MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 34, 1996

PLANE QUASI-STATIONARY THERMAL PROBLEMS WITH CONVECTIVE BOUNDARY CONDITION

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A fundamental solution is given for the temperature field due to a line heat source that moves at a constant speed over the surface of a convective half-space. On the basis of this solution approximate and exact solutions to the heat conductivity mixed boundary value problem for the quasi-static temperature field under convection conditions has been obtained. The influence of the coefficient of heat transfer and speed on the temperature distribution has been traced.

1. Introduction

In solution to practical problems of thermoelasticity such as thermal processes in grinding (cf Sipaĭlov (1978)) frictional heat generation due to sliding of the two bodies (cf Ling (1973); Barber and Comminou (1989)), laser and electron beam surface transformation (cf Festa et al. (1988) and (1990)), etc., the heat-affected region of the body surface is limited. More often it is assumed that the surface of the body outside this region is thermoinsulated. This assumption permits the corresponding fundamental solutions (cf Carslaw and Jaeger (1959); Rożnowski (1989)) to be used in the way enabling the boundary conditions outside the heat-affected area to be satisfied automatically. However, the convection and radiation conditions are more realistic. The solution to the thermal conduction problem for a line heat sources with constant power (plane-strain) moving at a constant speed on the convective half-space surface were found by Cameron et al. (1965) and by Yevtushenko, Ukhanska (1994). An simplified solution to this problem in the case of a fast-moving, arbitrarily distributed heat source was obtained by Ling and Yang (1971). These results were obtained for a large Peclet number; i.e., when the heat conduction in the direction of motion is neglected. In this paper the exact solution for the temperature distribution in a semi-infinite solid affected by a uniformly moving distributed heat source is presented. The results are obtained for all the values of Peclet number accepted within the framework of two-dimensional, quasi-static, uncoupled, thermoelasticity theory (cf Boley and Weiner (1960)). Noting that the solution to mixed-value problems for the steady temperature field under radiation conditions was obtained by Gladwell et al. (1983).

2. Instantaneous line source

We assume that the heat source of power q per unit length along the line $x = y = 0$ instantaneously acts at the instant $t = 0$ on the surface of the half-space $y \geq 0$, which is initially at zero temperature. If there is no heat flow across the surface plane $y = 0$, i.e.

$$\frac{\partial T}{\partial y} = 0 \qquad |x| < \infty \qquad y = 0 \tag{2.1}$$

the solution of the thermal-conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2.2}$$

at $t \geq 0$ has the form (cf Carslaw and Jaeger (1959))

$$T = \frac{q}{2\pi K t}\exp(-R^2) \qquad R^2 = \frac{r^2}{4kt} = \frac{x^2 + y^2}{4kt} \tag{2.3}$$

where

T – temperature
t – time
K – conductivity
k – diffusivity.

Examine the case when on the surface $y = 0$ the heat exchange is given according to Newton's law of cooling

$$\frac{\partial T}{\partial y} = \gamma T \qquad |x| < \infty \qquad y = 0 \tag{2.4}$$

where $\gamma = h/K$. h is the coefficient of heat exchange. The fundamental solution of Eq (2.2) satisfying the boundary condition (2.4) will be presented in the form

$$T = \frac{q}{2\pi K t}\left[\exp(-R^2) - \gamma\int_0^\infty \exp(-R'^2 - \gamma y')\, dy'\right] \tag{2.5}$$

where

$$R'^2 = \frac{r'^2}{4kt} = \frac{x'^2 + (y + y')^2}{4kt}$$

The first of term in Eq (2.5) represents the temperature (2.3) for the thermoinsulated surface $y = 0$ and the second term involves heat exchange with the surroundings and is the solution Eq (2.2) for the half-space with a line heat sources distributed in accordance with the exponential formula, i.e.

$$q' = \gamma q \exp(-\gamma x)$$

We show that the solution of Eq (2.5) satisfies the boundary condition (2.4). We rewrite Eq (2.5) in the form

$$T = \frac{q}{2\pi K t}\exp(-R^2)\left[1 - \gamma\int_0^\infty \exp\left(-\frac{2yy' + y'^2}{4kt} - \gamma y'\right) dy'\right] \tag{2.6}$$

The integral in Eq (2.6) at $y = 0$ will be transformed as

$$\begin{aligned} I_0(t) &= \int_0^\infty \exp\left(-\frac{y'^2}{4kt} - \gamma y'\right) dy' = \\ &= \exp(\gamma^2 kt)\int_0^\infty \exp\left[-\left(\frac{y'}{2\sqrt{kt}} - \gamma\sqrt{kt}\right)^2\right] dy' \end{aligned}$$

After the substitution

$$\xi = \frac{y'}{2\sqrt{kt}} + \gamma\sqrt{kt} \qquad\qquad dy' = 2\sqrt{kt}\, d\xi \tag{2.7}$$

we have

$$\begin{aligned} I_0(t) &= 2\sqrt{kt}\exp(\gamma^2 kt)\int_{\gamma\sqrt{kt}}^\infty \exp(-\xi^2)\, d\xi = \\ &= \sqrt{\pi kt}\exp(\gamma^2 kt)\operatorname{erfc}(\gamma\sqrt{kt}) \end{aligned} \tag{2.8}$$

where $\operatorname{erfc}(\cdot) = 1 - \operatorname{erf}(\cdot)$, $\operatorname{erf}(\cdot)$ is the probability integral (cf Abramowitz and Stegun (1965)).

Taking Eq (2.8) into account the temperature of the half-space surface $y = 0$ is

$$T\Big|_{y=0} = \frac{q}{2\pi Kt}\exp(-R^2)\Big[1 - \gamma\sqrt{\pi kt}\exp(\gamma^2 kt)\operatorname{erfc}(\gamma\sqrt{kt})\Big] \tag{2.9}$$

After differentiating Eq (2.6) with respect to y we find the derivative with respect to temperature along the normal to the surface $y = 0$ in the form

$$\frac{\partial T}{\partial y}\Big|_{y=0} = \frac{q}{2\pi Kt}\exp(-R^2)\frac{\gamma}{2kt}\int\limits_0^\infty y'\exp\Big(-\frac{y'^2}{4kt} - \gamma y'\Big)\,dy' \tag{2.10}$$

The integral in Eq (2.10)

$$I_1(t) = \int\limits_0^\infty y'\exp\Big(-\frac{y'^2}{4kt} - \gamma y'\Big)\,dy'$$

after substitution for variables from Eq (2.7) can be written as

$$I_1(t) = 2kt\Big[1 - \gamma\sqrt{\pi kt}\exp(\gamma^2 kt)\operatorname{erfc}(\gamma\sqrt{kt})\Big]$$

and from Eqs (2.10) we obtain

$$\frac{\partial T}{\partial y}\Big|_{y=0} = \frac{q}{2\pi Kt}\gamma\exp(-R^2)\Big[1 - \gamma\sqrt{\pi kt}\exp(\gamma^2 kt)\operatorname{erfc}(\gamma\sqrt{kt})\Big)\Big] \tag{2.11}$$

Putting Eqs (2.9) and (2.11) into the boundary condition (2.4) we obtain an identity.

3. Continuous moving source

Let the line heat source of power q move over the surface $y = 0$ of the half-space at a constant speed V in the direction of x-axis. The heat conduction equation in the coordinate system $0xy$ fixed the source takes the form

$$\frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2} + \frac{V}{k}\frac{\partial T_0}{\partial x} = \frac{1}{k}\frac{\partial T_0}{\partial t} \tag{3.1}$$

Satisfying this equation the temperature field will be modified and on the basis of Eq (2.5) at the time $t \geq 0$ under the boundary condition (2.4) we have

$$\begin{aligned} T_0(x,y) &= \frac{q}{2\pi K}\int\limits_0^t \frac{dt'}{t-t'}\Big\{\exp\Big[-\frac{[x+V(t-t')]^2+y^2}{4k(t-t')}\Big] + \\ &- \gamma\int\limits_0^\infty \exp\Big[-\frac{[x+V(t-t')]^2+(y+y')^2}{4k(t-t')} - \gamma y'\Big]\,dy'\Big\} \end{aligned} \tag{3.2}$$

Eq (3.2) represents the temperature field due to the moving line heat source which is continuously acting during the time t on the surface of the convective cooled half-space. Eq (3.2) can be rewritten as

$$T_0 = \frac{q}{2\pi K}(I_2 - \gamma I_3) \tag{3.3}$$

where

$$I_2 = \int\limits_0^t \frac{dt'}{t-t'}\exp\Big[-\frac{[x+V(t-t')]^2+y^2}{4k(t-t')}\Big] \tag{3.4}$$

$$I_3 = \int\limits_0^t \frac{dt'}{t-t'}\int\limits_0^\infty \exp\Big[-\frac{[x+V(t-t')]^2+(y+y')^2}{4k(t-t')} - \gamma y'\Big]\,dy' \tag{3.5}$$

By substituting for the variables

$$\xi = \frac{r^2}{4k(t-t')} \qquad\qquad dt' = \frac{t-t'}{\xi}d\xi$$

the integral I_2 (Eq (3.4)) will be written as

$$I_2 = \exp\Big(-\frac{Vx}{2k}\Big)\int\limits_{\frac{r^2}{4kt}}^\infty \frac{d\xi}{\xi}\exp\Big[-\Big(\xi + \frac{V^2r^2}{16k^2\xi}\Big)\Big] \tag{3.6}$$

In a quasi-stationary case $t \to \infty$ from Eq (3.6) we obtain

$$I_2 = 2\exp\Big(-\frac{Vx}{2k}\Big)\int_0^\infty \frac{d\eta}{\eta}\exp\Big[-\Big(\frac{V^2r^2\eta^2}{4k^2}+\frac{1}{4\eta^2}\Big)\Big] \tag{3.7}$$

where

$$\eta = \frac{2k\sqrt{\xi}}{Vr} \qquad\qquad 2\frac{d\eta}{\eta} = \frac{d\xi}{\xi}$$

Since (cf Abramowitz and Stegun (1965))

$$\int_0^\infty \frac{d\eta}{\eta}\exp\Big[-\Big(x^2\eta^2+\frac{1}{4\eta^2}\Big)\Big] = K_0(x)$$

from Eq (3.7) it follows

$$I_2 = 2\exp\Big(-\frac{Vx}{2k}\Big)K_0\Big(\frac{V}{2k}\sqrt{x^2+y^2}\Big) \tag{3.8}$$

where $K_0(\cdot)$ is a modified Bessel function of the second kind and zero order.

To calculate the integral I_3 (Eq (3.5)) we substitute for $y+y'=y''$ and change the order of integration. Then we have

$$I_3 = \exp(\gamma y)\int_y^\infty \exp(-\gamma y'')\int_0^t \frac{dt'}{t-t'}\exp\Big[-\frac{[x+V(t-t')]^2+y''^2}{4k(t-t')}\Big]\,dy'' \tag{3.9}$$

The inside integral in Eq (3.9) is equal to the integral I_2 of Eq (3.4) and, consequently, at $t \to \infty$ it coincides with the value of I_2 given by the formula (3.8). Therefore

$$I_3 = 2\exp\Big(-\frac{Vx}{2k}+\gamma y\Big)\int_y^\infty \exp(-\gamma y')K_0\Big(\frac{V}{2k}\sqrt{x^2+y'^2}\Big)\,dy' \tag{3.10}$$

Taking Eqs (3.8) and (3.10) into account, Eq (3.3) for the quasi-stationary temperature field in a half-space takes the form

$$\begin{aligned} T_0(x,y) \;=\; & \frac{q}{\pi K}\exp\Big(-\frac{Vx}{2k}\Big)\Big[K_0\Big(\frac{V}{2k}\sqrt{x^2+y^2}\Big) + \\ & - \gamma\exp(\gamma y)\int_y^\infty \exp(-\gamma y')K_0\Big(\frac{V}{2k}\sqrt{x^2+y'^2}\Big)\,dy'\Big] \end{aligned} \tag{3.11}$$

For $\gamma = 0$ from Eq (3.11) it follows a well-known result for the quasi-stationary temperature field in the half-space surface of which is thermoinsulated (cf Festa (1990)).

We shall analyse the influence of the convective heat exchange on the temperature field affected by a line heat source. Introducing dimensionless variables

$$\hat{X} = \frac{Vx}{2k} \qquad \hat{Y} = \frac{Vy}{2k} \tag{3.12}$$

we rewrite Eq (3.11) in the form

$$T_0(\hat{X}, \hat{Y}) = \frac{q}{\pi K} \exp(-\hat{X}) K_0(\sqrt{\hat{X}^2 + \hat{Y}^2}) D(\hat{X}, \hat{Y}, \beta) \tag{3.13}$$

$$D(\hat{X}, \hat{Y}, \beta) = 1 - \beta \exp(\beta \hat{Y}) \frac{\int_{\hat{Y}}^{\infty} \exp(-\beta \hat{Y}') K_0(\sqrt{\hat{X}^2 + \hat{Y}'^2})\, d\hat{Y}'}{K_0(\sqrt{\hat{X}^2 + \hat{Y}^2})} \tag{3.14}$$

where $\beta = 2\gamma k/V$.

The influence of the heat exchange is determined by the function $D(\hat{X}, \hat{Y}, \beta)$, Eq (3.14). We shall estimate the influence of the heat exchange on the surface temperature. In this case at $\hat{Y} = 0$, $\hat{X} \to 0$, $K_0(|\hat{X}|) \to \infty$ and the numerator of Eq (3.14) is limited. Hence, in the neighborhood of the source $D(\hat{X}, 0, \beta) \to 1$, i.e., at every method of cooling the influence of heat transfer is not practically reflected on the surface temperature. At a distance from the source the influence of the heat flow increases (the function $D(\hat{X}, 0, \beta)$ decreases when approaching zero), but the heat given back to the surroundings decreases because the difference between temperatures of surface of half-space and surrounding decreases appreciable.

In Fig.1 one can see dimensionless temperature fields $T_0^* = \pi K T/q$ for a line source without ($\beta = 0$, solid curves) and with cooling ($\beta = 1$, dashed curves) calculated from Eqs (3.13) and (3.14). It is clear that the influence of the heat exchange on the surface temperature of the half-space is reflected at some distance from the source. The deeper layers do not even feel the influence of cooling.

4. Continuous moving source

We consider a uniform strip heat source of a width $2a$ moving at the constant relative speed V in the direction of x-axis over the surface of a

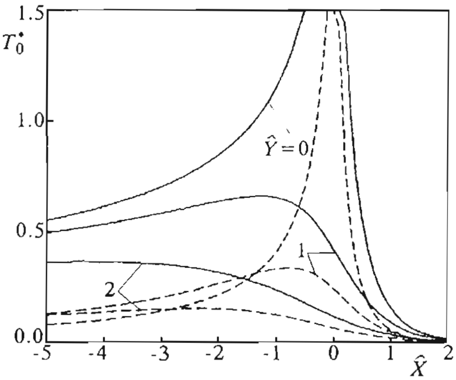

Fig. 1. The distribution of the dimensionless temperature T_0^* without cooling ($\beta = 0$, solid curve) and with cooling ($\beta = 1$, dashed curve)

semi-infinite body. The surface $y = 0$ of the half-space is convective cooled which is described by the boundary condition (2.4). The temperature field in the semi-infinite body is defined by integrating the solution of Eq (3.11) in space. We obtain

$$T_1(x,y) = \frac{q}{\pi K}\int\limits_{-a}^{a} \exp\Big(-\frac{V(x-x')}{2k}\Big)\Big[K_0\Big(\frac{V}{2k}\sqrt{(x-x')^2+y^2}\Big) + \tag{4.1}$$

$$-\gamma\exp(\gamma y)\int\limits_{y}^{\infty}\exp(-\gamma y')K_0\Big(\frac{V}{2k}\sqrt{(x-x')^2+y'^2}\Big)\,dy'\Big]\,dx'$$

The temperature (4.1) can be rewritten now in the dimensionless form

$$T_1(X,Y) = \frac{aq}{\pi K}\int\limits_{-1}^{1}\exp[-\mathrm{Pe}(X-X')]\Big[K_0\Big(\mathrm{Pe}\sqrt{(X-X')^2+Y^2}\Big) + \tag{4.2}$$

$$-\mathrm{Bi}\exp(\mathrm{Bi}Y)\int\limits_{Y}^{\infty}\exp(-\mathrm{Bi}Y')K_0\Big(\mathrm{Pe}\sqrt{(X-X')^2+Y'^2}\Big)\,dY'\Big]\,dX'$$

where $X = x/a$, $Y = y/a$, $\mathrm{Pe} = Va/2k$ is the Peclet number, $\mathrm{Bi} = \gamma a$ is Biot's parameter. The derivative $\partial T_1/\partial Y$ is given by

$$
\begin{aligned}
\frac{\partial T_1}{\partial Y} = \frac{aq}{\pi K} \int\limits_{-1}^{1} & \exp[-\mathrm{Pe}(X - X')]\Big[-K_1\Big(\mathrm{Pe}\sqrt{(X - X')^2 + Y^2}\Big) \cdot \\
& \cdot \frac{Y}{\sqrt{(X - X')^2 + Y^2}} + \mathrm{Bi} K_0\Big(\mathrm{Pe}\sqrt{(X - X')^2 + Y^2}\Big) + \\
& -\mathrm{Bi}^2 \exp(\mathrm{Bi}Y) \int\limits_{Y}^{\infty} \exp(-\mathrm{Bi}Y') K_0\Big(\mathrm{Pe}\sqrt{(X - X')^2 + Y'^2}\Big)\, dY'\Big]\, dX'
\end{aligned}
\tag{4.3}
$$

where $K_1(\cdot)$ is a modified Bessel function of the second kind and the first order.

When $|X| > 1$, then from Eq (4.3) it follows

$$
\frac{\partial T_1}{\partial Y} = BiT_1 \qquad |X| > 1 \qquad Y = 0 \tag{4.4}
$$

When $|X| < 1$, then at $X \to X'$, $Y - 0$

$$
K_1(\mathrm{Pe}|X - X'|) - \frac{1}{\mathrm{Pe}|X - X'|}
$$

and the first term in curly brackets within the integral sings in Eq (4.3) has a singularity. To open this singularity we use the solution for continuous line source (3.2) which has been integrated within the limits of $-a \leq x \leq a$. For $\gamma = 0$ we obtain

$$
T_1(x, y, t) = \frac{q}{2\pi K} \int\limits_{-a}^{a} \int\limits_{0}^{t} \frac{dx' dt'}{\sqrt{t - t'}} \exp\Big(-\frac{[x - x' + V(t - t')]^2 + y^2}{4k(t - t')}\Big) \tag{4.5}
$$

Putting

$$
u' = \frac{x - x' + V(t - t')}{2\sqrt{k(t - t')}} \qquad du' = -\frac{dx'}{2\sqrt{k(t - t')}}
$$

from Eq (4.5) we find

$$
T_1(x, y, t) = \frac{q\sqrt{\pi k}}{2\pi K} \int\limits_{0}^{t} \frac{dt'}{\sqrt{t - t'}} \exp\Big(-\frac{y^2}{4k(t - t')}\Big)[\mathrm{erf}(A_+) - \mathrm{erf}(A_-)] \tag{4.6}
$$

where

$$
A_{\pm} = \frac{x \pm a - V(t - t')}{2\sqrt{k(t - t')}}
$$

We note that Eq (4.6) coincides completely with the results of Festa (1988). The derivate of the temperature T_1 (4.5) with respect to y is

$$\frac{\partial T_1}{\partial y} = -\frac{q\sqrt{\pi k}}{2\pi K}\int\limits_0^t \frac{dt'}{\sqrt{t-t'}}\frac{y}{2k(t-t')}\exp\Big(-\frac{y^2}{4k(t-t')}\Big)[\mathrm{erf}(A_+)-\mathrm{erf}(A_-)] \tag{4.7}$$

Putting

$$\eta = \frac{y'}{2\sqrt{k(t-t')}} \qquad d\eta = \frac{ydt'}{4\sqrt{k(t-t')}(t-t')}$$

from Eq (4.7) we find

$$\frac{\partial T_1}{\partial y} = -\frac{q}{K\sqrt{\pi}}\int\limits_{\frac{y}{2\sqrt{\pi k}}}^{t} \exp(-\eta^2)[\mathrm{erf}(B_+)-\mathrm{erf}(B_-)]\,d\eta \tag{4.8}$$

where

$$B_\pm = \frac{x \pm a + \frac{Vy^2}{4k\eta^2}}{\frac{y}{\eta}}$$

At $y = 0$, $|x| < a$ we have $B_\pm = \pm\infty$. So far as $\mathrm{erf}(\pm\infty) = \pm 1$, from Eq (4.8) as $t \to \infty$ we obtain

$$\frac{\partial T_1}{\partial y} = -\frac{2q}{K\sqrt{\pi}}\int\limits_0^t \exp(-\eta^2)\,d\eta = -\frac{q}{K} \qquad |x| < a \qquad y = 0 \tag{4.9}$$

Therefore, taking Eq (4.9) into account from Eq (4.3) it follows

$$\frac{\partial T_1}{\partial Y} = -\frac{aq}{K} + \mathrm{Bi}T_1 \qquad |X| < 1 \qquad Y = 0 \tag{4.10}$$

Thus, the solution of Eq (4.2) satisfies the boundary conditons (4.4) and (4.10). The condition (4.10) shows that, besides inward heat flow, there is still an outward heat flow dependent on temperature. The solution of Eq (4.2) may be used for the analysis of temperature fields in the processes, in which heat exchange is realized all over the surface simultaneously with rapid inductive heating, electrospark alloy and etc.

5. Approximate solution to the heat conduction mixed problem

In frictional heat generation, grinding, etc., the heat exchange is realised on a free surface (condition (4.4)), and in the contact area a heat flow is known and the boundary condition are given in the form

$$\frac{\partial T_2}{\partial Y} = -\frac{aq}{K} \qquad |X| < 1 \qquad Y = 0 \tag{5.1}$$

Consider that it is possible to satisfy the boundary condition (5.1) proceeding from the solution of Eq (4.2) which, as shown, satisfies the conditions (4.4), (4.10). We replaced the temperature in the right-hand side of the condition (4.10) by the mean temperature in the heating region

$$\theta = \frac{1}{2}\int_{-1}^{1} T_1(X,0)\,dX$$

We have

$$\frac{\partial T_2}{\partial Y} = -\frac{aq}{K} + \theta \qquad |X| < 1 \qquad Y = 0 \tag{5.2}$$

The condition (5.2) coincides with the condition (5.1) with the accuracy of some multiplier λ, i.e.

$$-\frac{aq}{K} = \lambda\Big(-\frac{aq}{K} + \mathrm{Bi}\theta\Big)$$

Then

$$\lambda = \frac{1}{1 - \mathrm{Bi}\frac{\theta^*}{\pi}} > 1$$

where $\theta^* = \pi K\theta/(aq)$. The value of λ depends on the product of Biot's parameter Bi and the mean value of the temperature θ in the heating region. We find λ numerically. The dependence of the dimensionless parameter λ on the Peclet number Pe for several values of Biot's parameter Bi is shown in Fig.2.

The solution to the problem, which satisfies the condition of heat exchange in the heating region, can be written

$$T_2(X,Y) = \lambda T_1(X,Y) \tag{5.3}$$

where the function $T_1(X,Y)$ can be found from Eq (4.2).

The distribution of non-dimensional surface temperature $T_2^* = \pi KT_2/(qa)$ for $\mathrm{Pe} = 1.0$ is shown in Fig.3. The dashed curve indicates the case of full

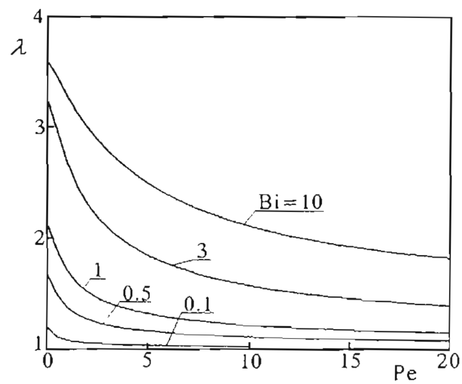

Fig. 2. The dependence of the dimensionless parameter λ from Peclet number at Bi=0.1; 0.5; 1; 3; 10

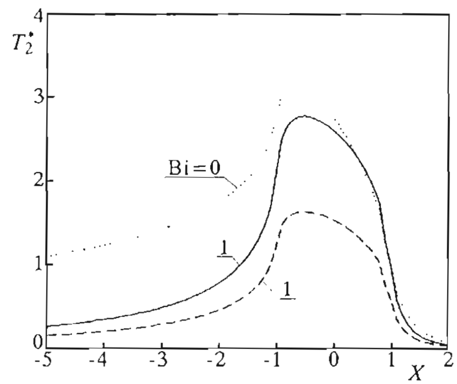

Fig. 3. The distribution of the dimensionless surface temperatures for Pe = 1 (dashed curve – T_1^* at Bi = 1; solid curve – T_2^* at Bi = 1; dotted curve – T_2^* at Bi = 0)

heat exchange all over the surface $y = 0$ of the half-space ($\lambda = 1$, Bi = 1, solution (4.2)). Multiplying the corresponding values of this curve by λ, which is 1.706 for this case, we obtain the temperature distribution corresponding to the solution of (5.3) (solid curve). The temperature distribution on the surface $y = 0$ without the heat exchange (Bi = 0, dotted curve) is given here for comparison. One can see that the heat flow lowers the maximum temperature insignificantly (in this case by 9%). The heat exchange affects mostly the surface temperature after the source. So at $X = -2; -3; -4; -5$ this temperature forms 46.3%; 35.5%; 29.1%; 24.7% from the corresponding temperature at Bi = 0. At a distance from the source the influence of the heat exchange becomes stronger though the absolute values of heat flows into the cooling surroundings decrease.

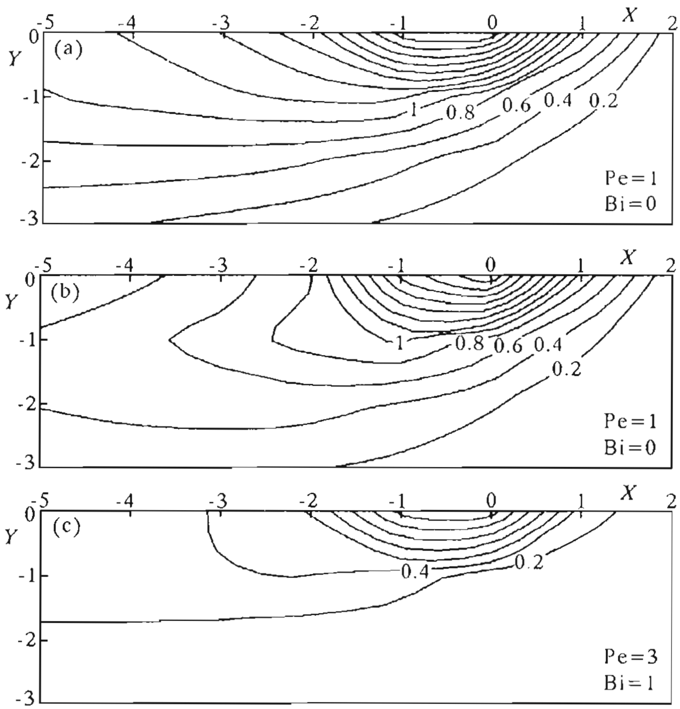

Fig. 4. The isothermas of the dimensionless temperature T_2^* in the half-space: (a) - Pe=1, Bi=0; (b) - Pe=1, Bi=1; (c) - Pe=3, Bi=1

Iso-temperature T_2^* contour plots are displayed in Fig.4. It can be stated that in the presence of the heat exchange the depth of the penetration of high temperatures decreases. The depth decreases of penetration with the increase in the Peclet number.

6. Exact solution to conduction mixed problem

The solution to the quasi-stationary boundary problem of conduction may be obtained by using directly the method of the compensation of heat losses (cf Sipaïlov (1978)). The essence of this method is as follows. The solution of Eq (4.2) as it was mentioned above, satisfies the condition of the heat exchange all over the surface $y = 0$ of the half-space. It means that the integration of the fundamental solution of Eq (3.11) has brought the heat exchange in the heating region. For the density heat flow q directed inside a half-space, it follows from Eq (4.10) that a heat flow has appeared by the heat exchange directed from the half-space to the cooling surroundings without the heating area. To remove this discrepancy it is necessary, within the limits of the contact area, to add the heat flow distributed according to the losses, i.e., as $\mathrm{Bi}T_1$ where T_1 is defined by Eq (4.2). This additional source locally compensates the heat losses in the source region but simultaneously it also brings in the partial heat losses affected by the solution construction. To compensate these new losses it is necessary to add one source defined $\mathrm{Bi}T^{(0)}$ where $T^{(0)}$ is the temperature distribution in the heating area from the first compensating source. This element will also have its losses but significantly smaller. The process of such compensations may be theoretically infinite. As a result we obtain an exact solution, i.e., completely compensated losses for cooling in the heating region satisfying the external heat exchange condition. The exact solution will be represented by a series with an infinite number of terms in the form

$$T_3(X,Y) = \lim_{N\to\infty} T^{(N)}(X,Y) \tag{6.1}$$

$$\begin{aligned} T^{(N)}(X,Y) &= T_1(X,Y) + \frac{\mathrm{Bi}}{\pi}\sum_{n=0}^{N-1}\Big\{\int_{-1}^{1} T^{(n)}[\mathrm{Pe}X-(X-X')]\cdot \\ &\cdot\exp[-\mathrm{Pe}(X-X')]\Big[K_0\Big(\mathrm{Pe}\sqrt{(X-X')^2+Y^2}\Big) + \\ &-\mathrm{Bi}\exp(\mathrm{Bi}Y)\int_Y^{\infty}\exp(-\mathrm{Bi}Y')K_0\Big(\mathrm{Pe}\sqrt{(X-X')^2+Y'^2}\Big)\,dY'\Big]\,dX'\Big\} \end{aligned} \tag{6.2}$$

where $T^{(0)}(X,Y) = T_1(X,Y)$.

Using Eq (6.1) can be shown that the heat exchange influences insignificantly the temperature distribution in the heat-affected area. The dimensionless temperature distribution $T_3^* = \pi KT_3/(qa)$ on the surface $y=0$ at Pe=1 (Bi=0, dotted curve) and the intensive heat exchange (Bi=1, solid line) is shown in Fig.5. The approximate temperature distribution (5.3) is given here for comparison (dashed curve). The drop of the maximum temperature in comparison with the case of thermoinsulated surface of the half-space is not large and does not exeed 5%. To reach a relative accuracy of calculations equal to 1% it was necessary to take seven terms of the series (6.2).

We note that some obstacles can be found when using Eq (6.1) and since the law of distribution of the heat flow in the heat-affected area does not

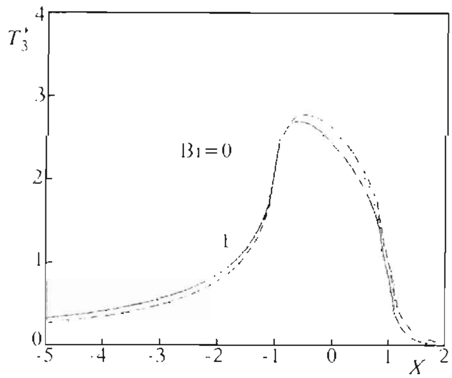

Fig. 5. The distribution of the dimensionless surface temperatures for Pe=1 (dashed curve - T_3^* at Bi=1, solid curve - T_3^* at Bi=1, dotted curve - T_3^* at Bi=0)

affect significantly the temperature distribution it is more convinient to use the approximate solution (5.3).

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Płaskie quasistacjonarne zagadnienia termiczne z konwektywnym warunkiem brzegowym

Streszczenie

W pracy podano rozwiązanie podstawowe zagadnienia rozkładu temperatury w półprzestrzeni wywołanego liniowymi źródłami ciepła poruszającymi się ze stalą prędkością po powierzchni ciała z uwzględnieniem konwekcji. Na podstawie otrzymanych rezultatów podano przybliżone lub dokładne rozwiązania mieszanych zagadnień brzegowych. Zbadano wpływ prędkości poruszania się źródła ciepła oraz własności termicznych materiału na rozkład temperatury w półprzestrzeni.

Manuscript received June 28, 1995; accepted to print August 22, 1995