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ON THE THERMALLY EXCITED STRESS WAVES IN A ROD RESTING ON ELASTIC FOUNDATION

MAREK SOKOŁOWSKI

ZBIGNIEW WESOŁOWSKI

Institute of Fundamental Technological Research, Warsaw

e-mail: msokol@ippt.gov.pl , zwesol@ippt.gov.pl

A semi-infinite rod $x > 0$ is attached to an elastic foundation reacting to axial displacements of the rod with forces proportional to the displacements. Temperature field $T(x)$ is applied to a portion of the rod at time $t = 0$. The stress disturbance produced by that temperature is divided into two waves which propagate along the rod, one of them being reflected from its end according to the support conditions. The solution is written in an integral form, and it is analyzed in approximate manner using both the analytical methods (integral transforms) and numerical integration.

Key words: stress waves, elastic rod, thermoelasticity

1. Basic equations

Consider a one-dimensional problem of a semi-infinite elastic rod $x > 0$ of constant cross-section A attached to the elastic foundation (Fig.1). Assume that the surface stresses τ exerted by the foundation on the rod are uniformly distributed around its periphery ℓ and equal $\tau(x,t) = ku(x,t)$, where $u(x,t)$ denotes axial displacement of the rod and k – the constant characterizing the foundation rigidity; thus, the horizontal force dX exerted on rod's element of length dx equals $dX = k\ell u(x,t)dx$. Since the corresponding force of inertia is $dF = -\varrho A\partial^2 u/\partial t^2 dx$ (ϱ – mass density), and axial forces acting on the cross-sections x and $x + dx$ equal $A\sigma$ and $-A[\sigma + (\partial\sigma/\partial x)dx]$ respectively, the condition of equilibrium of the element shown in Fig.1 yields the equation

$$A\frac{\partial\sigma}{\partial x} - k\ell u - \varrho A\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

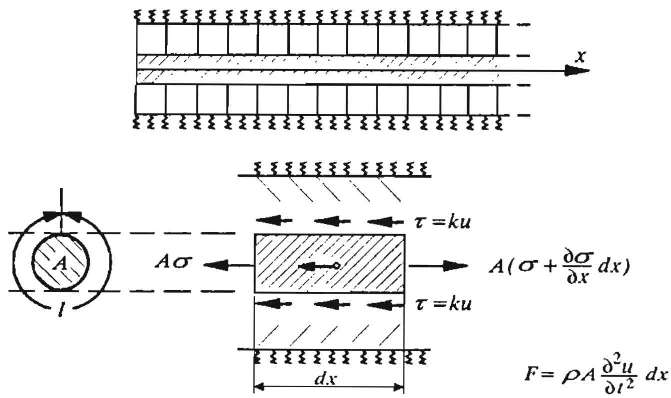


Fig. 1.

Making use of the well-known Duhamel-Neumann equations (cf e.g. Nowacki (1962)) reduced to the one-dimensional case

$$\sigma = E\Big(\frac{\partial u}{\partial x} - \beta T\Big) \tag{1.2}$$

(here β – linear thermal expansion coefficient), and introducing two reduced foundation elasticity coefficients

$$\kappa = \sqrt{\frac{k\ell}{\varrho A}} \qquad h = \sqrt{\frac{k\ell}{EA}}$$

Eq (1.1) may be rewritten in the form

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \kappa^2 u = -c^2\beta\frac{\partial T}{\partial x}\eta(t) \tag{1.3}$$

or

$$\frac{\partial^2 u}{\partial^2 x} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - h^2 u = \beta\frac{\partial T}{\partial x}\eta(t) \tag{1.4}$$

Here $c = \sqrt{E/\varrho}$ is the longitudinal elastic wave propagation speed and symbol $\eta(x)$ denotes the Heaviside step function

$$\eta(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1/2 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$$

Eq (1.4) is the nonhomogeneous Klein-Gordon differential equation known from mathematical physics (cf, e.g., Morse and Feshbach (1953)). Similar equations were also considered by Sokołowski and Wesołowski (1989), (1992), in connection with a slightly different problem of a two-component composite rod transmitting simple displacement pulses.

Eqs (1.3), (1.4) will be analyzed under two different boundary conditions at the end $x = 0$ of the rod:

Case (A) – end $x = 0$ of the rod is fixed

$$u(0,t) = 0 \tag{1.5}$$

Case (B) – end $x = 0$ of the rod is stress-free

$$\sigma(0,t) = 0 \tag{1.6}$$

The initial conditions for $u(t)$ are assumed to be homogeneous

$$u(x,0) = \dot{u}(x,0) = 0 \tag{1.7}$$

Assume now for simplicity that at time $t = 0$ temperature $T(x)$ is applied to a portion of the rod and that it is kept constant afterwards, i.e. $T = T(x)\eta(t)$, and consider Case (A) of the fixed end of the rod. Let us apply to Eq (1.4) double integral transforms: the Laplace transform $\mathcal{L}$ and the Fourier sine transform $\mathcal{F}_s$ (cf e.g. Bateman (1954))

$$\mathcal{L}\{u(x,t)\} = \bar{u}(x,p) = \int\limits_0^\infty u(x,t)\mathrm{e}^{-pt}\,dt$$

$$\mathcal{F}_s\{u(x,t)\} = u_s(\alpha,t) = \int\limits_0^\infty u(x,t)\sin\alpha x\;dx$$

It is known that under the initial conditions (1.7), the Laplace transform of the second derivative of u equals

$$\mathcal{L}\Big\{\frac{\partial^2 u}{\partial t^2}\Big\} = p^2\bar{u}(p)$$

and due to the boundary conditions (1.5), the Fourier transform

$$\mathcal{F}_s\Big\{\frac{\partial^2 u}{\partial x^2}\Big\} = -\alpha^2 u_s(\alpha)$$

If, in addition, $T(\infty) = 0$, we obtain from Eq (1.4) the formula for the double transform of $u(x,t)$

$$\bar{u}_s(\alpha,p) = c^2\beta\frac{\alpha T_c(\alpha)}{p(p^2+\kappa^2+\alpha^2c^2)} \tag{1.8}$$

symbol $T_c(\alpha)$ denoting the cosine Fourier transform of function $T(x)$

$$T_c(\alpha) = \int\limits_0^\infty T(x)\cos\alpha x\, dx \tag{1.9}$$

In Case (B) of a rod with a stress-free end $x = 0$ transforms $\mathcal{L}$ and $\mathcal{F}_c$ must be applied to Eq (1.4), this time leading to the result

$$\bar{u}_c(\alpha, p) = -c^2\beta\frac{\alpha T_s(\alpha)}{p(p^2 + \kappa^2 + \alpha^2c^2)} \tag{1.10}$$

under the additional assumption that $T(0) = 0$.

Apply now the inverse Laplace transform formula (cf Bateman (1954))

$$u(x,t) = \frac{1}{2\pi\mathrm{i}}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \bar{u}(x,p)\mathrm{e}^{pt}\, dp$$

to the solution (1.8); using the reduced time-variable $\tau = ct$, in view of the formula (cf Bateman (1954))

$$\mathcal{L}^{-1}\Big\{\frac{1}{p(p^2 + A^2)}\Big\} = \frac{1 - \cos At}{A^2}$$

the $\mathcal{F}_s$ – transform of $u(x,\tau)$ equals

$$u_s(\alpha,\tau) = \beta\frac{\alpha T_c(\alpha)(1 - \cos\Omega\tau)}{\alpha^2 + h^2} \tag{1.11}$$

where

$$\Omega = \sqrt{\alpha^2 + h^2}$$

Term $\cos\Omega\tau$ seriously complicates the following procedure of inversion of the transform Eq (1.11).

Application of the inverse $\mathcal{F}_s$ transform

$$u(x,\tau) = \frac{2}{\pi}\int\limits_0^\infty u_s(\alpha,\tau)\sin\alpha x\, d\alpha$$

to Eq (1.11) yields

$$u(x,\tau) = \frac{2\beta}{\pi}\int\limits_0^\infty \frac{\alpha T_c(\alpha)(1 - \cos\Omega\tau)}{\alpha^2 + h^2}\sin\alpha x\, d\alpha \tag{1.12}$$

Using Eq (1.2) we easily obtain the integral formulae for the stress

$$\sigma(x,\tau) = -\frac{2E\beta}{\pi}[F_1(x) + F_2(x,\tau)] \tag{1.13}$$

with the notations

$$F_1(x) = h^2 \int\limits_0^\infty \frac{T_c(\alpha)\cos\alpha x}{\alpha^2 + h^2}\,d\alpha$$
$$F_2(x,\tau) = \int\limits_0^\infty \frac{\alpha^2 T_c(\alpha)\cos\Omega\tau\cos\alpha x}{\alpha^2 + h^2}\,d\alpha \tag{1.14}$$

Solutions (1.12) are clearly seen to satisfy the boundary conditions (1.5) of Case (A), i.e. $u(0,\tau) = 0$.

To solve the other Case (B), the $\mathcal{F}_c$ transform (1.9) replaces the formerly used $\mathcal{F}_s$. A procedure similar to that used above in Eqs (1.8) ÷ (1.14) leads to the results

$$u(x,\tau) = -\frac{2\beta}{\pi}\int\limits_0^\infty \frac{\alpha T_s(\alpha)(1-\cos\Omega\tau)\cos\alpha x}{\alpha^2 + h^2}\,d\alpha \tag{1.15}$$

and

$$\sigma(x,\tau) = -\frac{2E\beta}{\pi}[F_3(x) + F_4(x,\tau)] \tag{1.16}$$

with

$$F_3(x) = h^2 \int\limits_0^\infty \frac{T_s(\alpha)\sin\alpha x}{\alpha^2 + h^2}\,d\alpha$$
$$F_4(x,\tau) = \int\limits_0^\infty \frac{\alpha^2 T_s(\alpha)\cos\Omega\tau\sin\alpha x}{\alpha^2 + h^2}\,d\alpha \tag{1.17}$$

Solution (1.16) evidently satisfies the boundary condition (1.6) of Case (B), $\sigma(0,\tau) = 0$.

Presentation of the solutions (1.12) ÷ (1.17) in a closed, explicit form is possible in certain special cases only; let us discuss these cases.

2. Explicit solutions of the problem

2.1. Statical solutions

In the statical cases of a semi-infinite rod resting on elastic foundation and transmitting temperature-exited longitudinal stresses, the problem is governed by the ordinary differential equation

$$\frac{d^2u(x)}{dx^2} - h^2u(x) = \beta\frac{dT(x)}{dx} \tag{2.1}$$

obtained from Eq (1.4) by disregarding the second left-hand term. Applying the $\mathcal{F}_s$ procedure in Case (A), we obtain

$$\sigma(x) = -\frac{2E\beta}{\pi}h^2\int\limits_0^\infty \frac{T_c(\alpha)\cos\alpha x}{\alpha^2+h^2}\,d\alpha \tag{2.2}$$

It is seen that this coincides exactly with the first term of solution (1.13)

$$\sigma(x) = -\frac{2E\beta}{\pi}F_1(x)$$

the second term F_2 representing the dynamic effects of the process. With $t\to\infty$ the term $F_2(x,\tau)$ in Eq (1.13) tends to zero for every $x<\infty$, and thus the final state of deformation and stress is identical with the statical solution (2.2). It may also be rewritten in a more convenient form

$$\sigma(x) = -E\beta h\Big[\mathrm{e}^{-hx}\int\limits_0^x T(\xi)\cosh h\xi\,d\xi + \cosh hx\int\limits_x^\infty T(\xi)\mathrm{e}^{-h\xi}\,d\xi\Big] \tag{2.3}$$

In the particular case when $T(x)=T_0$ for $0<x<b$ and $T(x)=0$ for $x>b$, i.e. $T(x)=T_0\eta(b-x)$

$$\sigma(x) = -E\beta T_0\begin{cases} 1-\mathrm{e}^{-bh}\cosh hx & x<b\\ \sinh hb\mathrm{e}^{-hx} & x>b\end{cases} \tag{2.4}$$

Diagrams of function $\sigma(x)$ made for several values of the foundation rigidity constant h are shown in Fig.2. It is seen that with increasing values of h, the $\sigma(x)$ – curves approach the value $-\mathrm{e}\beta T_0$ for $x<b$, and 0 for $x>b$; with a perfectly rigid foundation $h\to\infty$, displacements u vanish and, according to Eq (1.2), the stress $\sigma = E\beta T(x)$. At the same time, it is also

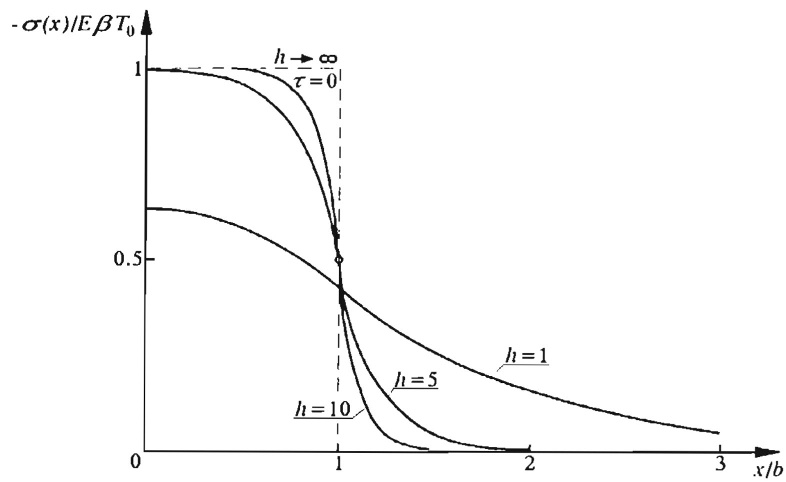


Fig. 2.

the initial value of stress $\sigma(x,0)$ of Eq (1.13), since simple transformation of Eq (1.14) at $\tau = 0$ yields, under arbitrary $T(x)$ – distribution

$$F_1(x,0) + F_2(x,0) = \int_0^\infty \Big[\frac{h^2}{\alpha^2 + h^2} + \frac{\alpha^2}{\alpha^2 + h^2}\Big] T_c \cos \alpha x \, d\alpha$$

and

$$\sigma(x,0) = -\frac{2E\beta}{\pi} \int_0^\infty T_c(\alpha) \cos \alpha x \, d\alpha = -E\beta T(x) \tag{2.5}$$

thus it may be concluded that the actual dynamic wave propagation process starts from the state (2.5) and ends up with (2.3), (2.4) at $t \to \infty$.

Similar reasoning applied to the Case (B) (stress-free end of the rod) leads to the following results.

The time-independent, statical solution for a semi-infinite rod with stress-free end $x = 0$ is (in the case of $T(x) = T_0 \eta(b - x)$)

$$\sigma(x) = -E\beta T_0 \begin{cases} 1 - e^{-hx} - e^{-hb} \sinh hx & x < b \\ e^{-hx}(\cosh hb - 1) & x > b \end{cases} \tag{2.6}$$

Both solutions (2.4) and (2.6) are continuous with their first derivatives at $x = b$.

2.2. Stress waves in a rod without foundation

In case of $h = 0$ (no foundation), the classical problem of stress wave propagation in a semi-infinite rod is described by Eq (1.3) reduced to the form (cf Nowacki (1963))

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -c^2 \beta \frac{\partial T}{\partial x} \eta(t) \tag{2.7}$$

Application of the Laplace and Fourier sine transforms (in Case (A)) leads to the known results for the displacement u

$$u(x,\tau) = \frac{2\beta}{\pi} \int\limits_0^\infty \frac{T_c(\alpha)}{\alpha}(1 - \cos\alpha\tau)\sin\alpha x \; d\alpha \tag{2.8}$$

and stress

$$\sigma(x,\tau) = -\frac{2E\beta}{\pi} \int\limits_0^\infty T_c(\alpha)\cos\alpha\tau\cos\alpha x \; d\alpha \tag{2.9}$$

Using the decomposition

$$\cos\alpha x \cos\alpha\tau = \frac{1}{2}[\cos\alpha(x+\tau) + \cos\alpha(x-\tau)]$$

and the known $\mathcal{F}^{-1}$ transform formula

$$\frac{2}{\pi}\int\limits_0^\infty T_c(\alpha)\cos\alpha X \; d\alpha = \begin{cases} T(X) & \text{for } X > 0 \\ T(-X) & \text{for } X < 0 \end{cases}$$

we obtain the known result (cf Nowacki (1963))

$$\begin{aligned} \sigma(x,\tau) &= -\frac{1}{2}E\beta[T(x+\tau) + T(x-\tau)] \qquad && x > \tau \\ \sigma(x,\tau) &= -\frac{1}{2}E\beta[T(x+\tau) + T(\tau-x)] \qquad && x < \tau \end{aligned} \tag{2.10}$$

If $T(x)$ happens to be an even function of x, the upper formula (2.10) is sufficient. The stress disturbance $-E\beta T(x)$ is divided into two halves propagating in opposite directions at speed c; one of the waves is reflected from the fixed end of the rod.

In case of a free end $x = 0$, the solution assumes the form

$$\begin{aligned} \sigma(x,\tau) &= -\frac{1}{2}E\beta[T(x+\tau) + T(x-\tau)] \qquad && x > \tau \\ \sigma(x,\tau) &= -\frac{1}{2}E\beta[T(x+\tau) - T(\tau-x)] \qquad && x < \tau \end{aligned} \tag{2.11}$$

The halfwave travelling to the left changes its sign upon reflection from the stress-free end of the rod. Here again, if $T(x)$ is an odd function of x, the upper formula of Eq (2.11) represents the complete solution.

Several stages of the wave propagation process described by Eqs (2.10) and (2.11) are depicted in Fig.3.

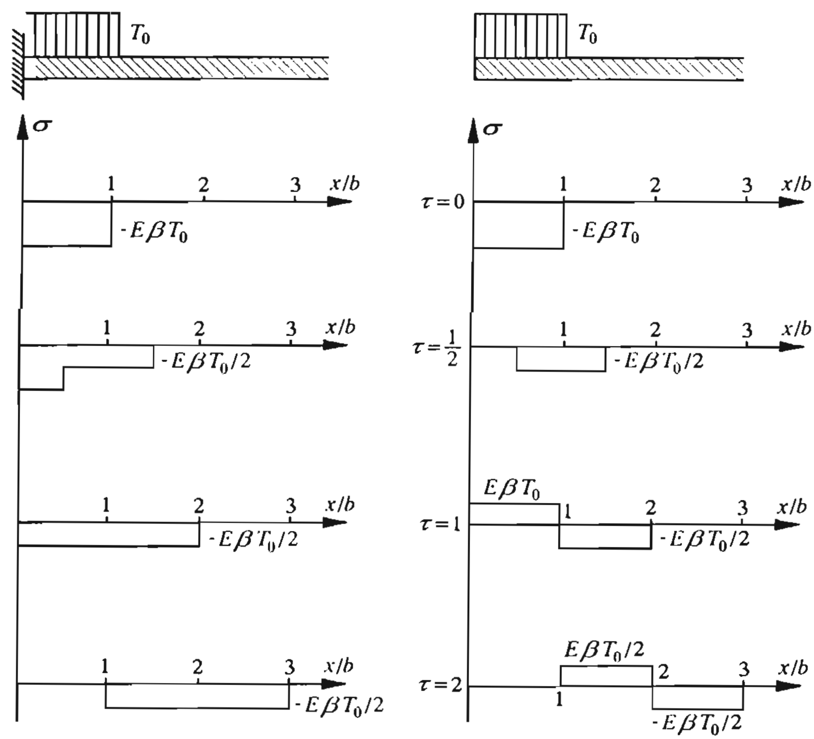


Fig. 3.

2.3. General case

Let us now return to the general solution derived in Section 1 Eq (1.13), concerning Case (A), presents the stress $\sigma(x,\tau)$ as a sum of the static solution (2.2) and the dynamic term involving the integral $(1.14)_2$

$$F_2(x,\tau) = \int\limits_0^\infty \frac{\alpha^2 T_c(\alpha)\cos\Omega\tau\cos\alpha x}{\alpha^2+h^2}\,d\alpha \tag{2.12}$$

with $\Omega = \sqrt{\alpha^2+h^2}$. Consider, for the sake of simplicity, the case when $T(x) = T_0\eta(b-x), T_c = T_0(\sin \alpha b)/\alpha$. Using the trigonometric decomposition formula

$$\sin \alpha b \cos \alpha x = \frac{1}{2}[\sin \alpha(b+x) + \sin \alpha(b-x)] \tag{2.13}$$

Eq (2.12) is written as the sum

$$F_2(x,\tau) = \frac{T_0}{2}\Big[\int\limits_0^\infty \frac{\alpha \cos \Omega\tau \sin \alpha(b+x)\;d\alpha}{\alpha^2+h^2} + \int\limits_0^\infty \frac{\alpha \cos \Omega\tau \sin \alpha(b-x)\;d\alpha}{\alpha^2+h^2}\Big]$$

According to Bateman (1954), p 85, the integral

$$\int\limits_0^\infty \frac{\alpha \cos \Omega\tau \sin \alpha X}{\alpha^2+h^2}\;d\alpha = \frac{\pi}{2}e^{-hX} \tag{2.14}$$

for positive values of $X > \tau$. The result cannot be, however, written in an explicit form for $x < \tau$. Substituting this result into the preceding formula for F_2 we conclude that

$$F_2 = \begin{cases} \frac{T_0\pi}{4}\Big[e^{-h(b+x)} + e^{-h(b-x)}\Big] & \text{for} \quad b > x\;, \quad b-x > \tau \\ \frac{T_0\pi}{4}\Big[e^{-h(x+b)} - e^{-h(x-b)}\Big] & \text{for} \quad b < x\;, \quad x-b > \tau \end{cases} \tag{2.15}$$

Thus, for $0 < x < b-\tau$

$$F_2 = \frac{T_0\pi}{2}e^{-hb}\cosh hx$$

and for $x > b+\tau$

$$F_2 = -\frac{T_0\pi}{2}e^{-hx}\sinh hb$$

For $b-\tau < x < b+\tau$, F_2 cannot be written in an explicit form.

Returning now to Eq (1.13) and using the result (2.4) (static solution corresponding to $T(x) = T_0\eta(b-x)$), we obtain

$$\sigma(x,\tau) = -E\beta T_0 \begin{cases} 1 & \text{for} \quad x < b-\tau \\ 0 & \text{for} \quad x > b+\tau \end{cases} \tag{2.16}$$

In the region $b-\tau < x < b+\tau$, due to the disturbances originating from point $x = b$, explicit formula for stress $\sigma(x,t)$ is not known. However, outside that region the stress preserves its original values and equals $-E\beta T_0$ for $x < b-\tau$

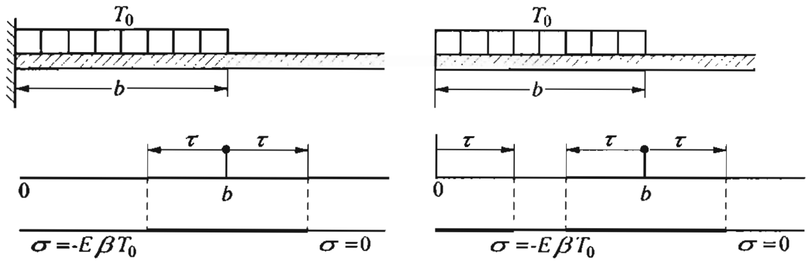


Fig. 4.

and zero for $x > b+\tau$, out of the reach of the waves propagating from $x = b$; at small values of $\tau < b$, the initial stress remains unchanged (cf Fig.4).

A similar approach can be applied to Case (B) of the stress-free end of the rod. Stress (1.16) also constitutes a sum of the static and dynamic solution F_3 and F_4, and integration in Eq $(1.17)_2$ with $T_s(\alpha) = T_0(1-\cos\alpha b)/\alpha$ is possible under the condition that $\tau < x < b-\tau$ and $b+\tau < x$, and leads there to the results

$$\sigma(x,\tau) = -E\beta T_0 \begin{cases} 1 & \text{for } \tau < x < b-\tau \\ 0 & \text{for } b+\tau < x \end{cases}$$

In contrast to the results (2.16), the regions in which the stresses remain unknown includes the neighbourhood of the rod end $x = 0$. The extents of those regions in Cases (A) and (B) are shown in Fig.4.

3. Approximate evaluation of stresses

Let us now try to evaluate the integral (2.12) in an approximate manner. The considerations will be confined to Case (A) and temperature $T(x) = T_0\eta(b-x)$.

Formula (2.12) determining the time-dependent part of the solution is

$$F_2 = T_0 \int_0^\infty \frac{\alpha\cos\Omega\tau\sin\alpha b\cos\alpha x}{\alpha^2+h^2}\,d\alpha \tag{3.1}$$

It is easily verified that, in a certain range of values of the parameter h and variable τ the following approximation of function $\cos\Omega\tau$ is possible

$$\cos\tau\sqrt{\alpha^2+h^2} \approx \cos\alpha\tau - \frac{1}{2}\tau h^2\frac{\sin\alpha\tau}{\alpha} \tag{3.2}$$

The error resulting from such an approximation is small provided both h and τ are small enough; the error is of the order of $\tau^4h^4/24$ for small values of α and decreases at increasing values of α. Thus with $h\tau < 1$, the approximation should prove to be effective in evaluation of the improper integral (3.1).

Let us substitute the approximation (3.2) into the integrand of Eq (3.1). Denoting by $\widetilde{F}_2$ the resulting approximation of F_2, we obtain

$$\widetilde{F}_2 = T_0\Big[\int\limits_0^\infty \frac{\alpha\cos\alpha\tau\sin\alpha b\cos\alpha x}{\alpha^2+h^2}\,d\alpha - \frac{1}{2}\tau h^2\int\limits_0^\infty \frac{\sin\alpha\tau\sin\alpha b\cos\alpha x}{\alpha^2+h^2}\,d\alpha\Big] \tag{3.3}$$

Simple trigonometric transformations lead, at $\tau < b$, to

$$\begin{aligned}\widetilde{F}_2 &= \frac{T_0}{2}\Big[\int\limits_0^\infty \frac{\alpha\sin\alpha(b+\tau)\cos\alpha x}{\alpha^2+h^2}\,d\alpha + \int\limits_0^\infty \frac{\alpha\sin\alpha(b-\tau)\cos\alpha x}{\alpha^2+h^2}\,d\alpha + \\ &- \frac{\tau h^2}{2}\int\limits_0^\infty \frac{\cos\alpha(b-\tau)\cos\alpha x}{\alpha^2+h^2}\,d\alpha + \frac{\tau h^2}{2}\int\limits_0^\infty \frac{\cos\alpha(b+\tau)\cos\alpha x}{\alpha^2+h^2}\,d\alpha\Big]\end{aligned} \tag{3.4}$$

For $\tau > b$, the second right-hand integral in Eq (3.4) must be written in the form

$$-\int\limits_0^\infty \frac{\alpha\sin\alpha(\tau-b)\cos\alpha x}{\alpha^2+h^2}\,d\alpha$$

Applying now the known formulae (cf Bateman (1954), pp 19, 21)

$$\begin{aligned}\int\limits_0^\infty \frac{\alpha\sin\alpha Y\cos\alpha x}{\alpha^2+h^2}\,d\alpha &= \frac{\pi}{2}\begin{cases} e^{-hY}\cosh hx & x < Y \\ -e^{-hx}\sinh hY & x > Y\end{cases} \\ \int\limits_0^\infty \frac{\cos\alpha Y\cos\alpha x}{\alpha^2+h^2}\,d\alpha &= \frac{\pi}{2h}\begin{cases} e^{-hY}\cosh hx & x < Y \\ e^{-hx}\cosh hY & x > Y\end{cases}\end{aligned} \tag{3.5}$$

the approximate formula for the stress $\widetilde{\sigma}(x,\tau)$ assumes the form

$$\widetilde{\sigma}(x,\tau) = \sigma_{st}(x) - \frac{2E\beta}{\pi}\widetilde{F}_2(x,\tau) \tag{3.6}$$

Here σ_{st} is the static solution (2.4) derived in the preceding section. On using the results (3.5) and taking into account the behaviour of the solutions in various regions of variability of x and τ, we obtain the final form of our approximate results.

- **Case I:** $\tau < b$

For $0 < x < b - \tau$

$$\tilde{\sigma} = -E\beta T_0\Big[1 - f(h\tau)\mathrm{e}^{-hb}\cosh hx\Big] \tag{3.7}$$

For $b - \tau < x < b$

$$\tilde{\sigma} = -E\beta T_0\Big[1 - A_1\cosh hx - A_2\mathrm{e}^{-hx}\Big] \tag{3.8}$$

For $b < x < b + \tau$

$$\tilde{\sigma} = -E\beta T_0\Big[A_3\cosh hx - A_4\mathrm{e}^{-hx}\Big] \tag{3.9}$$

For $b + \tau < x < \infty$

$$\tilde{\sigma} = -E\beta T_0 f(h\tau)\sinh hb\mathrm{e}^{-hx} \tag{3.10}$$

- **Case II:** $b < \tau < 2b$

For $0 < x < \tau - b$

$$\tilde{\sigma} = -E\beta T_0\Big[1 - A_5\cosh hx\Big] \tag{3.11}$$

For $\tau - b < x < b$ – Eq (3.8) holds true.

In the remaining intervals Eqs (3.9) and (3.10) remain unchanged.

- **Case III:** $2b < \tau$

For $0 < x < b$ formula (3.11) holds true.

For $b < x < \tau - b$

$$\tilde{\sigma} = -E\beta T_0\sinh hb\Big[\mathrm{e}^{-hx} - \mathrm{e}^{-h\tau}\Big(1 + \frac{h\tau}{2}\Big)\cosh hx\Big] \tag{3.12}$$

For $\tau - b < x < \tau + b$ Eq (3.9) holds true.

For $\tau + b < x$ Eq (3.10) holds true.

In Eqs (3.7)÷(3.12) the following auxiliary notations have been introduced

$$
\begin{aligned}
A_1 &= \Big[1 - \frac{1}{2}\Big(1 + \frac{1}{2}h\tau\Big)e^{-h\tau}\Big]e^{-hb}\\
A_2 &= \frac{1}{2}\Big[\sinh h(b-\tau) + \frac{1}{2}h\tau\cosh h(b-\tau)\Big]\\
A_3 &= \frac{1}{2}\Big(1 + \frac{1}{2}h\tau\Big)e^{-h(b+\tau)}\\
A_4 &= \sinh hb - \frac{1}{2}\sinh h(b-\tau) - \frac{1}{4}h\tau\cosh h(b-\tau)\\
f(h\tau) &= 1 - \cosh h\tau + \frac{1}{2}h\tau\sinh h\tau
\end{aligned}
\tag{3.13}
$$

Simple inspection of Eqs (3.7), (3.10) shows that in regions where, according to Eq (2.16), the stress should be equal to $-E\beta T_0$ and 0, respectively, the approximate formulae yield $-E\beta T_0[1 - f(h\tau)e^{-hb}\cosh hx]$ and $-E\beta T_0 f(h\tau)\sinh hb e^{-hx}$. It is easily verified, however, that function $f(h\tau)$ in Eq $(3.13)_5$ for $h\tau < 1$ is small of the order of $h^4\tau^4/24$; upon multiplication by functions $e^{-hb}\cosh hx$ and $\sinh hb e^{-hx}$, which are both less than unity in the relevant intervals, the results obtained prove to be a good approximation of the accurate ones.

Finally, another interesting conclusion may be drawn from the approximate formulae (3.7)÷(3.13). Points $x = b+\tau$ and $x = b-\tau$ (or $x = \tau-b$) determine the positions of the stress wave fronts. The corresponding stress jumps are calculated by substituting (in Case I, for instance) the value $x = b + \tau$ into the formulae (3.10) and (3.9) and subtracting the results. The other stress jump occurs at $x = b - \tau$ and it is evaluated by subtracting the stresses $\widetilde{\sigma}$ given by Eqs (3.8) and (3.7). In both cases (and also for $\tau > b$, Eqs (3.11), (3.12)) the jumps are exactly equal to $\pm\frac{1}{2}E\beta T_0$, i.e. they are the same as those in the classical solutions shown in Fig.3, Eqs (2.10) and (2.11). A similar phenomenon was also observed by Sokołowski and Wesołowski (1989), (1992), where the value of the displacement jumps at the wave front propagating along the composite rod remained constant throughout the process.

This observation is also confirmed by numerical integration. The results are illustrated in Fig.5 ÷ Fig.7 showing the process of stress wave propagation. The graphs are based on the approximate relations derived in this section and on the results of numerical integration. The diagrams in Fig.5 and Fig.6 (based on the approximate formulae derived above) are drawn in distorted scale to demonstrate better the tendency of variation of the stress pulses, with invariable values of the stress jumps at the wave fronts.

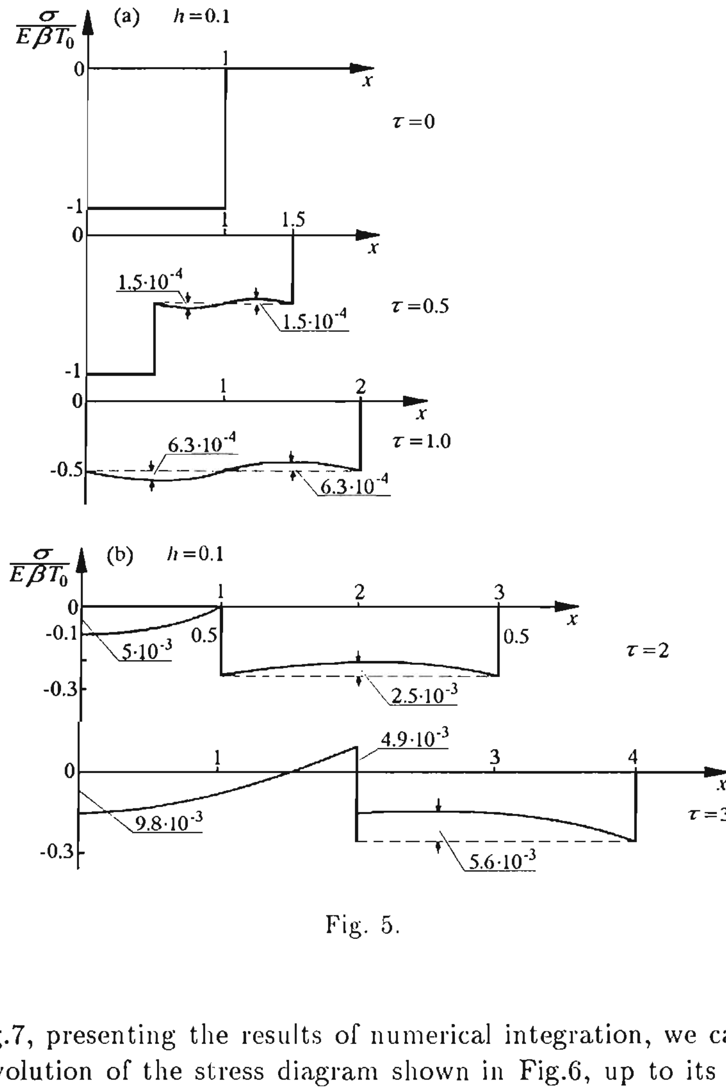
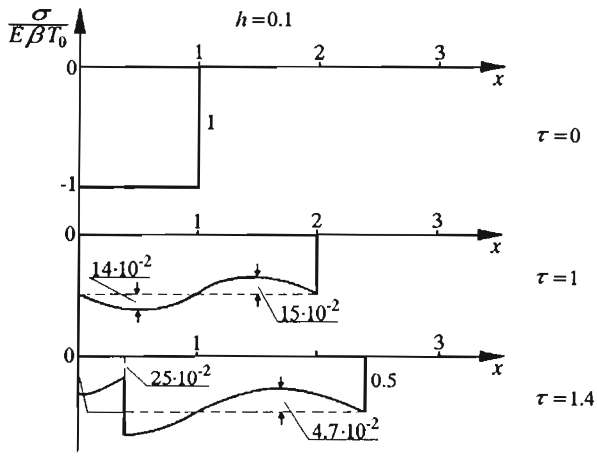


Fig. 5.

In Fig.7, presenting the results of numerical integration, we can observe further evolution of the stress diagram shown in Fig.6, up to its final state identical with the static solution (2.4). The results illustrated in Fig.8 are also based on numerical integration of the formula (1.13). Actually, the diagrams present the stress $\Delta_h\sigma(x,\tau)$ being the difference

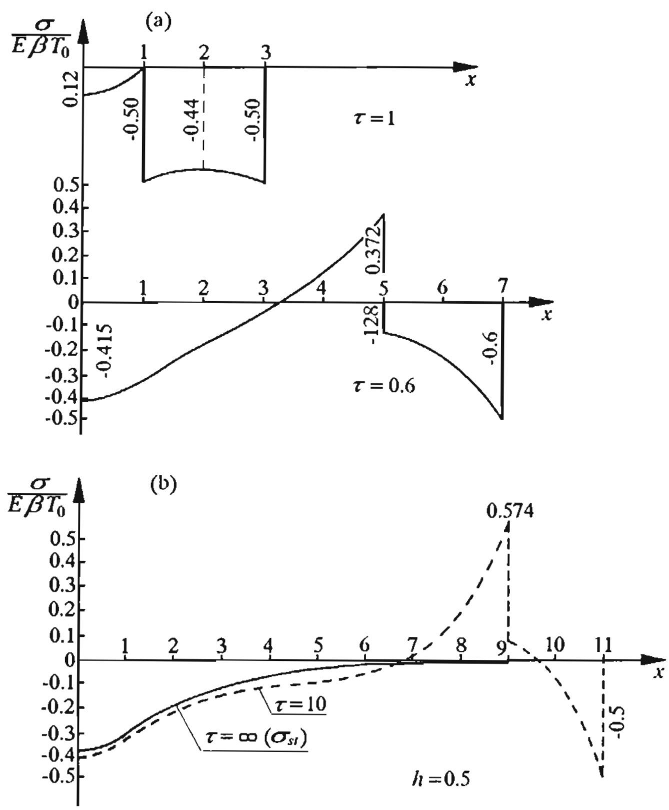
$$\Delta_h\sigma(x,\tau) = \sigma(x,\tau)_{h=5} - \sigma(x,\tau)_{h=0}$$

between the stress (1.13) and that occurring in a foundation-free rod (2.9). Also here the effect of propagation of disturbances from the point $x = b$ (cf Fig.4) may be observed.



$\frac{\sigma}{E\beta T_0}$
$h=0.1$
$\tau=0$
$\tau=1$
$\tau=1.4$
$14\cdot10^{-2}$
$15\cdot10^{-2}$
$25\cdot10^{-2}$
$4.7\cdot10^{-2}$
0.5
x

Fig. 6.



(a)
$\frac{\sigma}{E\beta T_0}$
$\tau=1$
0.12
-0.50
-0.44
-0.50
0.372
-128
-0.415
-0.6
$\tau=0.6$
(b)
0.574
$\tau=10$
$\tau=\infty\ (\sigma_{st})$
-0.5
$h=0.5$
x

Fig. 7.

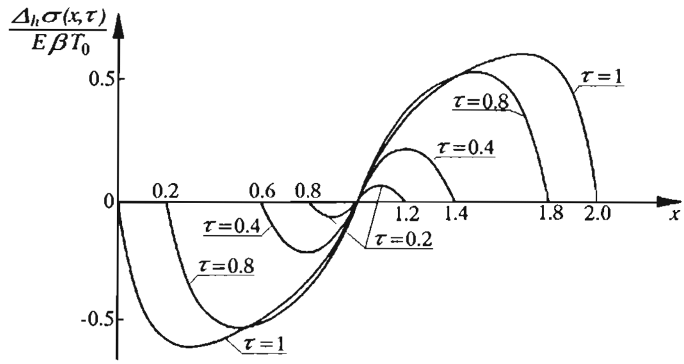


Fig. 8.

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References

1. BATEMAN H., 1954, *Tables of Integral Transforms*, vol. 1, McGraw-Hill Book Co., Inc., New York-Toronto-London
2. MORSE P.M., FESHBACH H., 1953, *Method of Theoretical Physics*, McGraw-Hill Book Co., Inc., New York-Toronto-London
3. NOWACKI W., 1962, *Thermoelasticity*, Pergamon Press – Polish Scientific Publishers, Warszawa
4. NOWACKI W., 1963, *Dynamics of Elastic Systems*, Chapman and Hall Ltd., London
5. SOKOŁOWSKI M., WESOŁOWSKI Z., 1989, Elastic Wave Propagation in a Two-Component Composite Structure, *Arch. Mech.*, **41**, 1
6. SOKOŁOWSKI M., WESOŁOWSKI Z., 1992, Dynamic Effects of Fracture in Two-Component Elastic Rod, *J. Tech. Phys.*, **33**, 3-4

O termicznie wzbudzanych falach naprężenia w pręcie spoczywającym na sprężystym podłożu

Streszczenie

Półnieskończony pręt $x > 0$ jest przymocowany do sprężystego podłoża działającego na pręt siłami proporcjonalnymi do jego przemieszczeń osiowych. Do części pręta przykłada się w chwili $t = 0$ pole temperatury $T(x)$. Wywołane w ten sposób zaburzenie stanu naprężenia rozprzestrzenia się w postaci dwóch fal sprężystych w obu kierunkach pręta; jedna z tych fal ulega odbiciu od końca pręta stosownie do warunków jego podparcia. Rozwiązanie zostało przedstawione w postaci całkowej i zanalizowane w sposób przybliżony w oparciu o metody analityczne (teoria transformacji całkowych) oraz numeryczne.

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