathsf{A}^{-1}\boldsymbol{K} + \mathsf{A}^{-1}\boldsymbol{r} \tag{3.10}$$

Here

$$K_i = F_i - \Big(A_{ij.k} - \frac{1}{2}A_{jk.i}\Big)\dot{\boldsymbol{q}}^j\dot{\boldsymbol{q}}^k \tag{3.11}$$

and $A_{ij.k} = \partial A_{ij}/\partial q^k$.

**Remark 3.2.** *Suppose that $f_\alpha(\boldsymbol{q}) = 0$ and denote by $\boldsymbol{\mathcal{R}}^\alpha$ the associated reaction in the physical space $\mathbb{R}^3$. Then we may write*

$$\boldsymbol{r}^\alpha = G_{\boldsymbol{q}}^{\alpha *}\boldsymbol{\mathcal{R}}^\alpha \qquad G_{\boldsymbol{q}}^\alpha \boldsymbol{u} = \boldsymbol{U}^\alpha \qquad (\alpha \text{ not summed!})$$

*where $G_{\boldsymbol{q}}^\alpha : \mathbb{R}^m \to \mathbb{R}^3$ and $G_{\boldsymbol{q}}^{\alpha *} : \mathbb{R}^3 - \mathbb{R}^m$ is the transpose of $G_{\boldsymbol{q}}^\alpha$ (cf Jean and Moreau (1991), (1992)).*

□

We set

$$\boldsymbol{r} = \sum_{\alpha \in J(\boldsymbol{q})} \boldsymbol{r}^\alpha$$

In the case od smooth frictionless motions, but in the presence of nonsmooth constraints, we have

$$-\boldsymbol{r} \in N(\boldsymbol{q}) \tag{3.12}$$

Thus Lagrange's equations take the form of an *inclusion*

$$\begin{gathered} -\mathsf{A}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{K}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \in N(\boldsymbol{q}) \\ (\text{Lebesgue} - \text{AE in } I) \qquad \forall t \in I : \ \boldsymbol{q}(t) \in \Phi \end{gathered} \tag{3.13}$$

If $C \subset \mathbb{R}^m$ is a closed convex subset then $\psi_C$ denotes its indicator function (cf Ekeland and Temam (1976); Rockafellar (1970)). Moreover

$$\partial\Psi_C(x) = \begin{cases} 0 & \text{if} \quad x \in \operatorname{int}C \\ N(x) & \text{if} \quad x \in C/\operatorname{int}C \\ \emptyset & \text{if} \quad x \neq C \end{cases} \tag{3.14}$$

stands for the subdifferential of $\Psi_C$ at $x$. We have

**Theorem 3.3.** *A smooth motion, with initial data* $\boldsymbol{q}(t_0) \in \Phi$, *is a solution of Eq (3.13), if and only if the velocity function* $\boldsymbol{u}$ *associated with* $\boldsymbol{q}$ *through Eq (3.1), satisfies Lebesgue - AE in* $I$ *the differential inclusion*

$$-\mathbf{A}(q)\dot{u} + \boldsymbol{K}(t, \boldsymbol{q}, \boldsymbol{u}) \in \partial\Psi_{V(\boldsymbol{q})}(\boldsymbol{u}) \tag{3.15}$$

**Remark 3.4.**

(i) *$N(\boldsymbol{q})$ is the polar cone of $V(\boldsymbol{q})$*

$$N(\boldsymbol{q}) := [V(\boldsymbol{q})]^* = \Big\{\boldsymbol{y} \in \mathbb{R}^m | \boldsymbol{y} \cdot \boldsymbol{v} \leq 0, \ \forall \boldsymbol{v} \in V(\boldsymbol{q})\Big\} \tag{3.16}$$

(ii) $\partial\Psi_{V(\boldsymbol{q})}(\boldsymbol{u}) \subset N(\boldsymbol{q})$

(iii) *The initial data:* $\boldsymbol{q}(t_0) = \boldsymbol{q}_0 \in \Phi$, $\boldsymbol{u}(t_0) = \boldsymbol{u}_0 \in V(\boldsymbol{q}_0)$

(iv) *For smooth motions we have:* $\forall t \in I \ \ \boldsymbol{u}(t) \in V(\boldsymbol{q}(t)) \cap -V(\boldsymbol{q}(t)) =$ *a linear subspace of* $\mathbb{R}^m$ *orthogonal to* $N(\boldsymbol{q}(t))$.

For instance, one can easily envisage cases where $V(\boldsymbol{q}(t)) \cap - V(\boldsymbol{q}(t)) = \{0\}$ or $V(\boldsymbol{q}(t)) \cap -V(\boldsymbol{q}(t)) = \mathbb{R}^1$.

### 3.3. Lagrange's equations: nonsmooth motions

The velocity function $\boldsymbol{u}$ is no longer locally absolutely continuous; now $\boldsymbol{u} \in \operatorname{lbv}(I, \mathbb{R}^m)$, i.e., $\boldsymbol{u}$ has locally bounded variation in time, see Section 2. With such a function an $\mathbb{R}^m$ - valued measure $d\boldsymbol{u}$ on the interval $I$ is associated. As we already know from Section 2, $d\boldsymbol{u}$ is a differential measure of $\boldsymbol{u}$. For every compact subinterval $[\sigma, \tau] \subset I$ we have

$$\int_{[\sigma,\tau]} d\boldsymbol{u} = \boldsymbol{u}^+(\tau) - \boldsymbol{u}^-(\sigma) \tag{3.17}$$

and particularly

$$\int_{\{t_s\}} d\boldsymbol{u} = \boldsymbol{u}^+(t_s) - \boldsymbol{u}^-(t_s) \tag{3.18}$$

Thus $d\boldsymbol{u}$ depends on the function $\boldsymbol{u}$ only through $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$.

In order to formulate Lagrange's equations for nonsmooth motions, let us first suppose that $\boldsymbol{u}$ is absolutely continuous; then $d\boldsymbol{u} = \boldsymbol{u}'_t dt$, $\boldsymbol{u}'_t \in L^1_{loc}(I, dt, \mathbb{R}^m)$. Lagrange's equations may be written in the form

$$A_{ij}(\boldsymbol{q})u'^j_t + [A_{ij,k}(\boldsymbol{q}) - \frac{1}{2}A_{jk,i}(\boldsymbol{q})]u^j u^k = c_i = F_i + r_i$$

or equivalently as an *equality of measures* on $I$

$$A_{ij}(\boldsymbol{q})du^j + [A_{ij,k}(\boldsymbol{q}) - \frac{1}{2}A_{jk,i}(\boldsymbol{q})]u^j u^k dt = c_i dt \tag{3.19}$$

Suppose now that $\boldsymbol{u} \in \mathrm{lbv}(I, \mathbb{R}^m)$. We replace $c_i dt$ in Eq (3.19) by some real measures $dC_i$; thus

$$dC_i = F_i(t, \boldsymbol{q}, \boldsymbol{u})dt + dR_i$$

Here $F_i$ are given forces and $dR_i$ are the components of the *contact impulsion* $d\boldsymbol{R}$, an $\mathbb{R}^m$ – valued measure on $I$. For $\boldsymbol{u} \in \mathrm{lbv}(I\mathbb{R}^m)$ Eq (3.18) is generalized to

$$\mathsf{A}(q)d\boldsymbol{u} - \boldsymbol{K}(t, \boldsymbol{q}, \boldsymbol{u})dt = d\boldsymbol{R} \tag{3.20}$$

where $K_i(t, \boldsymbol{q}, \boldsymbol{u}) = F_i(t, \boldsymbol{q}, \boldsymbol{u}) - [A_{ij,k}(\boldsymbol{q}) - \frac{1}{2}A_{jk,i}(\boldsymbol{q})]u^j u^k$. The equality of $\mathbb{R}^m$ – valued measures on $I$ defined by Eq (3.20) was proposed by Moreau (cf Jean and Moreau (1991), (1992); Monteiro Marques (1987); Moreau (1983), (1986)) as governing the *dynamics of nonsmooth motions*. We observe that Eq (3.20) is quite general; it applies to nonsmooth motions without and with friction as well as to collisions.

We recall that for smooth motions

$$-\boldsymbol{r}(t) \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}(t)) \qquad \forall t \in I$$

and $d\boldsymbol{R} = \boldsymbol{r}dt$, $\boldsymbol{r} \in L^1_{loc}(I, dt, \mathbb{R}^m)$. For nonsmooth motions one introduces

**Definition 3.5.** The set of constraints is said to be *frictionless and soft* if the total contact impulsion admits a representation $d\boldsymbol{r} = \boldsymbol{R}'_\mu d\mu$, where $d\mu$ denotes a nonnegative real measure on $I$ and $\boldsymbol{R}'_\mu \in L^1_{loc}(I, d\mu, \mathbb{R}^m)$, such that for every $t \in I$

$$-\boldsymbol{R}'_\mu(t) \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}^+(t)) \tag{3.21}$$

□

**Remark 3.6.**

**(i)** *For nonsmooth motions $d\mu$ is in general different from the Lebesgue measure $dt$.*

**(ii)** *Contact law (3.21) holds for each $t \in I$ if and only of the same is true after replacing $d\mu$ by another nonnegative real measure relative to which $d\boldsymbol{R}$ possesses a density function.*

A straightforward generalization of Eq (3.15) to nonsmooth frictionless and soft motions is given by

$$-\mathsf{A}(\boldsymbol{q})\boldsymbol{u}'_{\mu} + \boldsymbol{K}(t,\boldsymbol{q},\boldsymbol{u})t'_{\mu} \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}^{+}(t)) \tag{3.22}$$

By using Remark 3.6.(ii), we may write differential inclusion (3.22) in the form of a *measure differential inclusion*

$$-\mathsf{A}(\boldsymbol{q})d\boldsymbol{u} + \boldsymbol{K}(t,\boldsymbol{q},\boldsymbol{u})dt \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}^{+}(t)) \tag{3.23}$$

The following statement interrelates $\boldsymbol{u}^{+}(t)$ and $\boldsymbol{u}^{-}(t)$.

**Proposition 3.7.** *For $t \in I$, different from the possible right end of this interval, and any motion satisfying (3.23) we have*

$$\boldsymbol{u}^{+}(t) = \operatorname{prox}\Big(\boldsymbol{u}^{-}(t), V(\boldsymbol{q}(t))\Big) \tag{3.24}$$

*Here the proximation is to be understood in the sense of the kinetic metric, i.e., the Euclidean metric, defined in $\mathbb{R}^m$ by the matrix $\mathsf{A}(q(t))$.*

□

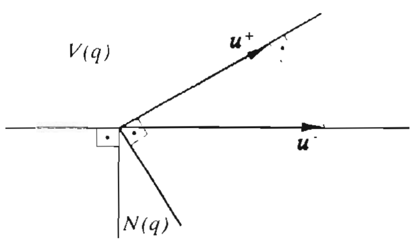


Fig. 1.

For the definition and properties of the proximation (prox) the reader should refer to the book by Ekeland and Temam (1976). Fig.1 provides a simple illustration of Eq (3.24).

**Remark 3.8.** *The frictionless nonsmooth motions without softness are characterized by*

$$-\boldsymbol{R}'_{\mu}(t) \in N(\boldsymbol{q}(t)) \tag{3.25}$$

*This relation is more general than*

$$-\boldsymbol{R}'_{\mu}(t) \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}^{+}(t)) \tag{3.26}$$

*since the right-hand side of Eq (3.26) is contained in $N(\boldsymbol{q}(t))$.*

We are now in position to formulate the initial value problem for a system, whose nonsmooth motions are frictionless and soft. We assume that $I = [0,T]$.

**Problem 1.** Find a function $\boldsymbol{u} \in \mathrm{lbv}(I, \mathbb{R}^m)$, such that if $\boldsymbol{q}$ is defined by

$$\boldsymbol{q}(t) = \boldsymbol{q}_0 + \int_0^t \boldsymbol{u}^+(\tau)\, d\tau\ , \qquad t \in I \tag{3.27}$$

then they satisfy

$$\boldsymbol{q}(0) = \boldsymbol{q}_0 \in \Phi \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in V(\boldsymbol{q}_0) \tag{3.28}$$

$$\boldsymbol{q}(t) \in \Phi \qquad \boldsymbol{u}(t) \in V(\boldsymbol{q}(t)) \qquad t \in I \tag{3.29}$$

$$-\mathsf{A}(\boldsymbol{q})d\boldsymbol{u} + \boldsymbol{K}(t,\boldsymbol{q},\boldsymbol{u})dt \in \partial\Psi_{V(\boldsymbol{q}(t))}(\boldsymbol{u}^+(t)) \tag{3.30}$$

□

Now we pass to motions with friction. By $\mathcal{R}$ we denote the reaction force at a point of contact; $C$ is a friction cone. For instance, $C$ is usually generated by the Coulomb condition. Anisotropic friction conditions, however, are not precluded. Further let $\boldsymbol{V}$ denote the (physical) relative velocity between two bodies of the system or between an element and a constraint. Following Moreau (1986), the friction law is assumed in the form

$$-\boldsymbol{V} \in \mathrm{proj}_{\mathcal{T}}\partial\Psi_C(\mathcal{R}) \tag{3.31}$$

Here $\mathcal{T}$ stands for the tangent space at the point of contact (in the physical space).

Let $\alpha \in \{1,2\}$ and consider motions of one point only in the space $\mathbb{R}^3$. We formulate

**Problem 2.** Find $\boldsymbol{u} \in \mathrm{lbv}(I, \mathbb{R}^3)$, such that for $\boldsymbol{q}$ defined by Eq (3.27) the following conditions are satisfied

$$\boldsymbol{q}(0) = \boldsymbol{q}_0 \in \Phi \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in V(\boldsymbol{q}_0) \tag{3.32}$$

$$\boldsymbol{q}(t) \in \Phi \qquad \boldsymbol{u}(t) \in V(\boldsymbol{q}(t)) \qquad t \in I \tag{3.33}$$

$$\boldsymbol{q}(t) \in \mathrm{int}\Phi \Rightarrow \mathcal{R}'_\mu(t) = 0 \tag{3.34}$$

$$[\boldsymbol{q}(t) \in \mathrm{bdry}\Phi,\ \boldsymbol{u}^+(t) \in \mathrm{int}V(\boldsymbol{q}(t))] \Rightarrow \mathcal{R}'_\mu(t) = 0 \tag{3.35}$$

$$\begin{gathered}[\boldsymbol{q}(t) \in \mathrm{bdry}\Phi,\ \boldsymbol{u}^+(t) \in \mathrm{bdry}V(\boldsymbol{q}(t)),\ \boldsymbol{u}^+(t)\cdot\nabla f_\alpha(\boldsymbol{q}(t)) = 0] \Rightarrow \\ \Rightarrow -\boldsymbol{u}^+(t) \in \mathrm{proj}_{\mathcal{T}_\alpha(\boldsymbol{q}(t))}\partial\Psi_{C_\alpha(\boldsymbol{q}(t))}(\mathcal{R}'_\mu(t))\end{gathered} \tag{3.36}$$

$$\mathsf{A}(\boldsymbol{q}(t))d\boldsymbol{u} - \boldsymbol{K}(t,\boldsymbol{q}(t),\boldsymbol{u}(t))dt = \mathcal{R}'_\mu d\mu \tag{3.37}$$

□

Jean and Moreau (1992), Monteiro Marques (1987) consider only the case $\alpha = 1$. The existence result proved by Monteiro Marques (1987) for $\alpha = 1$ does not apply directly to Problem 2. Fig.2 provides an illustration of friction law (3.35).

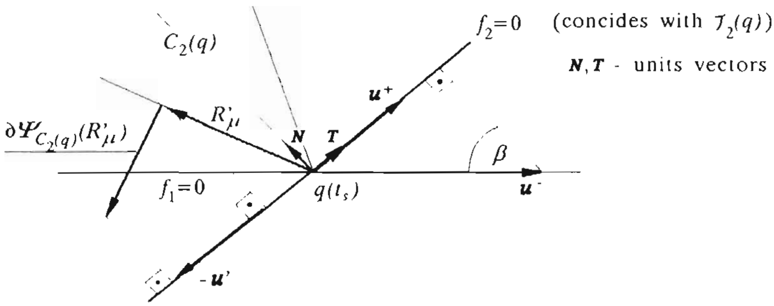


Fig. 2.

**Example.** Consider plane motions of a point with mass $m$, see Fig.2. For $\mathcal{R}'_{\mu N} \leq 0$ the Coulomb condition has the form

$$|\mathcal{R}'_{\mu T}| - \nu \mathcal{R}'_{\mu N} \leq 0 \tag{3.38}$$

where $\nu$ is the coefficient of friction. Eq (3.37) reduces to

$$m d\boldsymbol{u} = \boldsymbol{\mathcal{R}}'_\mu d\mu \tag{3.39}$$

Setting

$$S_N = \int \mathcal{R}'_{\mu N}\, d\mu \qquad\qquad \boldsymbol{S}_T = \int \boldsymbol{\mathcal{R}}'_{\mu T}\, d\mu \tag{3.40}$$

from Eq (3.38) we obtain

$$|\boldsymbol{S}_T| - \nu S_N \leq 0 \tag{3.41}$$

We recall that $d\mu$ is a nonnegative measure. Further we have

$$\int_{\{t_s\}} m\, d\boldsymbol{u} = m \int_{\{t_s\}} d\boldsymbol{u} = m[\boldsymbol{u}^+(t_s) - \boldsymbol{u}^-(t_s)] \tag{3.42}$$

Thus Eq (3.38) yields (cf Gryboś (1969))

$$m\boldsymbol{u}^+ - m\boldsymbol{u}^- = \boldsymbol{S} = S_N \boldsymbol{N} + \boldsymbol{S}_T \tag{3.43}$$

Simple considerations yield

$$\boldsymbol{u}^+(t_s) = |\boldsymbol{u}^-(t_s)|(\sin\beta - \nu\cos\beta)\boldsymbol{T} \tag{3.44}$$

provided that $\sin\beta - \nu\cos\beta \geq 0$.

### 3.4. Approximations

Let $I = [0,T]$. We shall now propose a sequence $\{\boldsymbol{u}_n\}$ of approximations of the velocity and the corresponding sequence $\{\boldsymbol{q}_n\}$ for Problems 1 and 2.

**3.4.1.** For each positive integer $n$, let us set $h = (h_n) = t/n$ and $t_{n,i} = i/h = iT/n$ $(0 \le i \le n)$. In the case of Problem 1 we introduce two finite sequences $\{\boldsymbol{q}_{n,i}\}$ and $\{\boldsymbol{u}_{n,i}\}$ of elements of $\mathbb{R}^m$ as follows

$$\boldsymbol{q}_{n,0} = \boldsymbol{q}_0 \tag{3.45}$$

$$\boldsymbol{u}_{n,0} = \operatorname{proj}\Big(\boldsymbol{u}_0 + h\mathsf{A}^{-1}(\boldsymbol{q}_{n,0})K(t_{n,0},\boldsymbol{q}_{n,0},\boldsymbol{u}_0), V(\boldsymbol{q}_{n,0})\Big) \tag{3.46}$$

$$\boldsymbol{q}_{n,i+1} = \boldsymbol{q}_{n,i} + h\boldsymbol{u}_{n,i} \tag{3.47}$$

$$\boldsymbol{u}_{n,i+1} = \operatorname{proj}\Big(\boldsymbol{u}_{n,i} + h\mathsf{A}^{-1}(\boldsymbol{q}_{n,i+1})K(t_{n,i+1},\boldsymbol{q}_{n,i+1},\boldsymbol{u}_{n,i}), V(\boldsymbol{q}_{n,i+1})\Big) \tag{3.48}$$

Next, $\boldsymbol{u}_n$ is defined by

$$\boldsymbol{u}_n(t) = \boldsymbol{u}_{n,i} \qquad \text{if} \qquad t \in [t_{n,i}, t_{n,i+1}) , \quad 0 \le i \le n-1 \tag{3.49}$$

and $\boldsymbol{u}_n(T) = \boldsymbol{u}_{n,n}$. Then $\boldsymbol{q}_n$ is readily obtained by integration

$$\boldsymbol{q}_n(t) = \boldsymbol{q}_0 + \int_0^t \boldsymbol{u}_n(\tau)\, d\tau \tag{3.50}$$

We observe that

$$\boldsymbol{u}_n(t_{n,i}) = \boldsymbol{u}_{n,i} \qquad\qquad \boldsymbol{q}_n(t_{n,i}) = \boldsymbol{q}_{n,i} \tag{3.51}$$

**3.4.2.** Before passing to the formulation of the time-discretization algorithm for Problem 2, we note that Eq (3.36) requires $\boldsymbol{R}'_\mu(t) \in C_\alpha(\boldsymbol{q}(t))$, $d\mu$ – almost everywhere. If shock happens at an instant $t$, the measure $d\boldsymbol{R}$ has an atom at $t$ that equals to $(\boldsymbol{u}^+(t) - \boldsymbol{u}^-(t))\delta_t$; moreover $t$ is also an atom of the positive measure $d\mu$. The right-hand side of Eq (3.36) is a cone, hence this relation is equivalent to

$$-\,\boldsymbol{u}^+(t) \in \operatorname{proj}_{T_\alpha(\boldsymbol{q}(t))}\partial\Psi_{C_\alpha(\boldsymbol{q}(t))}(\boldsymbol{u}^+(t) - \boldsymbol{u}^-(t)) \tag{3.52}$$

which in turn is equivalent to

$$\boldsymbol{u}^+(t) = \operatorname{proj}\Big(0, [\boldsymbol{u}^-(t) + C_\alpha(\boldsymbol{q}(t))] \cap T_\alpha(\boldsymbol{q}(t))\Big) \tag{3.53}$$

To prove Eq (3.53) one may use Fig.2; the proof is then straightforward.

We will now generalize the time-discretization algorithm, for $\alpha = 1$ proposed by Monteiro Marques (1987). Let the initial data $\boldsymbol{q}_0$ and $\boldsymbol{u}_0$ be such that $\boldsymbol{q}_0 \in \Phi$, $\boldsymbol{u}_0 \in V(\boldsymbol{q}_0)$. For every $n \geq 1$ we define a sequence of approximations

$$h = h_n = 2^{-n}T \tag{3.54}$$

$$t_{n,i} = ih = i2^{-n}T \qquad (i = 0, ..., 2^n) \tag{3.55}$$

$$\boldsymbol{u}_{n,0} = \boldsymbol{u}_0 \qquad \boldsymbol{q}_{n,0} = \boldsymbol{q}_0 \tag{3.56}$$

$$\boldsymbol{q}_{n,i} = \boldsymbol{q}_{n,i-1} + h\boldsymbol{u}_{n,i-1} \qquad (1 < i \leq 2^n) \tag{3.57}$$

$$\boldsymbol{v}_{n,i} = \boldsymbol{u}_{n,i-1} + h\mathsf{A}^{-1}(\boldsymbol{q}_{n,i})K(t_{n,i}, \boldsymbol{q}_{n,i}, \boldsymbol{u}_{n,i-1}) \qquad (1 < i \leq 2^n) \tag{3.58}$$

$$\boldsymbol{u}_{n,i} = \begin{cases} \boldsymbol{v}_{n,i} & \text{if} \quad \boldsymbol{v}_{n,i} \in V(\boldsymbol{q}_{n,i}) \\ P(\boldsymbol{v}_{n,i}, \boldsymbol{q}_{n,i}) := \operatorname{proj}\Big(0, [\boldsymbol{v}_{n,i} + C_\alpha(\boldsymbol{q}_{n,i})] \cap \mathcal{T}_\alpha(\boldsymbol{q}_{n,i})\Big) \\ \qquad\qquad\qquad\quad \text{if} \quad f_\alpha(\boldsymbol{q}_{n,i}) \geq 0 \ \wedge \ \boldsymbol{v}_{n,i} \neq V(\boldsymbol{q}_{n,i}) \end{cases} \tag{3.59}$$

$$\boldsymbol{u}_n(t) = \begin{cases} \boldsymbol{u}_{n,i} & \text{if} \quad t \in I_{n,i} = [t_{n,i}, t_{n,i+1}) \,, \qquad 0 \leq i < 2^n \\ \boldsymbol{u}_{n,2^n} & \text{if} \quad t = T = t_{n,2^n} \end{cases}$$

$$\boldsymbol{q}_n(t) = \boldsymbol{q}_0 + \int_0^t \boldsymbol{u}_n(\tau)\, d\tau = \boldsymbol{q}_{n,i} + (t - t_{n,i})\boldsymbol{u}_{n,i} \,, \qquad t \in I_{n,i} \tag{3.60}$$

We observe that relation (3.53) has been used to define Eq $(3.59)_2$.

## 4. Concluding remarks

In the case of nonsmooth constraints, described by functions $f_\alpha$, smooth motions are rather an exception than a rule, even if friction is neglected. Coulomb friction law (3.31) incorporates anisotropic friction. In the last case the coefficient of friction is not the same in different directions (cf Zmitrowicz (1981)). More general friction condition can be used instead of Eq (3.31). For instance, the subdifferential on the right-hand side of Eq (3.31) may be replaced by Clarke's subdifferential, which occurs when $C$ is not a convex set. It would also be interesting to incorporate in our schemes the case where the friction coefficient depends on the velocity.

## Reference

1. EKELAND T., TEMAM R., 1976, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam
2. GRYBOŚ R., 1969, *Theory of Impact in Discrete Mechanical Systems*, PWN, Warszawa, in Polish
3. JAEGER J., 1992, Elastic Impact with Friction, Thesis, Technical University of Delft
4. JEAN M., MOREAU J.J., 1991, Dynamics of Elastic or Rigid Bodies with Frictional Contact: Numerical Approach, *Preprint* No 91-1, *USTL*, Montpellier
5. JEAN M., MOREAU J.J., 1992, Unilaterality and Dry Friction in the Dynamics of Rigid Body Collections, *Proc. Contact Mech. Int. Symp.*, A.Curnier (edit.), Presse Pol. et Univ. Romandes, Lausanne, 31-48
6. MONTEIRO MARQUES M.D.P., 1987, Inclusões diferenciais e choques inelásticos, Thesis, Faculdade de Ciencias de Lisboa
7. MOREAU J.J., 1983, Liaisons unilatérales sans frottement et chocs inélastiques, *C.R. Acad. Sci. Paris*, Série II, **296**, 1473-1476
8. MOREAU J.J., 1986, Une formulation du contact à frottement sec; application an calcul numérique, *C.R. Acad. Sci. Paris*, Série II, **302**, 799-801
9. MOREAU J.J., 1988, Bounded Variation in Time, *Topics in Nonsmooth Mechanics*, J.J.Moreau, P.D.Panagiotopoulos, G.Strang (edit.), Birkhäuser Verlag, Basel, 1-74
10. ROCKAFELLAR R.T., 1970, *Convex Analysis*, Princeton University Press, Princeton
11. ZHONG Z.H., MACKERLE J., 1994, Contact-Impact Problems: a Review with Bibliography, *Appl. Mech. Reviews*, **47**, 55-76
12. ZMITROWICZ A., 1981, A Theoretical Model of Anisotropic Dry Friction, *Wear*, **73**, 9-39

## Dynamiczne zagadnienia kontaktowe układów z niegładkimi więzami

Streszczenie

Celem pracy jest przedstawienie pewnych zastosowań mechaniki ukladów materialnych z niegładkimi więzami jednostronnymi. Rozpatrzono zagadnienia bez tarcia i z tarciem dla układów o skończonej liczbie stopni swobody. Sformułowano problemy początkowe, przy czym uwzględniono zderzenia.

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