MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 34, 1996

BIFURCATION AND POSTBIFURCATION OF RODS

PÉTER B. BÉDA

Department of Mechanics, Technical University of Budapest

e-mail: bedap@kme.bme.hu

In studying elastic bifurcations the general way is to start with a variational setting. After then, the uniqueness of linear part is to be considered to find the bifurcation points and the curvature shows whether the equilibrium curve is sub- or supercritical. These methods offer no possibility to investigate even a small neighbourhood of the bifurcation point being important for possible secondary bifurcations. If we need this information, we should approach the problem in the other way. The equilibrium equation should be studied in a local form and some kind of reduction process should be realised to end up with an algebraic equation called the bifurcation equation. This equation describes the nonlinear behavior of the system in the vicinity of the bifurcation point giving information on the secondary bifurcations. Moreover it enables us to find also the effects of imperfections. In the paper the connections and advantages of both methods are discussed. As an example the buckling process in a rod is presented.

1. Energy method

In the case of elastic body investigation of the bifurcations is based generally on the energy equation

$$\delta W_e + \delta W_i + \delta W_j = 0 \tag{1.1}$$

where δW_e, δW_i denote the virtual works of the external and internal forces, respectively, and $\delta W_j = \delta \int_{t_0}^{t} T\, dt$, where T is the kinetic energy. In a static conservative system both the internal and external forces have potentials U, V and

$$\delta W_e = -\sum_{i=1}^{n} \frac{\partial U}{\partial u_i} \delta u_i \qquad\qquad \delta W_i = -\sum_{i=1}^{n} \frac{\partial V}{\partial u_i} \delta u_i$$

where $u = \{u_i\}$, $i = 1, \ldots, n$ are generalized coordinates. The sum $E = U + V$ is called potential energy of the system. In a system depending on a parameter λ, $E = E(u, \lambda)$. For the equilibrium

$$E_{,u}(u, \lambda)\delta u = 0 \tag{1.2}$$

Introducing a parameter τ the equilibrium curve can be given in the form $u = u(\tau)$, $\lambda = \lambda(\tau)$ for u and λ satisfying Eq (1.2). The tangent of this curve can be calculated by differentiating with respect to τ at certain u_0, λ_0

$$\delta u\Big(E_{,uu}(u_0, \lambda_0)u_\tau + E_{,u\lambda}(u_0, \lambda_0)\lambda_\tau\Big) = 0 \tag{1.3}$$

When both $E_{,uu}$ and $E_{,u\lambda}$ are regular at (u_0, λ_0), this point is called the regular one. When $E_{,uu}$ is singular, then there exists a zero eigenvalue. In the case of a simple eigenvalue there is a single eigenvector X satisfying

$$\delta u E_{,uu}(u_0, \lambda_0)X = 0 \tag{1.4}$$

When $\delta u E_{,u\lambda}(u_0, \lambda_0)X \neq 0$ Eq (1.3) can be solved for $\lambda_\tau = 0$, that is, the equilibrium curve "turns back". This point is called a turning point. In the case $\delta u E_{,u\lambda}(u_0, \lambda_0)X = 0$ there is no unique tangent and the point is called a bifurcation point.

The equilibrium curve can be studied at bifurcation points by defining the trivial curve in the form $u^0(\lambda)$. The nontrivial equilibrium curve is searched for as $u = u^0(\lambda) + v$. Then by expanding into a power series $\lambda = \lambda_0 + \lambda_1\tau + \ldots$, $v = v_1\tau + \ldots$ and substituting into Eq (1.3) and into its higher order derivatives after comparing the coefficients of each term τ^i of the power series expansions a set of equations is obtained. It can be solved systematically for λ_i, v_i $i = 1, \ldots, n$. For the linear part we can write $v_1 = X$. Then the second derivative of Eq (1.2) with respect to τ is

$$\delta u\Big(E_{,uu}(u_0, \lambda_0)u_{\tau\tau} + E_{,uuu}(u_0, \lambda_0)u_\tau u_\tau + 2E_{,uu\lambda}(u_0, \lambda_0)u_\tau\lambda_\tau +$$

$$+E_{,u\lambda\lambda}(u_0, \lambda_0)\lambda_\tau^2 + E_{,u\lambda}(u_0, \lambda_0)\lambda_{\tau\tau}\Big) = 0$$

Assuming that there is no higher order term in λ, then the first $E_{,u\lambda\lambda}(u_0, \lambda_0) = 0$. In terms of the power series expansions λ_1 can be obtained for $\delta u = X$ at $\tau = 0$ as

$$\lambda_1 = -\frac{1}{2}\frac{E_{,uuu}(u_0, \lambda_0)[X, X, X]}{E_{,uuu}(u_0, \lambda_0)[(u^0)_\lambda, X, X] + E_{,uu\lambda}(u_0, \lambda_0)[X, X]}$$

Quite similarly the other terms can also be calculated. In local investigation only the first nonzero coefficient is important, thus first few λ_i can be given in a general form (cf Nguyen (1993)) and for a real mechanical system the values of parameters should be substituted into these expressions.

2. Static bifurcation theory

The bifurcation theory is applied, when the state of equilibrium is described as a solution u of the equation of balance depending on a parameter λ

$$G(u, \lambda) = 0 \tag{2.1}$$

In Eq (2.1) the left-hand side can be calculated from Eq (1.1) in the usual way (cf Gantmacher (1967)) and end up in, e.g., Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_i} - \frac{\partial T}{\partial u_i} - Q_{u_i}$$

where Q_{u_i} are the generalized forces. When all the forces have potentials as in the previous part $Q_{u_i} = -\partial U/\partial u_i = -\partial V/\partial u_i$, and the investigation is restricted to statics, the right-hand side of Eq (2.1) is the same as the multiplyer of δu in Eq (1.2).

Static bifurcation theory investigates maintence of the uniqueness of the state of equilibrium under quasistatic variation of a selected bifurcation parameter. When there is a critical value of the parameter, at which the uniqueness is lost, the structure of the set of solutions changes. This case is called the static bifurcation (cf Chow and Hale (1982)). Generally such problems occur, when the linear part of Eq (2.1) turns to be singular at the considered solution u_0 for some values λ_0 of the bifurcation parameter. Thus the derivative D_uG of G is singular at u_0, that is, $D_uG(u_0, \lambda_0)\big(\equiv E_{,uu}(u_0, \lambda_0)\big)$ has a nontrivial kernel. Finding a basis of the nontrivial kernel the so-called critical eigenmodes (cf Troger and Steindl (1990)) are obtained giving some information on the arising nontrivial solutions. When there is a unique zero eigenvalue with multiplicity one, the basis of the nontrivial kernel can be identified with the eigenvector X given in Section 1.

Having detected the bifurcation point the interest focuses on the post- or prebifurcation behavior. The essence of the investigation of post- or prebifurcation is to find out, how many solutions do we have before and after the bifurcation. When doing it instead of the point (u_0, λ_0) a small neighbourhood of it should be considered in searching for the nontrivial solutions.

Now the Liapunov-Schmidt reduction should be applied (cf Golubitsky and Schaeffer (1985)). Firstly, the linearized part is defined by $L : U \to U$, $L := D_uG(u_0, \lambda_0)$. It is assumed that:

(a) Kernel L is a finite dimensional subspace of U

(b) Range L is a closed subspace of U of finite codimension

(c) L is self-adjoint.

For operators revealing properties (a) and (b) the $U = \mathrm{Ran}L \oplus \mathrm{Ker}L$ decomposition is possible. Now Eq (2.1) can be splitted into the equivalent pair of equations

$$
\begin{aligned}
PG(u,\lambda) &= 0 \\
(I-P)G(u,\lambda) &= 0
\end{aligned}
\tag{2.2}
$$

where $P:\ U \to \mathrm{Ran}L$ is a projection operator into L range. The same splitting can also be made for the variable u

$$
u = v + w \tag{2.3}
$$

where $v \in \mathrm{Ker}L$, $w \in \mathrm{Ran}L$. Substituting Eq (2.3) into Eq $(2.2)_1$, the equation

$$
PG(v+w,\lambda) = 0 \tag{2.4}
$$

can be solved in the $\mathrm{Ran}L$ applying the Inverse Function Theory and w can be written as

$$
w = \psi(v,\lambda) \tag{2.5}
$$

in a neighbourhood of the origin. Then Eqs (2.3) and (2.5) should be substituted into Eq $(2.2)_1$ and the bifurcation equation in the form

$$
(I-P)G\Big(v+\psi(v,\lambda),\lambda\Big) = 0 \tag{2.6}
$$

is obtained. Let us denote

$$
\Phi(v,\lambda) = (I-P)G\Big(v+\psi(v,\lambda),\lambda\Big)
$$

In the case of a unique zero eigenvalue with multiplicity one an algebraic equation equivalent to Eq (2.6) can be defined. At first the function

$$
g:\ R\times R \to R \qquad\qquad g(q,\lambda) := \langle X, \Phi(qX,\lambda)\rangle \tag{2.7}
$$

should be defined, where q is a scalar coordinate and $\langle\cdot,\cdot\rangle$ is a scalar product. Then the equation

$$
g(q,\lambda) = 0 \tag{2.8}
$$

shows the changes in the number of equilibria and the postbuckling behavior of the system. Generally the function (2.7) is searched for in terms of as a Taylor expansion.

3. Case study 1: The postbifurcation of an Euler elastica

In this part we show two methods applied to the investigation of the postbuckling of an Euler elastica having length L under the thrusting force λ. Having the arc coordinate s, the stiffness k, while u is the angle between tangent vectors of the center line in loaded and unloaded rods, respectively, the potential energy of the system is

$$E(u,\lambda) = \frac{1}{2}\int_0^L k u_{,s}^2(s)\, ds + \lambda \int_0^L \cos u(s)\, ds$$

The left-hand side of the equation of equilibrium (1.2) is

$$E_{,u}(u,\lambda)\delta u = \int_0^L k u_{,s}(s)\delta u_{,s}(s)\, ds - \lambda \int_0^L \delta u \sin u(s)\, ds$$

The trivial equilibrium is $u(s) \equiv 0$. For the critical point from Eq (1.4) substituting for the eigenvector X

$$\delta u E_{,uu}(u_0,\lambda_0)X = \int_0^L k X_{,s}(s)\delta u\, ds - \lambda_0 \int_0^L \delta u X\, ds = 0$$

Having done partial integration

$$\Big[k X_{,s}(s)\delta u\Big]_0^L - k\int_0^L X_{,ss}(s)\delta u\, ds - \lambda_0 \int_0^L \delta u X\, ds = 0$$

Thus for the boundary conditions $X_{,s}(0) = X_{,s}(L) = 0$ the equation

$$\frac{d^2X}{ds^2} + \lambda_0 X = 0 \tag{3.1}$$

is obtained. The solution is $l\lambda_0 = \pi^2 k/L^2$ and $X(s) = \cos(\pi s/L)$. It means that the system has a bifurcation at the critical load $\pi^2 k/L^2$. Now the postbuckling can be investigated by substituting into the general expressions mentioned in Section 1

$$E_{,uu\lambda}[X,\delta u]=\int_0^L \delta u X\cos u\, ds$$

$$E_{,uu\lambda}(u_0,\lambda_0)[X,X]=\int_0^L \cos^2\frac{\pi s}{L}\, ds \tag{3.2}$$

$$E_{,uuu}[X,X,\delta u]=-\lambda\int_0^L \delta u X^2\sin u\, ds$$

$$E_{,uuu}(u_0,\lambda_0)[X,X,X]=0 \tag{3.3}$$

$$E_{,uuuu}(u_0,\lambda_0)[X,X,X,X]=-\lambda_0\int_0^L \delta u\cos^4\frac{\pi s}{L}\, ds \tag{3.4}$$

and $\lambda_1 = 0$, $\lambda_2 = \lambda_0/3$ is obtained. It shows that the postbuckling is supercritical.

To demonstrate how the bifurcation theory works, the Kirchhoff rod equation

$$\frac{d^2u}{ds^2}+\lambda\sin u=0 \tag{3.5}$$

is used, where u is the same angle as before. Let the boundary conditions be accepted in the form given above $u(0) = u(L) = 0$. The linearized form of Eq (3.5) is $(d^2u/ds^2) + \lambda u = 0$ having the same form as Eqs (3.2). The least eigenvalue and its eigenvector is obviously the same as Eqs (3.2), thus the decomposition (2.3) is $u = q\cos(\pi s/L) + w$. The scalar product of Eq (2.7) can be defined by $\langle a,b\rangle := \frac{2}{L}\int_0^L ab\, ds$. From Eq (2.5) can easily be proven that w is at least of order three in q, thus by truncating the Taylor expansion of q at third order, Eq (2.4) can be neglected and

$$\begin{aligned} g(q,\lambda) &= \frac{2}{L}\Big(q\int_0^L \frac{\pi^2}{L^2}\cos\frac{\pi s}{L}\sin\pi s L\, ds + \\ &+ \lambda q\int_0^L \cos^2\frac{\pi s}{L}\, ds-\frac{q^3}{6}\int_0^L\cos^4\frac{\pi s}{L}\, ds+\ldots\Big)\end{aligned}$$

The bifurcation equation is

$$0 = \lambda q - \frac{3}{4}q^3 + \dots$$

having always the trivial solution $q \equiv 0$, a nontrivial solution $\lambda = \frac{3}{4}q^2$ exists for $\lambda \geq 0$ showing the supercritical nature. We notice that the right-hand sides of Eqs (3.2) ÷ (3.4) are obviously the same as the appropriate coefficients in the bifurcation equation.

4. Case study 2: Secondary buckling

The second method reduces the equilibrium equation being a differential equation to the algebraic one. The identity is local. It is valid in a small neighbourhood of the trivial solution and of the critical parameter value. After this reduction investigation of the postbifurcation consists in the study of bifurcation equation.

Below, the case of a twisted elastica will be considered under terminal thrust. This problem has two possible postbuckling behaviors (cf Béda et al. (1992)). These are called the super- and the subcritical cases, respectively (cf Triger and Steindl (1990)). There is a critical twist at which the behavior of the system changes from the supercritical to subcritical type. Let us introduce a parameter α representing the distance from this critical twist. The steps of reduction are the same when λ is a vector, components of which are load and imperfection parameter α. In this case the bifurcation equation is

$$g(q. \lambda, \alpha) = 0$$

Assuming $q_0 = \lambda_0 = 0$, for derivatives of the (perfect) system

$$\frac{\partial g}{\partial q}\Big|_0 = \frac{\partial^2 g}{\partial q^2}\Big|_0 = \frac{\partial^3 g}{\partial q^3}\Big|_0 = \frac{\partial^4 g}{\partial q^4}\Big|_0 = 0$$

$$\frac{\partial^5 g}{\partial q^5}\Big|_0 = \delta \neq 0$$

The local bifurcation equation is

$$\lambda q + \delta q^5 + o(|q|^7) = 0$$

Nontrivial solution exists, when $\lambda\delta < 0$, that is, the sub- or supercritical behavior depends on the sign of δ. When there is a small imperfection, a nonzero third order derivative should exist $\left.\frac{\partial^3 g}{\partial q^3}\right|_0 = 6\alpha \neq 0$, where $\alpha \ll 1$ and the local imperfect bifurcation equation is

$$\lambda q + \alpha q^3 + \delta q^5 + o(|q|^7) = 0 \tag{4.1}$$

Consider now a two parameter bifurcation with λ, α and study the change in number of solutions in a small neighbourhood of the origin by quasi-static variation of the parameter. To simplify the problem, the following equation

$$\lambda q + \alpha q^3 + q^5 + o(|q|^7) = 0 \tag{4.2}$$

is considered, where parameters λ, α are obtained from Eq (4.1) dividing if $\delta \neq 0$. For the sake of simplicity the same notation is used as in Eq (4.1). Now, as in multiparameter bifurcation problems, we are seeking for the values of (λ, α), at which the number of the solutions changes. For this reason we differentiate Eq (4.2) with respect to q

$$\lambda + 3\alpha q^2 + 5q^4 + o(|q|^6) = 0 \tag{4.3}$$

When there is a change in the number of solutions of Eq (4.2), both Eqs (4.2) and (4.3) should be satisfied.

In the case $q = 0$, Eq (4.2) is an identity and Eq (4.3) is satisfied at

$$\forall\, |\alpha| \ll 1 \qquad \lambda = 0 \tag{4.4}$$

When $q \neq 0$, in a sufficiently small neighbourhood of the origin, from Eq (4.3) it follows

$$\lambda = -3\alpha q^2 - 5q^4 + o(|q|^6)$$

and substituting into Eq (4.2) yields

$$\alpha = -2q^2 + o(|q|^4) \tag{4.5}$$

Substituting it into the formula for λ

$$\lambda = q^4 + o(|q|^6) \tag{4.6}$$

in a small neighbourhood of the origin. From Eqs (4.5) and (4.6) it follows that the number of the solutions of Eq (4.2) changes when

$$\lambda = \frac{\alpha^2}{4} \qquad\qquad \alpha < 0 \tag{4.7}$$

While Eq (4.4) represents the curve of the primary bifurcation (the same condition as the perfect system), the other curve represented by Eq (4.7) can be interpreted as a secondary bifurcation. The curves of primary and secondary bifurcations, represented by Eqs (4.4) and (4.7), respectively, divide the parameter plane (λ, α) into three regions. The number of solutions of Eq (4.2) changes when crossing one of these curves. From Eq (4.2) we have

$$q(\lambda + \alpha q^2) + o(|q|^5) = 0$$

thus when crossing Eq (4.4) for $\alpha > 0$ the number of the local solutions decreases, and for $\alpha < 0$ increases by two, when λ increases. Moreover, at $\alpha = 0$, when $\lambda < 0$, the number of solutions is three, and when $\lambda \geq 0$, only one solution exists.

The number of solutions to Eq (4.7) and on the negative part of Eq (4.4) is three and on the positive part of Eq (4.4) is two, then the possible numbers of solutions in a small neighbourhood of the origin are 1,3,5, respectively. The results are shown in Fig.1.

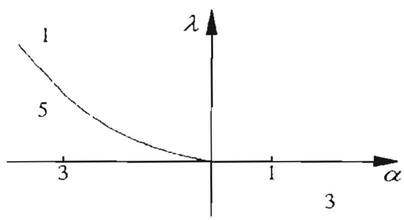

Fig. 1.

5. Conclusion

This paper deals with two methods of investigation of the static bifurcation. We have seen that the calculation steps in both methods are practically the same. The first one needs less abstract mathematics and gives clearer physical interpretation. Moreover, having expressions for the general case the solution to the problem consists in simple calculations with the current values of parameters.

For more complicated problems a higher mathematical abstraction can be helpful. Deriving a locally equivalent algebraic equation, as it has been done by using the second method, gives a start for the further investigation. In the second case study we solved a complicated problem and got results for a secondary buckling.

Acknowledgement
This work was supported by the OTKA 5-736.

References

1. BÉDA P.B., STEINDL A., TROGER H., 1992, Postbuckling of a Twisted Rod under Terminal Thrust, *Dynamics and Stability of Systems*, **7**, 4, 219-232
2. CHOW S-N., HALE J., 1982, *The Methods of Bifurcation Theory*, Springer, New York
3. GANTMACHER F., 1967, *Lectures in Analytical Mechanics*, Mir Publ. Moscow
4. GOLUBITSKY M., SCHAEFFER D.G., 1985, *Singularities and Groups in Bifurcation Theory*, **1**, Springer, New York
5. NGUYEN Q.S., 1993, *Stabilité des Structures Elastiques*, École Polytechnique, Palaiseau
6. TROGER H., STEINDL A., 1990, *Nonlinear Stability and Bifurcation Theory, an Introduction for Scientists and Engineers*, Springer, New York

Bifurkacje i post-bifurkacje prętów

Streszczenie

W ogólnym przypadku analizę bifurkacji sprężystych należy rozpocząć od opracowania modelu wariacyjnego. Następnie, w celu znalezienia punktów bifurkacji, powinna być rozpatrzona jednoznaczność części liniowej. Krzywizna wskazuje na równowagę pod- lub nadkrytyczną. Metoda taka nie daje jednak możliwości badania nawet małego otoczenia punktu bifurkacji, co jest ważne w badaniu bifurkacji wtórnych. Aby uzyskać takie informacje należy zastosować inną metodę. Równania równowagi rozważane są tylko w postaci lokalnej, a w wyniku redukcji otrzymujemy równanie algebraiczne, zwane równaniem bifurkacji. Równanie to przedstawia nieliniowe zachowanie się układu w pobliżu punktu bifurkacji, dając równocześnie informację o bifurkacjach wtórnych. Umożliwia ponadto znalezienie konsekwencji wad. W pracy przedstawiono powiązania i porównania obu metod. Jako przykład rozpatrzono wyboczenie pręta.

Manuscript received September 26, 1994; accepted for print June 28, 1995