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Fractionalization of Hankel Type Integral Transforms and Their Relevance

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Abstract. In this paper, the fractionalization of certain types of Hankel transforms is suggested. Barut-Girardello type transforms are then introduced along with the relevant fractional order forms. Finally some further generalizations are suggested.

1. Introduction

The theory of Hankel transforms is very vast and it is studied by many researchers in recent as well as in past. The forward and inverse transforms are completely symmetric and resemble the Fourier transform, with the complex exponent as kernal being replaced by the bessel function of first kind $J_{\alpha-\beta}$ of order $\alpha-\beta \geq -\frac{1}{2}$.

The formal equivalence between the zeroth-order Hankel transform and the Abel transform followed by the Fourier transform is used as basis for developing fast algorithms for the computation of the zeroth-order Hankel transform [5]. Algorithms for the computation of the Hankel transform of integer order $n > 0$ have been proposed. On the basis of the general relation involving the Hankel transform of integer order $n > 0$ and the Abel transform, whose kernel is modulated by the Chebyshev polynomial of the first kind of order n, followed by the Fourier sine or cosine transform according to whether n is odd or even [5,13].

It has been evidenced in [18] the formal equivalence between the Hankel transform of order $\alpha-\beta$ and the Erdelyi-Kober fractional integral of order $(\alpha-\beta+\frac{1}{2})$ followed by the Fourier cosine transform, with both acted on function and the resulting transform being modulated by properly $\alpha-\beta$ - dependent power functions of the inherent variables.

Notably such as equivalence suggests a tool for the optical computation of Erdelyi-Kober type fractional integrals of order $(n+\frac{1}{2})$, n integer, through the optical implementation of the Hankel and ID Fourier transforms. In a sense, the Erdelyi-Kober type fractional integrals of order $n+\frac{1}{2}$

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can be regarded as the $(2n+1)$-plane Abe1 transform on $\mathbb{R}^m$, with $0 < 2n+1 < m$.

Various forms of Hankel like integral transforms have been considered in detail in a series of papers [6,7,10–12,22]. We will be concerned here with certain Hankel type transform having relevance among others, in connection with the solution to evolution problems involving the Bessel type differential operators $x^{\alpha+3\beta-1}\left(\frac{\partial}{\partial x}\right)x^{2(\alpha-\beta)+1}\left(\frac{\partial}{\partial x}\right)x^{-3\alpha-\beta}$ and $x^{-3\alpha-\beta}\left(\frac{\partial}{\partial x}\right)x^{2(\alpha-\beta)+1}\left(\frac{\partial}{\partial x}\right)x^{\alpha+3\beta-1}$ with $\alpha-\beta \geq -\frac{1}{2}$ and $-2(\alpha+\beta)$ real parameter [6,11]. We introduced the fractional order forms of such Hankel-type transforms by following the lines of the fractionalization of the conventional Hankel transform [9,16].

Work of Torre [17] motivated us to prepare this paper.

2. Hankel type transforms:

The first and second Hankel-type transforms of Bessel order $\alpha-\beta$, depending on an arbitrary real parameter $-2(\alpha+\beta)$, respectively defined by the operational relations [6,11].

$$\tilde{f}_{1,\alpha,\beta}(y) = \left[h_{1,\alpha-\beta,-2(\alpha+\beta)}f\right](y) = y^{1-4(\alpha+\beta)}\int_0^\infty (xy)^{2(\alpha+\beta)}J_{\alpha-\beta}(xy)f(x)dx \tag{1}$$

$$\tilde{f}_{2,\alpha,\beta}(y) = [h_{2,\alpha-\beta,-2(\alpha+\beta)}f](y) = \int_0^\infty x^{1-4(\alpha+\beta)}(xy)^{2(\alpha+\beta)}J_{\alpha-\beta}(xy)f(x)dx \tag{2}$$

where $J_{\alpha-\beta}$ is the Bessel type function of the first kind and order $(\alpha-\beta) \geq \frac{-1}{2}$. Here $f \in L^2(\mathbb{R}^+)$ – the space of the complex-valued functions which are Lebesgue integrable on $\mathbb{R}^+ = (0,+\infty)$.The transform $[h_{2,\alpha-\beta,-2(\alpha+\beta)}f](y)$ for $\alpha = \frac{-1}{3}\beta$ was originally considered in [15], where the relevant condition for its inversion were established. Also (1) and (2) relate to the Hankel type – Clifford transforms [10].

For suitable values of α,β the Able transforms can be framed within the formalism, developed in [20, 21] concerning the integral transforms associated with complex linear transformations in quantum mechanics, which maps the position and momentum operators to canonically conjugate, but not necessarily Hermitian operator. Thus according to that formalism, the Able transforms can be seen as the radial parts of n-dimentional linear cannonical transformations, specifically representing a $\pi/2$-rotation for each pair of the cannonically conjugate operators in the respective n-component position and momentum operator vectors. Precisely, $n = 4(\alpha+\beta)$ for (1) and $n = 2[1-2(\alpha+\beta)] = 2-4(\alpha+\beta)$ for (2).

The order $\alpha-\beta$ of the Bessel type function relates to the eigen value $\lambda = -l(l+n-2)$, $l = 0,1,2,\ldots$ of the angular momentum; specifically, it turns out that $l = \alpha-\beta-\frac{n}{2}+1$, and so $l = -(\alpha+3\beta-1)$ for (1) and $l = 3\alpha+\beta$ for (2), thus respectively yeilding $\lambda = 3(\alpha^2+\beta^2)+10\alpha\beta-4(\alpha+\beta)+1$ and $\lambda = 3(\alpha^2+\beta^2)+10\alpha\beta$. When $\alpha+\beta = \frac{1}{4}$, $n = 1$ in both cases, and accordingly both transforms yield the conventional Hankel type transform. The symmetry of (1) and (2) reflects into the relation between the respective integral kernels $K_{1,\alpha-\beta,-2(\alpha+\beta)}(x,y)$ and $K_{2,\alpha-\beta,-2(\alpha+\beta)}(y,x)$; i.e. $K_{1,\alpha-\beta,-2(\alpha+\beta)}(x,y) = y^{1-2(\alpha+\beta)}\,x^{2(\alpha+\beta)}\,J_{\alpha-\beta}(xy) = K_{2,\alpha-\beta,-2(\alpha+\beta)}(y,x)$.

As a consequence of well known orthogonality relation of the Bessel functions, both transforms (1) and (2) are self reciprocal.

$$h^{-1}_{1,\alpha-\beta,-2(\alpha+\beta)} = h_{1,\alpha-\beta,-2(\alpha+\beta)}, \qquad h^{-1}_{2,\alpha-\beta,-2(\alpha+\beta)} = h_{2,\alpha-\beta,-2(\alpha+\beta)}. \tag{3}$$

Interestingly, the adjoint operator of $h_{1,\alpha-\beta,-2(\alpha+\beta)}$ is $h_{2,\alpha-\beta,-2(\alpha+\beta)}$ and so

$$h^{*}_{1,\alpha-\beta,-2(\alpha+\beta)} = h_{2,\alpha-\beta,-2(\alpha+\beta)}, \qquad h^{*}_{2,\alpha-\beta,-2(\alpha+\beta)} = h_{1,\alpha-\beta,-2(\alpha+\beta)}. \tag{4}$$

Also one can prove for the operator $h_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $h_{2,\alpha-\beta,-2(\alpha+\beta)}$ the Parsevel equalities [12]

$$\begin{gathered}\int_0^\infty x^{-1+4(\alpha+\beta)}\, f^*(x)\, g(x)dx = \int_0^\infty x^{-1+4(\alpha+\beta)}\, \tilde{f^*}_{1,\alpha-\beta,-2(\alpha+\beta)}\, \tilde{g}_{1,\alpha-\beta,-2(\alpha+\beta)}(x)dx, \\ \int_0^\infty x^{1-4(\alpha+\beta)}\, f^*(x)\, g(x)dx = \int_0^\infty x^{1-4(\alpha+\beta)}\, \tilde{f^*}_{2,\alpha-\beta,-2(\alpha+\beta)}\, \tilde{g}_{2,\alpha-\beta,-2(\alpha+\beta)}(x)dx\end{gathered} \tag{5}$$

both containing a weight function i.e. $x^{-1+4(\alpha+\beta)}$ and $x^{1-4(\alpha+\beta)}$ respectively. A mixed Parsevel relation holds as well, which writes as

$$\int_0^\infty f^*(x)\, g(x)dx = \int_0^\infty \tilde{f^*}_{2,\alpha-\beta,-2(\alpha+\beta)}\, \tilde{g}_{2,\alpha-\beta,-2(\alpha+\beta)}(x)dx. \tag{6}$$

Note that it does not contain any weight function and involves both transforms [12].
Relations (4) and (6) express the complementary of (1) and (2).
For $\alpha+\beta = \frac{1}{4}$, we recover the conventional Hankel type transform of Bessel order $\alpha-\beta$;

$$\hat{h}_{1,\alpha-\beta,\frac{-1}{2}} = \hat{h}_{2,\alpha-\beta,\frac{-1}{2}} \equiv \hat{h}_{\alpha-\beta} \quad with$$

$$\left[\hat{h}_{\alpha-\beta,-2(\alpha+\beta)} f\right](y) \equiv \tilde{f}_{\alpha-\beta}(y) = \int_0^\infty (xy)^{2(\alpha+\beta)} J_{\alpha-\beta}(xy) f(x)dx = \int_0^\infty K_{\alpha,\beta}(x,y) f(x)dx \tag{7}$$

the latter expressing the transform in terms of the kernel $K_{\alpha,\beta}(x,y) = (xy)^{2(\alpha+\beta)}\, J_{\alpha-\beta}(xy) = K_{\alpha,\beta}(y,x)$.
Now we evidence the similarity transformations like structure of the transforms (1) and (2), being indeed

$$\begin{aligned}\left[\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= (y)^{-2(\alpha+\beta)+1/2} \int_0^\infty (x)^{2(\alpha+\beta)-1/2}\, K_{\alpha,\beta}(x,y)\, f(x)dx \\ &= (y)^{-2(\alpha+\beta)+1/2} \left[\hat{h}_{\alpha,\beta}(x)^{2(\alpha+\beta)-1/2} f\right](y) \\ \left[\hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= (y)^{2(\alpha+\beta)-1/2} \int_0^\infty (x)^{-2(\alpha+\beta)+1/2}\, K_{\alpha,\beta}(x,y)\, f(x)dx \\ &= (y)^{2(\alpha+\beta)-1/2} \left[\hat{h}_{\alpha,\beta}(x)^{-2(\alpha+\beta)+1/2} f\right](y)\end{aligned} \tag{8}$$

and so for the kernels $K_{1,\alpha-\beta,-2(\alpha+\beta)}(x,y) = (y)^{-2(\alpha+\beta)+1/2} K_{\alpha,\beta}(x,y)x^{2(\alpha+\beta)-1/2} = K_{2,\alpha-\beta,-2(\alpha+\beta)}(y,x)$. It can be easily verified that [6, 11]

$$\begin{aligned} \left[\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)}\, \hat{B}^{*}{}_{\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= -y^2 \left[\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y) \\ \left[\hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)}\, \hat{B}_{\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= -y^2 \left[\hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y) \end{aligned} \tag{9}$$

where $\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}$ is the Bessel type differential operator

$$\begin{aligned} \hat{B}_{\alpha-\beta,-2(\alpha+\beta)} &= x^{\alpha+3\beta-1}\, D_x x^{2(\alpha-\beta)+1}\, D_x x^{-3\alpha-\beta} \\ &= D_x^2 + [1-4(\alpha+\beta)]\, \frac{1}{x}\, D_x + \frac{(3\alpha+\beta)(\alpha+3\beta)}{x^2} \end{aligned} \tag{10}$$

whose adjoint is then

$$\begin{aligned} \hat{B}^{*}_{\alpha-\beta,-2(\alpha+\beta)} &= x^{-3\alpha-\beta}\, D_x x^{2(\alpha-\beta)+1}\, D_x x^{\alpha+3\beta-1} \\ &= D_x^2 + [4(\alpha+\beta)-1]\, \frac{1}{x}\, D_x + \frac{(\alpha+3\beta-1)(3\alpha+\beta-1)}{x^2} \end{aligned} \tag{11}$$

Notice that for $\alpha+\beta = \frac{1}{4}$ both operators turn into the self-adjoint operator $\hat{B}_{\alpha,\beta}$, being

$$\hat{B}_{\alpha-\beta,-\frac{1}{2}} = \hat{B}^{*}_{\alpha-\beta,-\frac{1}{2}} = D_x^2 + \frac{64\,\alpha\beta+3}{16\,x^2} \equiv \hat{B}_{\alpha-\beta}. \tag{12}$$

From (9) it follows, for instance, that the solution of the differential equation [6, 11].

$$k\, \frac{\partial}{\partial\tau}\, h(x,\tau) = \hat{B}^{*}_{\alpha-\beta,-2(\alpha+\beta)} \tag{13}$$

satisfying the initial condition $h(x,0) = f(x)$ can be written into the transform conjugate y-space as $\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)}(y,\tau) = e^{-\frac{y^2}{k\tau}}\, \hat{f}_{1,\alpha,\beta}(y)$ for any value of the arbitrary constant k. Then transforming back to the x-space, one obtains

$$h(x,\tau) = \frac{k}{2\tau}\, x^{1-4(\alpha+\beta)} \int_0^\infty (xy)^{2(\alpha+\beta)-(\frac{k}{4\tau})(x^2+y^2)}\, I_{\alpha-\beta}\left(\frac{k}{2\tau}xy\right) f(y)dy \tag{14}$$

under the condition that $|arg(\frac{\tau}{k})| \le \frac{\pi}{4}$, which for both τ and k real turns into $\frac{\tau}{k} > 0$.

3. Hankel-type transforms of fractional order:

It is well known that equation (13) has the formal solution $h(x,\tau) = e^{\frac{\tau}{k}}\, \hat{B}^{*}_{\alpha,\beta} f(x)$. Equation (14) yields an explicit functional representation of the exponential operator $e^{b\, \hat{B}^{*}_{\alpha-\beta,-2(\alpha+\beta)}}$:

$$\left[e^{b\, \hat{B}^{*}_{\alpha-\beta,-2(\alpha+\beta)}}\right] f(y) = \frac{1}{2b}\, y^{1-4(\alpha+\beta)} \int_0^\infty (xy)^{2(\alpha+\beta)}\, e^{-(\frac{1}{4b})(x^2+y^2)}\, I_{\alpha-\beta}\left(\frac{xy}{2b}\right) f(x)dx \tag{15}$$

where $I_{\alpha-\beta}$ denotes the modified Bessel function of the first kind of order $\alpha-\beta$: $J_{\alpha-\beta}(ix) = i^{\alpha-\beta}\, I_{\alpha-\beta}(x)$.

In particular, setting $b = \frac{i}{2}$ we obtain a representation of $\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)}$ in the form of a symmetric fractional product of the exponential of the generators of the $su(1,1)$ algebra:

$$\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)} = i^{\alpha-\beta+1} e^{-(\frac{i}{2})x^2} e^{(\frac{i}{2})\, \hat{B}^{*}_{\alpha-\beta,-2(\alpha+\beta)}} e^{-(\frac{i}{2})x^2} \tag{16}$$

In this connection, we may note that the operators

$$\hat{K}_+^{(1)} = \frac{1}{2}x^2, \quad \hat{K}_-^{(1)} = -\frac{1}{2}\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{K}_3^{(1)} = -\frac{i}{2}\left(x\frac{d}{dx} + 2(\alpha+\beta)\right) \tag{17}$$

conjugate a non self-adjoint one variable realization of the $su(1,1)$ algebra generators according to the inherent commutation relations

$$\left[\hat{K}_+^{(1)}, \hat{K}_-^{(1)}\right] = 2i\hat{K}_3^{(1)}, \quad \left[\hat{K}_\pm^{(1)}, \hat{K}_3^{(1)}\right] = \pm i\hat{K}_\pm^{(1)}. \tag{18}$$

Thus (16) can be formally be rewritten in terms of the operators $\hat{K}_+^{(1)}$ and $\hat{K}_-^{(1)}$, and further recast in the single exponential form

$$\hat{h}_{\alpha-\beta,-2(\alpha+\beta)} = i^{\alpha-\beta+1}\, e^{-i\frac{\pi}{2}}\left[\hat{K}_+^{(1)} + \hat{K}_-^{(1)}\right]. \tag{19}$$

This is on account of disentanglement relation for the $su(1,1)$ algebra generators

$$e^{-i\phi\left[\hat{K}_+^{(1)}+\hat{K}_-^{(1)}\right]} = e^{-i\tan(\phi/2)\hat{K}_+^{(1)}}\, e^{-i\sin\phi \hat{K}_-^{(1)}}\, e^{-i\tan(\phi/2)\hat{K}_+^{(1)}} \tag{20}$$

holding for $-\pi < \phi < \pi$. Expressing (19) corresponds to the value $\phi = (\pi/2)$.
Exploiting the integral transform representation (15) of the centred operator in equation (20), we obtain an expression for the operator $e^{-i\phi[\hat{K}_+^{(1)}+\hat{K}_-^{(1)}]}$ in the form of a Hankel -type integral transform. Then writing $\phi = a(\pi/2)$ and multiplying both sides by $e^{i(a\pi/2)(\alpha-\beta+1)}$, one ends up on the L.H.S. with the a^{th} power of the operator $i^{\alpha-\beta+1}e^{-i\frac{\pi}{2}[\hat{K}_+^{(1)}+\hat{K}_-^{(1)}]}$ and corresponding on the R.H.S. with a^{th} power of the first Hankel-type transform, $\hat{H}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$, or the first Hankel-type transform of fractional order a. Accordingly, we can write

$$\begin{aligned}
&\left[\hat{H}^a_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y) \\
&= \frac{e^{i(\alpha-\beta+1)(\phi-\pi/2)}}{\sin\phi}\, x^{1-4(\alpha+\beta)} \int_0^\infty (xy)^{2(\alpha+\beta)}\, e^{\frac{i}{2}\cot\phi[x^2+y^2]}\, J_{\alpha-\beta}\left(\frac{xy}{\sin\phi}\right) f(x)dx \\
&= [e^{i\phi(\alpha-\beta+1)}\, e^{-i\phi\left[\hat{K}_+^{(1)}+\hat{K}_-^{(1)}\right]} f](y) = \tilde{f}^{(a)}_{1,\alpha,\beta}(y)
\end{aligned} \tag{21}$$

where $\phi = a(\pi/2)$.
We can interpret it as the functional representation of the operator associated with the equation

$$\begin{aligned}
&i\frac{\partial}{\partial\tau}h(x,\tau) \\
&= -\frac{1}{2}\left\{\frac{\partial^2}{\partial x^2} - (1-4(\alpha+\beta))\frac{1}{x}\frac{\partial}{\partial x} + [1-2(\alpha+\beta)^2 - (\alpha-\beta)^2]\frac{1}{x^2} - x^2 + 2(\alpha-\beta+1)\right\} h(x,\tau)
\end{aligned} \tag{22}$$

with the relevant initial condition $h(x,0) = f(x)$.The evolution variable is here measured in units of $(\pi/2) : \tau = a(\pi/2)$, and conventionally denoted by ϕ.
we can also develop similar results in relation with the second Hankel-type transform $\hat{H}_{2,\alpha-\beta,-2(\alpha+\beta)}$,the inherent $su(1,1)$ algebra generators being the adjoint of equation (17), namely,

$$\hat{K}_+^{(2)} = \frac{1}{2}x^2, \quad \hat{K}_-^{(2)} = -\frac{1}{2}\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{K}_3^{(2)} = -\frac{i}{2}\left(x\frac{d}{dx} - 2(\alpha+\beta) + 1\right). \tag{23}$$

Now we introduce the second Hankel-type transform of fractional order a, $\hat{h}^{a}_{2,\alpha-\beta,-2(\alpha+\beta)}$ as

$$\begin{aligned}
&\left[\hat{h}^{a}_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y) \\
&= \frac{e^{i(\alpha-\beta+1)(\phi-\pi/2)}}{\sin\phi} \int_0^\infty x^{1-4(\alpha+\beta)}\,(xy)^{2(\alpha+\beta)}\; e^{\frac{i}{2}\cot\phi[x^2+y^2]}\; J_{\alpha-\beta}\left(\frac{xy}{\sin\phi}\right) f(x)dx \qquad (24)\\
&= \left[e^{i\phi(\alpha-\beta+1)}\; e^{-i\phi\left[\hat{K}^{(2)}_{+}+\hat{K}^{(2)}_{-}\right]} f\right](y) = \tilde{f}^{(a)}_{2,\alpha-\beta,-2(\alpha+\beta)}(y), \qquad \phi = a(\pi/2).
\end{aligned}$$

This yields the functional representation of the evolution operator for the equation

$$\begin{aligned}
&i\frac{\partial}{\partial\tau} h(x,\tau) \\
&= -\frac{1}{2}\left\{\frac{\partial^2}{\partial x^2} + (1-2(\alpha+\beta))\frac{1}{x}\frac{\partial}{\partial x} + \left[-2(\alpha+\beta)^2-(\alpha-\beta)^2\right]\frac{1}{x^2} - x^2 + 2(\alpha-\beta+1)\right\} h(x,\tau) \qquad (25)
\end{aligned}$$

with the initial condition $h(x,0) = f(x)$.The evolution variable parameterized as $\tau = a(\pi/2)$. For $\alpha+\beta = \frac{1}{4}$, both equations (21) and (24) yields the expression of the conventional Hankel transform of fractional order, originally introduced by Namia [9] and further investigated in [16, 21].In particular, for $\alpha+\beta = \frac{1}{4}$, equations (17) and (23) turn into the same set of self-adjoint operators

$$\hat{K}_{+} = \frac{1}{2}x^2, \quad \hat{K}_{-} = -\frac{1}{2}\hat{B}_{\alpha-\beta}, \quad \hat{K}_3 = -\frac{i}{2}\left(x\frac{d}{dx}+\frac{1}{2}\right) \qquad (26)$$

which pertain to the conventional Hankel transform (7).Thus we have

$$\hat{h}_{\alpha-\beta} = i^{\alpha-\beta+1} e^{i\left(\frac{\pi}{2}\right)\left[\frac{d^2}{dx^2}-(\alpha-\beta)^2-\frac{(1/4)}{x^2}-\frac{x^2}{2}\right]} \qquad (27)$$

and corresponding for the transform of fractional order a

$$\hat{h}^{a}_{\alpha-\beta} = e^{i\left(\frac{a\pi}{2}\right)(\alpha-\beta+1)} e^{i\left(\frac{a\pi}{2}\right)\left[\frac{d^2}{dx^2}-(\alpha-\beta)^2-\frac{(1/4)}{x^2}-\frac{x^2}{2}\right]} \qquad (28)$$

whose fractional equation is [9, 16, 21]

$$\left[\hat{H}^{a}_{\alpha-\beta} f\right](y) \equiv \tilde{f}^{(a)}_{\alpha-\beta}(y) = \frac{e^{i(\alpha-\beta+1)(\phi-\pi/2)}}{\sin\phi} \int_0^\infty (xy)^{\left(\frac{1}{2}\right)} e^{\frac{i}{2}\cot\phi[x^2+y^2]}\; J_{\alpha-\beta}\left(\frac{xy}{\sin\phi}\right) f(x)dx \qquad (29)$$

since (1) and (2), also $\hat{h}^{a}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^{a}_{2,\alpha-\beta,-2(\alpha+\beta)}$ can be framed for suitable values of α, β, within the formalism of [20, 21], the relevant Canonical transformation being now the rotation by the angle ϕ for each pair of corresponding canonically conjugate position and momentum operators in the relevant n-component operator vectors.

4. Properties of $\hat{h}^{a}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^{a}_{2,\alpha-\beta,-2(\alpha+\beta)}$:

The fractionalization of order a of an integral transform $\hat{T}$ is intended to produce an integral transform $\hat{T}^a$ which satisfy specific properties; more precisely we need that

(1) $\hat{T}^a$ is continuous with respect to the order,i.e. $\hat{T}^b \to \hat{T}^a$ as $b \to a$

(2) $\hat{T}^a$ obeys the semigroup property, so that composing two fractional transform of order a_1 and a_2 yeilds the fractional transform of order $a_1 + a_2$

$$\hat{T}^{a_1}\hat{T}^{a_2} = \hat{T}^{a_2}\hat{T}^{a_1} = \hat{T}^{a_1+a_2},$$

(3) $\hat{T}^a$ reduces to the identity operator with $a = 0$ and to the ordinary transform for $a = 1$; in symbols : $\hat{T}^0 = \hat{1}$ and $\hat{T}^1 = \hat{T}$.

From the procedure we followed to introduced the fractional order transforms $\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$, it is clear that both transforms satisfy the above properties.Thus the additivity properties follows from

$$\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)} = \left[\hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)}\right]^a, \quad \hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)} = \left[\hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)}\right]^a \tag{30}$$

which in turn implies that

$$\hat{h}^0_{1,\alpha-\beta,-2(\alpha+\beta)} = \hat{1}, \quad \hat{h}^0_{2,\alpha-\beta,-2(\alpha+\beta)} = \hat{1}. \tag{31}$$

In our case, the ordinary transform $\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ are recovered for $a = \pm 1$:

$$\begin{aligned} \hat{h}^1_{1,\alpha-\beta,-2(\alpha+\beta)} &= \hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{h}^{-1}_{1,\alpha-\beta,-2(\alpha+\beta)} = \hat{h}_{1,\alpha-\beta,-2(\alpha+\beta)} \\ \hat{h}^1_{2,\alpha-\beta,-2(\alpha+\beta)} &= \hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{h}^{-1}_{2,\alpha-\beta,-2(\alpha+\beta)} = \hat{h}_{2,\alpha-\beta,-2(\alpha+\beta)}. \end{aligned} \tag{32}$$

This confirms to the self-reciprocal property (3) of the ordinary transform. We have

$$\hat{h}^{-a}_{1,\alpha-\beta,-2(\alpha+\beta)} = \left[\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}\right]^{-1}, \quad \hat{h}^{-a}_{2,\alpha-\beta,-2(\alpha+\beta)} = \left[\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}\right]^{-1}. \tag{33}$$

In fact, both transforms are periodic with respect to order parameter a i.e.

$$\hat{h}^{a+2j}_{1,\alpha-\beta,-2(\alpha+\beta)} = \hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{h}^{a+2j}_{2,\alpha-\beta,-2(\alpha+\beta)} = \hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)} \tag{34}$$

so that a can be taken in [-1,1].

Intrestingly, for the adjoint operator, we find the cross-relations:

$$\left[\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}\right]^* = \hat{h}^{-a}_{2,\alpha-\beta,-2(\alpha+\beta)}, \quad \left[\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}\right]^* = \hat{h}^{-a}_{1,\alpha-\beta,-2(\alpha+\beta)} \tag{35}$$

which reproduce (4) for $a = \pm 1$.

Parsevel's equalities separately pertaining to $\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ are easily proceed i.e.

$$\begin{aligned} \int_0^\infty x^{-1+4(\alpha+\beta)} f(x)^* g(x) dx &= \int_0^\infty x^{1+2(\alpha+\beta)} \left[\tilde{f}^{(a)}_{1,\alpha-\beta,-2(\alpha+\beta)}(x)\right]^* \tilde{g}^{(a)}_{1,\alpha-\beta,-2(\alpha+\beta)}(x) dx \\ \int_0^\infty x^{1-2(\alpha+\beta)} f(x)^* g(x) dx &= \int_0^\infty x^{1-4(\alpha+\beta)} \left[\tilde{f}^{(a)}_{2,\alpha-\beta,-2(\alpha+\beta)}(x)\right]^* \tilde{g}^{(a)}_{2,\alpha-\beta,-2(\alpha+\beta)}(x) dx \end{aligned} \tag{36}$$

as well as the mixed Parsevel's relation:

$$\int_0^\infty f(x)^* g(x) dx = \int_0^\infty \left[\tilde{f}^{(a)}_{1,\alpha-\beta,-2(\alpha+\beta)}(x)\right]^* \tilde{g}^{(a)}_{2,\alpha-\beta,-2(\alpha+\beta)}(x) dx. \tag{37}$$

For $\alpha+\beta=\frac{1}{4}$, the above equalities turn into the energy preserving relation of the conventional transform:

$$\int_0^\infty f(x)^* g(x) dx = \int_0^\infty \left[\tilde{f}^{(a)}_{\alpha-\beta}(x)\right]^* \tilde{g}^{(a)}_{\alpha-\beta}(x) dx.$$

Operational rules similar to (9) can be stated for the fractional transforms, involving of course appropriate Bessel-type differential operators. Indeed we see that

$$\begin{aligned} \left[\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)} \;\; \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} f\right](y) &= -\frac{y^2}{\sin^2\phi}\left[\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y), \\ \left[\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)} \;\; \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} f\right](y) &= -\frac{y^2}{\sin^2\phi}\left[\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y), \end{aligned} \tag{38}$$

with

$$\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a} = -2\hat{K}^{(2)}_- - 2\cot^2\phi \hat{K}^{(2)}_+ - 4\cot\phi \hat{K}^{(2)}_3, \tag{39}$$

and in particular

$$\hat{B}_{\alpha-\beta,a} \equiv \hat{B}_{\alpha-\beta,-(\frac{1}{2}),a} = \hat{B}^*_{\alpha-\beta,-(\frac{1}{2}),a} = -2\hat{K}_- - 2\cot^2\phi\hat{K}_+ - 4\cot\phi\hat{K}_3. \tag{40}$$

Relation (38) gives the relevance of the fractional transforms for the solution of differential equations involving the operators $\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}$ and $\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}$ like, for instance, the evolution equation

$$k\frac{\partial}{\partial\tau} h(x,\tau) = \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}\, h(x,\tau), \tag{41}$$

or, more in general, the following

$$k\frac{\partial}{\partial\tau} h(x,\tau) = P(\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a})\, h(x,\tau) \tag{42}$$

involving polynomial function of $\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}$ (or the adjoint involving a polynomial of $\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}$).

Then considering equation (41), we note that the solution turns out to be

$$h(x,\tau) = \left[\hat{h}^{-a}_{1,\alpha-\beta,-2(\alpha+\beta)} \;\; \hat{h}^a_{\alpha-\beta,-2(\alpha+\beta),a}(y,\tau)\right](x,\tau) \tag{43}$$

Before giving details of the expression of $h(x,\tau)$ from the above scheme, let us note that the operators $\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}$ and $\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}$ arise from the adjoint transformation respectively of $\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}$ and $\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}$ through the operator $\hat{K}^{(2)}_+ = \hat{K}^{(1)}_+ = (\frac{x^2}{2})$. In other words:

$$\begin{aligned} \hat{B}_{\alpha-\beta,-2(\alpha+\beta),a} &= e^{-i\cot(\phi)\left(\frac{x^2}{2}\right)}\hat{B}_{\alpha-\beta,-2(\alpha+\beta)} e^{i\cot(\phi)\left(\frac{x^2}{2}\right)} \\ \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} &= e^{-i\cot(\phi)\left(\frac{x^2}{2}\right)}\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)} e^{i\cot(\phi)\left(\frac{x^2}{2}\right)} \end{aligned} \tag{44}$$

which can be recast as

$$\begin{aligned} \hat{B}_{\alpha-\beta,-2(\alpha+\beta),a} &= x^{\alpha+3\beta-1}\hat{D}_a x^{2(\alpha-\beta)+1}\hat{D}_a x^{-3\alpha-\beta} \\ \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} &= x^{-3\alpha-\beta}\hat{D}_a x^{2(\alpha+\beta)+1}\hat{D}_a x^{\alpha+3\beta-1} \end{aligned} \tag{45}$$

where $\hat{D}_a$ being a linear combination of the Heisenberg operators x and $-i\frac{\partial}{\partial x}$ as

$$\hat{D}_a = e^{-i\cot(\phi)\left(\frac{x^2}{2}\right)}\frac{\partial}{\partial x}e^{i\cot(\phi)\left(\frac{x^2}{2}\right)} = \frac{i}{\sin\phi}\left[\cos(\phi)x - i\sin(\phi)\frac{\partial}{\partial x}\right] \tag{46}$$

The exponential operators $e^{b\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}}$ and $e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}}$ arise from the same adjoint transformations of $e^{b\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}}$ and $e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}}$ respectively; viz

$$\begin{aligned} e^{b\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}} &= e^{-i\cot(\phi)\left(\frac{x^2}{2}\right)}e^{b\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}}e^{i\cot(\phi)\left(\frac{x^2}{2}\right)} \\ e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a}} &= e^{-i\cot(\phi)\left(\frac{x^2}{2}\right)}e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}}e^{i\cot(\phi)\left(\frac{x^2}{2}\right)} \end{aligned} \tag{47}$$

which on account of (15) yields an explicit functional expression for both operators.Accordingly, the solution of equation (41) can be written as

$$h(x,\tau) = \frac{k}{2\tau}x^{1-4(\alpha+\beta)}\int_0^\infty (xy)^{2(\alpha+\beta)}e^{-(\frac{k}{4\tau})(x^2+y^2)}e^{-(\frac{i}{2})\cot(\phi)(x^2-y^2)}I_{\alpha-\beta}\left(\frac{k}{2\tau}xy\right)f(y)dy \tag{48}$$

under the same condition specified in connection with equation (14). One can also note that the similarity transformation like link between the operators $\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a}$ and $\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}$ suggests to recover equation(48) from(41) transforming $h(x,\tau)$ to $h(x,\tau) = h(x,\tau)e^{i\cot(\phi)\left(\frac{x^2}{2}\right)}$. In fact the fractional transforms $\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ are linked to $\hat{h}^a_{\alpha-\beta}$ through the same similarity transformation holding between the ordinary transforms; viz.

$$\hat{h}^a_{1,\alpha-\beta,-2(\alpha+\beta)} = x^{-2(\alpha+\beta)+\frac{1}{2}}\ \hat{h}^a_{\alpha-\beta}\ x^{2(\alpha+\beta)-\frac{1}{2}}$$

$$\hat{h}^a_{2,\alpha-\beta,-2(\alpha+\beta)} = x^{2(\alpha+\beta)-\frac{1}{2}}\ \hat{h}^a_{\alpha-\beta}\ x^{-2(\alpha+\beta)+\frac{1}{2}}$$

as a straightforward consequence of the relations

$$\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a} = x^{2(\alpha+\beta)-\frac{1}{2}}\ \hat{B}_{\alpha-\beta,a}\ x^{-2(\alpha+\beta)+\frac{1}{2}}$$

$$\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} = x^{-2(\alpha+\beta)+\frac{1}{2}}\ \hat{B}_{\alpha-\beta,a}\ x^{2(\alpha+\beta)-\frac{1}{2}}.$$

5. Barut-Girardello-type transformations:

The functional expression (15) of $e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}}$ resembles the Barut-Girardello-type transform. The Barut-Girardello-type transform of Bessel order $\alpha-\beta$ is defined by [3].

$$\left[\hat{G}_{\alpha-\beta}f\right](y) = \sqrt{2}\int_0^\infty (xy)^{1/2}e^{-(1/2)(x^2+y^2)}I_{\alpha-\beta}(\sqrt{2}xy)f(x)dx. \tag{49}$$

As a straightforward generalization of the notion of coherent stakes associated with the Heisenberg algebra, such generalized coherent stakes were introduced as eigenstates of the lowering operator of the aforementioned algebra in the relative discrete representations $D^{\pm}(k)$, $k = -1/2, -1, -3/2, \ldots$

Then taking $2b = \frac{1}{\sqrt{2}}$ in equation (15) and multiplying the integrand function by $e^{-(1/2)(x^2+y^2)}e^{(1/2)(x^2+y^2)}$, we end up expression

$$\left[e^{-(1/\sqrt{2})\hat{K}_-^{(1)}}f\right](y) = e^{-(y^2/2)(\sqrt{2}-1)}\sqrt{2}\,y^{1-4(\alpha+\beta)}\int_0^\infty (xy)^{2(\alpha+\beta)}\,e^{-(1/2)(x^2+y^2)}$$
$$I_{\alpha-\beta}(\sqrt{2}xy)\,e^{-(x^2/2)(\sqrt{2}-1)}f(x)dx.$$

This allows us to define the first Barut-Girardello-type transform of Bessel order $\alpha-\beta$, depending on a real parameter $-2(\alpha+\beta)$, through the expression

$$\left[\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}f\right](y) = \sqrt{2}\,y^{1-4(\alpha+\beta)}\int_0^\infty (xy)^{2(\alpha+\beta)}\,e^{-(1/2)(x^2+y^2)}\,I_{\alpha-\beta}(\sqrt{2}xy)f(x)dx. \quad (50)$$

Thus we may write

$$\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)} = e^{(\sqrt{2}-1)\hat{K}_+^{(1)}}\,e^{\hat{K}_-^{(1)}/\sqrt{2}}\,e^{(\sqrt{2}-1)\hat{K}_+^{(1)}}$$

which can eventually be imposed into the single exponential form:

$$\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)} = e^{(\pi/4)\left[\hat{K}_+^{(1)}-\hat{K}_-^{(1)}\right]}. \quad (51)$$

It clearly states that $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$ can be regarded as the evolution operator $e^{-i\tau\hat{H}}$, associated with the dynamical problem ruled by the Hamiltonian operator

$$\hat{H} = \hat{K}_+^{(1)} - \hat{K}_-^{(1)} = \frac{1}{2}\left[x^2 + \hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}\right]$$

and evaluated at the purely imaginary value $\tau = i(\pi/4)$ of the evolution variable.

Now we can define the second Barut-Girardello-type transform of Bessel order $\alpha-\beta$ as

$$\left[\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}f\right](y) = \sqrt{2}\int_0^\infty x^{1-4(\alpha+\beta)}\,(xy)^{2(\alpha+\beta)}\,e^{-(1/2)(x^2+y^2)}I_{\alpha-\beta}(\sqrt{2}xy)f(x)dx \quad (52)$$

for which the following operational relatoins can be stated as:

$$\begin{aligned}\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)} &= e^{(\sqrt{2}-1)\hat{K}_+^{(2)}}\,e^{\hat{K}_-^{(2)}/\sqrt{2}}\,e^{(\sqrt{2}-1)\hat{K}_+^{(2)}} \\ &= e^{(\pi/4)\left[\hat{K}_+^{(2)}-\hat{K}_-^{(2)}\right]}, \quad (53)\end{aligned}$$

involving of course the operators (23). Accordingly, $\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}$ can be interpreted as the evolution operator operator $e^{-i\tau\hat{H}}$, associated with the dynamical problem ruled by the Hamiltonian operator

$$\hat{H} = \hat{K}_+^{(2)} - \hat{K}_-^{(2)} = \frac{1}{2}\left[x^2 + \hat{B}_{\alpha-\beta,-2(\alpha+\beta)}\right]$$

and evaluated at the same complex value $\tau = i(\pi/4)$ of the evolution variable as $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$.

Both definitions (50) and(52) can be recast into the comprehensive expression

$$\left[\hat{\mathcal{G}}_{j,\alpha-\beta,-2(\alpha+\beta)}f\right](y) = \int_0^\infty K^{(BG)}_{j,\alpha-\beta,-2(\alpha+\beta)}(x,y)f(x)dx, \qquad j = 1,2. \quad (54)$$

in terms of the kernels

$$K^{(BG)}_{1,\alpha-\beta,-2(\alpha+\beta)}(x,y) = \sqrt{2}y^{1-2(\alpha+\beta)}(x)^{2(\alpha+\beta)}I_{\alpha-\beta}(\sqrt{2}xy)e^{-(1/2)(x^2+y^2)} = K^{(BG)}_{2,\alpha-\beta,-2(\alpha+\beta)}(y,x)$$

which relate to the kernel $K^{(BG)}_{\alpha-\beta}(x,y) = \sqrt{2} I_{\alpha-\beta}(\sqrt{2}xy) e^{-(1/2)(x^2+y^2)}$ of the conventional transform (49) through the same similarity transformation like relation holding between $K_{1,\alpha-\beta,-2(\alpha+\beta)}(x,y)$, $K_{2,\alpha-\beta,-2(\alpha+\beta)}(y,x)$ and $K_{\alpha-\beta}(x,y)$.

As the Hankel-type transforms, the transforms $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}$ are adjoint to each other:

$$\hat{\mathcal{G}}^*_{1,\alpha-\beta,-2(\alpha+\beta)} = \hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}, \quad \hat{\mathcal{G}}^*_{2,\alpha-\beta,-2(\alpha+\beta)} = \hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}. \tag{55}$$

However, they are not self reciprocal; the respective inverse transforms can be easily obtained from the corresponding factored representation in equations (51) and (53) which yield

$$\begin{aligned}\left[\hat{\mathcal{G}}^{-1}_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= (-1)^{\alpha-\beta+1} \sqrt{2}\, y^{1-4(\alpha+\beta)} \int_0^\infty (xy)^{2(\alpha+\beta)}\, e^{(1/2)(x^2+y^2)}\, I_{\alpha-\beta}(\sqrt{2}xy) f(x) dx \\ &= \left\{\left[\hat{\mathcal{G}}^{-1}_{2,\alpha-\beta,-2(\alpha+\beta)}\right]^* f\right\}(y).\end{aligned}$$

Operational relations similar to (9) can be deduced for $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}$. In fact:

$$\begin{aligned}\left[\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}\ \hat{I}^*_{\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= 2y^2\ \left[\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y), \\ \left[\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}\ \hat{I}_{\alpha-\beta,-2(\alpha+\beta)} f\right](y) &= 2y^2\ \left[\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y),\end{aligned} \tag{56}$$

with the differential operator $\hat{I}_{\alpha-\beta,-2(\alpha+\beta)}$ being

$$\hat{I}_{\alpha-\beta,-2(\alpha+\beta)} = 2\left[\hat{K}^{(2)}_+ - \hat{K}^{(2)}_- + 2i\hat{K}^{(2)}_3\right]. \tag{57}$$

It can be easily seen that

$$\begin{aligned}\hat{I}_{\alpha-\beta,-2(\alpha+\beta)} = e^{-(x^2/2)}\ \hat{B}_{\alpha-\beta,-2(\alpha+\beta)}\ e^{-(x^2/2)} &= x^{2(\alpha+\beta)-(\alpha-\beta)-1} \\ &\times \left(x + \frac{\partial}{\partial x}\right)\ x^{2(\alpha-\beta)+1} \left(x + \frac{\partial}{\partial x}\right)\ x^{-(3\alpha+\beta)}\end{aligned} \tag{58}$$

Even though equation (56) correspond to equation (9) , pertaining to the Hankel transform, involve operators $\hat{I}_{\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{I}^*_{\alpha-\beta,-2(\alpha+\beta)}$ comprise also the operators $\hat{K}_+$ and $\hat{K}_3$ of the corresponding algebras.

6. Barut-Girardello-type transforms of fractional order:

We may introduce fractional order versions of the transforms $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}$. Let us consider, the disentanglement relation for the $su(1,1)$ algebra generators

$$e^{\zeta\,[\hat{K}_+-\hat{K}_-]} = e^{\tan(\zeta/2)\,\hat{K}_+}\ e^{-\sin(\zeta)\,\hat{K}_-}\ e^{\tan(\zeta/2)\,\hat{K}_+}, \tag{59}$$

holding for $-\pi < \zeta < \pi$. The expressions above obtained for $\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}$ correspond to value $\zeta = (\pi/4)$ with appropriate set of operations (21) and (23) being respectively involved.

Let us refer in particular to the operators (17) so that on account of (15) one ends up with

$$\left[e^{\zeta\,[\hat{K}_+^{(1)}-\hat{K}_-^{(1)}]}f\right]=\frac{1}{\sin\zeta}\,y^{1-4(\alpha+\beta)}\int_0^\infty(xy)^{2(\alpha+\beta)}\,e^{-(1/2)\cot(\zeta)(x^2+y^2)}\,I_{\alpha-\beta}\left(\frac{xy}{\sin\zeta}\right)f(x)dx$$

Then,writing $\zeta=(a\pi/4)$, one can obtains the a^{th} power of (51), with the first Barut-Girardello-type transforms of fractional order a being accordingly defined by the functional expression:

$$\left[\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)}f\right](y)=\frac{1}{\sin(\phi/2)}\,y^{1-4(\alpha+\beta)}\int_0^\infty(xy)^{2(\alpha+\beta)}\,e^{-\cot(\phi/2)(x^2+y^2)}\,I_{\alpha-\beta}\left(\frac{xy}{\sin(\phi/2)}\right)f(x)dx \tag{60}$$

with $\phi=(a\pi/2)$, as before

The second Barut-Girardello-type transforms of fractional order a is similarly introduced through

$$e^{(a\pi/4)\left[\hat{K}_+^{(2)}-\hat{K}_-^{(2)}\right]}=\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}, \tag{61}$$

the relevant functional expression being then:

$$\left[\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}f\right](y)=\frac{1}{\sin(\phi/2)}\int_0^\infty x^{1-2(\alpha+\beta)}(xy)^{2(\alpha+\beta)}\,e^{-(1/2)\cot(\phi/2)(x^2+y^2)}\,I_{\alpha-\beta}\left(\frac{xy}{\sin(\phi/2)}\right)f(x)dx. \tag{62}$$

The ordinary transforms are recovered, of course with $a=1$, while for $\alpha+\beta=\frac{1}{4}$, we obtain the conventional Barut-Girardello-type transforms of fractional order a, $\hat{\mathcal{G}}^a_{\alpha-\beta}$, introduced in [16]. $\hat{\mathcal{G}}^a_{1,\alpha-\beta,1/4}=\hat{\mathcal{G}}^a_{2,\alpha-\beta,1/4}\equiv\hat{\mathcal{G}}^a_{\alpha-\beta}$, with

$$\left[\hat{\mathcal{G}}^a_{\alpha-\beta}f\right](y)=\frac{1}{\sin(\phi/2)}\int_0^\infty\sqrt{xy}\,e^{-(1/2)\cot(\phi/2)(x^2+y^2)}\,I_{\alpha-\beta}\left(\frac{xy}{\sin(\phi/2)}\right)f(x)dx. \tag{63}$$

The fractional transforms $\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ are cyclic with respect to order a, being

$$\hat{\mathcal{G}}^{a+8j}_{1,\alpha-\beta,-2(\alpha+\beta)}=\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)},\qquad\hat{\mathcal{G}}^{a+8j}_{2,\alpha-\beta,-2(\alpha+\beta)}=\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}, \tag{64}$$

which allows us to limit the values of a to the interval $a\in[-4,4]$.The operational relations (55) can be generalized to $\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ for which we obtain

$$\begin{aligned}\left[\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)}\,\hat{I}^*_{\alpha-\beta,-2(\alpha+\beta),a}f\right](y)&=\frac{y^2}{\sin^2(\phi/2)}\,\left[\hat{\mathcal{G}}_{1,\alpha-\beta,-2(\alpha+\beta)}f\right](y),\\ \left[\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}\,\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a}f\right](y)&=\frac{y^2}{\sin^2(\phi/2)}\,\left[\hat{\mathcal{G}}_{2,\alpha-\beta,-2(\alpha+\beta)}f\right](y).\end{aligned} \tag{65}$$

The differential operator $\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a}$ is given by

$$\begin{aligned}\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a}&=2\cot^2(\phi/2)\,\hat{K}_+^{(2)}-\hat{K}_-^{(2)}+4i\cot(\phi/2)\,\hat{K}_3^{(2)}\\&=e^{[1-\cot(\phi/2)](x^2/2)}\,\hat{I}_{\alpha-\beta,-2(\alpha+\beta)}\,e^{-[1-\cot(\phi/2)](x^2/2)}.\end{aligned} \tag{66}$$

By using equation (57) we have

$$\begin{aligned}\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a} &= e^{-\cot(\phi/2)(x^2/2)}\ \hat{B}_{\alpha-\beta,-2(\alpha+\beta)}\ e^{\cot(\phi/2)(x^2/2)} \\ &= e^{2(\alpha+\beta)-(\alpha-\beta)-1}\hat{I}_a x^{2(\alpha-\beta)+1}\hat{I}_a x^{-2(\alpha+\beta)-(\alpha-\beta)} \qquad (67)\end{aligned}$$

with

$$\hat{I}_a = e^{-\cot(\phi/2)(x^2/2)}\ \frac{\partial}{\partial x} e^{\cot(\phi/2)(x^2/2)} = \frac{1}{\sin^2(\phi/2)}\left[\cos(\phi/2)x + \sin(\phi/2)\frac{\partial}{\partial x}\right]. \qquad (68)$$

Therefore, $\hat{\mathcal{G}}^a_{1,\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{\mathcal{G}}^a_{2,\alpha-\beta,-2(\alpha+\beta)}$ are of relevance in connection with evolution equations like

$$\begin{aligned}k\,\frac{\partial}{\partial\tau}\,h(x,\tau) &= P(\hat{I}^*_{\alpha-\beta,-2(\alpha+\beta),a})\ h(x,\tau), or \\ k\,\frac{\partial}{\partial\tau}\,h(x,\tau) &= P(\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a})\ h(x,\tau)\end{aligned}$$

involving polynomial function of $\hat{I}_{\alpha-\beta,-2(\alpha+\beta),a}$ and $\hat{I}^*_{\alpha-\beta,-2(\alpha+\beta),a}$ respectively.

7. Generalized Hankel transforms:

The $\mathcal{H}$ and $\mathcal{G}$ transform discussed above are associated with Hamiltonian operators involving a linear combination of the generators $\hat{K}_+$ and $\hat{K}_-$ of the relevant $su(1,1)$ algebra realizations in a form that naturally suggests an arbitrary respectively with the attractive and repulsive radial quantum mechanics oscillator.

The dynamical symmetry of the linear quantum mechanical oscillator is that of the $su(1,1)$ algebra, whose generator are defined in terms of the position and momentum operators are defined in terms of the position and momentum operators $\hat{x}$ and $\hat{p} = i\left(\frac{d}{dx}\right)$ $(h = 1)$ through the self-adjoint quadrantic expressions

$$\begin{aligned}\hat{K}_+ &= \frac{1}{2}\,\hat{x}^2 = \frac{1}{2}\,x^2, \qquad \hat{K}_- = -\frac{i}{2}\,\hat{p}^2 = -\frac{1}{2}\,\frac{d^2}{dx^2}, \\ \hat{K}_3 &= \frac{1}{4}\,(\hat{x}\hat{p} + \hat{p}\hat{x}) = -\frac{i}{2}\,\left(x\frac{d}{dx} + \frac{1}{2}\right).\end{aligned}$$

Thus the conventional Hankel Transform of any order a, being associated with the sum operator $\hat{K}_1 = \hat{K}_+ + \hat{K}_-$, turns out to be linked to the dynamics of the alternative radial oscillator, the relevant $\hat{K}_-$ generator (12) being the radical part of the 2D Laplacian operator. In fact, as noted earlier, the Hankel transform of integer Bessel order can be regarded as the radial part of the 2D Fourier transform of rotationally symmetric function, when polar co-ordinates are adopted.In other words we have

$$\left[\hat{\mathcal{F}}^a\ f(\zeta,\eta)\right](x,y) = e^{-im\phi}\ e^{-im\theta}\ \left[\hat{\mathcal{H}}^a_m\ \rho^{1/2}\ g(\rho)\right](r), \qquad m = 0,1,2,\ldots \qquad (69)$$

where (ρ,ϕ) and (r,θ) are polar co-ordinates respectively in the function and transform domain and f is a rotationally symmetric function: $f(\zeta,\eta) = g(\rho)\ e^{im\phi}$.

Evidently (69) can be generalised to the transform (21) and (24) as

$$\left[\hat{\mathcal{F}}^a\, f(\zeta,\eta)\right](x,y) = e^{-im\phi}\, e^{-im\theta}\, r^{-1+2(\alpha+\beta)}\, \left[\hat{\mathcal{H}}^a_{1,m}\, \rho^{1-2(\alpha+\beta)}\, g(\rho)\right](r),$$

$$\left[\hat{\mathcal{F}}^a\, f(\zeta,\eta)\right](x,y) = e^{-im\phi}\, e^{-im\theta}\, r^{-2(\alpha+\beta)}\, \left[\hat{\mathcal{H}}^a_{2,m,-2(\alpha+\beta)}\, \rho^{2(\alpha+\beta)}\, g(\rho)\right](r)$$

for non-negative integers m and rotationally symmetric functions.

Likewise, the Barut-Girardello transform resorting to the difference operator $\hat{K}_2 = \hat{K}_+ + \hat{K}_-$, can be associated with dynamics of the repulsive radical oscillator.Therefore Barut-Girardello transform of integer Bessel order can be regarded as the radial part of the 2D-Bargman transform $\hat{B}^a$ [2, 16, 20], the inherent relation being similar to (69) i.e.

$$\left[\hat{B}^a\, f(\zeta,\eta)\right](x,y) = e^{-im\theta}\, r^{-1/2}\, \left[\hat{\mathcal{G}}^a_m\, \rho^{1/2}\, g(\rho)\right](r), \qquad m = 0, 1, 2, \ldots \tag{70}$$

with the same meaning of the symbols as in equation (69).

The generalization of (70) to the transforms of the first and second type is obvious.

Following the correspondence of the Hankel to the Fourier transform, we may introduce a generalized fractional Hankel transforms as the operator associated with evolution equation driven by a generic operator belonging to the $su(1,1)$ algebra namely

$$\hat{H}^{(1,2)} = a\hat{K}_+^{(1,2)} + b\hat{K}_-^{(1,2)} + c\hat{K}_3^{(1,2)} + d(v+1)\hat{1}$$

being its pertinent to the algebra realization (17) or (23).

We exploit the disentanglement scheme

$$e^{-i\tau[a\hat{K}_+^{(1,2)} + b\hat{K}_-^{(1,2)} + c\hat{K}_3^{(1,2)} + d(v+1)\hat{1}]} = e^{-id\tau(\alpha-\beta+1)}\, e^{A\hat{K}_+^{(1,2)}}\, e^{C\hat{K}_3^{(1,2)}}\, e^{-i\phi\hat{K}_1^{(1,2)}},$$

giving the operator $e^{-i\tau\hat{H}^{(1,2)}}$ in three-term factored form, apart from the phase factor $e^{-id\tau(v+1)}$.

We may introduce the generalized Hankel-type transforms of first and second type ,depending on the parameter $\mathcal{P}$, m and γ,

$$\left[\hat{\mathcal{H}}^{a,\mathcal{P},m,\gamma}_{1,\alpha-\beta,-2(\alpha+\beta)} f\right](y) = \frac{e^{i(\alpha-\beta+1)(\gamma-\pi/2)}}{m\sin(\phi)} e^{-i\mathcal{P}(y^2/2)}\, y^{1-4(\alpha+\beta)} \int_0^\infty (xy)^{2(\alpha+\beta)} \times e^{(i/2)\cot(\phi)(x^2+(y^2/m^2))}\, J_{\alpha-\beta}\left(\frac{xy}{m\sin(\phi/2)}\right) f(x)dx \tag{71}$$

$$\left[\hat{\mathcal{H}}^{a,\mathcal{P},m,\gamma}_{2,\alpha-\beta,-2(\alpha+\beta)} f\right](y) = \frac{e^{i(\alpha-\beta+1)(\gamma-\pi/2)}}{m\sin(\phi)}\, e^{-i\mathcal{P}(y^2/2)} \int_0^\infty x^{1-4(\alpha+\beta)}\, (xy)^{2(\alpha+\beta)} \times e^{(i/2)\cot(\phi)(x^2+(y^2/m^2))}\, J_{\alpha-\beta}\left(\frac{xy}{m\sin(\phi/2)}\right) f(x)dx.$$

The above relations reproduce (21) and (24) respectively, for $b = a$, $d = -a$, $c = 0$ and $a\tau = \phi$; also relations like (9) can be deduced for the generalized transform.

In addition , generalized Borut-Girardello type transforms can be introduced on the basis of a

disentanglement scheme involving the operators $\hat{K}_2^{(1,2)}$ instead of $\hat{K}_1^{(1,2)}$.
As a conclusion, we note that from equation (71) we may recover with $\mathcal{P} = ib/m^2, m^2 = 1 - b^2, \tan(\phi) = -ib$ and $\gamma = 0$, the integral transformations corresponding to the exponential forms $e^{b\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}}$ and $e^{b\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}}$ (see (15) and the relevant adjoint expression, easily deducible) which can respectively be regarded as the first and second Weiestrass-Gauss integral transforms of Bessel order $\alpha-\beta$ depending on the real parameter $-2(\alpha+\beta)$. They generalize to $\alpha+\beta = 1/4$ the expression of the radical Weiestrass-Gauss integral transform, for which we have [16]

$$\left[\hat{W}_{\alpha-\beta,b} f\right](y) = \left[e^{-(b/2)\hat{B}_{\alpha-\beta}} f\right](y) = \frac{1}{b}\int_0^\infty (xy)^{(1/2)}\, e^{-(1/2b)(x^2+y^2)}\, I_{\alpha-\beta}\left(\frac{xy}{b}\right) f(x)dx$$

for any real parameter $\beta > 0$.
$\hat{W}_{\alpha-\beta,b}$ arises as the transfer operator associated with the radial part of heat conduction like equations.It can be in fact considered as the radial part of 2D Fresnel transform for real parameters is the optical operator for free propagation. A linear Weiestrass-Gauss integral transform has also been introduced [19] the relation of $\hat{W}_{m,b},\ m = 0, 1, 2, \dots$ to it is evidently similar to (69), when rationally symmetric function are involved.

Remark. *[(i)]*

(1) *If we take $\alpha = \frac{\nu}{2} - \frac{\mu}{4},\ \beta = -\frac{\mu}{4} - \frac{\nu}{2}$ throught this paper then all the results studied in this paper reduce to the results studied in Torre [17].*
(2) *Authors claim that results of this paper are stronger than that of Torre [17].*

8. Conclusions:

Following the scheme already applied to other type of transforms, like for instance, the Fourier transform , we have introduced the fractional forms of two adjoint self-reciprocal variants of the Hankel type transform, which, as noted are of interest in connection with evolution problems ruled by the Bessel-type differential operators,

$$\hat{B}_{\alpha-\beta,-2(\alpha+\beta)} = x^{\alpha+3\beta-1} D_x x^{2(\alpha-\beta)+1}\, D_x x^{-3\alpha-\beta}$$

and

$$\hat{B}_{\alpha-\beta,-2(\alpha+\beta)} = x^{-3\alpha-\beta} D_x x^{2(\alpha-\beta)+1} D_x x^{\alpha+3\beta-1}.$$

The fractional order transform relate to evolution problems ruled the operators

$$\hat{B}_{\alpha-\beta,-2(\alpha+\beta),a} = x^{\alpha+3\beta-1}\, \mathcal{L}_a\left(x, \left(\frac{\partial}{\partial x}\right)\right) x^{2(\alpha-\beta)+1}\, \mathcal{L}_a\left(x, \left(\frac{\partial}{\partial x}\right)\right) x^{-3\alpha-\beta}$$

and

$$\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta),a} = x^{-3\alpha-\beta}\, \mathcal{L}_a\left(x, \left(\frac{\partial}{\partial x}\right)\right) x^{2(\alpha-\beta)+1}\, \mathcal{L}_a\left(x, \left(\frac{\partial}{\partial x}\right)\right)\, x^{\alpha+3\beta-1}$$

where $\mathcal{L}_a\left(x, \left(\frac{\partial}{\partial x}\right)\right)$ is linear a-depending combination of x and $\left(\frac{\partial}{\partial x}\right)$. Since a ranges from -1 to 1, it is evident that the set of evolution problems inherent in the transforms has been greatly enlarge by the fractionalization.

In general we have shown that introduced transforms can be regarded as the evolution operators associated with evolution problems having an underlying $su(1,1)$ symmetry, the specific realization of the algebra resorting to $\hat{B}_{\alpha-\beta,-2(\alpha+\beta)}$ and $\hat{B}^*_{\alpha-\beta,-2(\alpha+\beta)}$ as the relative ladder operators.

Evidently, transforms of complex fractional order can be considered although of course the space of functions on which they can meaningfully be applied must be carefully investigated. Disregarding here this aspect of the question, we simply note that due to the additivity with respect to the order, we may write

$$\hat{\mathcal{H}}^{a_R+ia_I}_{j,\alpha-\beta,-2(\alpha+\beta)} = \hat{\mathcal{H}}^{a_R}_{j,\alpha-\beta,-2(\alpha+\beta)} + \hat{\mathcal{H}}^{ia_I}_{j,\alpha-\beta,-2(\alpha+\beta)}, \quad j = 1,2.$$

where a_R and a_I respectively denote the real and imaginary part of the order $a = a_R + ia_I$.Thus $\hat{\mathcal{H}}^{a_R}_{j,\alpha-\beta,-2(\alpha+\beta)}$, $j = 1,2$ have just the expression considered in this paper, while $\hat{\mathcal{H}}^{ia_I}_{j,\alpha-\beta,-2(\alpha+\beta)}$, $j = 1,2$ are from then easily deducible replacing ϕ with $i\phi$.

As earlier mentioned, the Hankel transform is of interest within the context of the fractional calculus [17]. Following the arguments in [18], for the transforms of our concern we find that

$$\begin{aligned} \left[\hat{\mathcal{H}}_{1,\alpha-\beta,-2(\alpha+\beta)}f\right](y) &= \frac{2^{1-(\alpha-\beta)}\, x^{-\alpha-3\beta+1}}{\sqrt{\pi}} \int_0^\infty \cos(y\tau) \left[\hat{\mathcal{K}}^{\alpha-\beta+1/2}\, g_1\right](\tau) d\tau, \\ \left[\hat{\mathcal{H}}_{2,\alpha-\beta,-2(\alpha+\beta)}f\right](y) &= \frac{2^{1-(\alpha-\beta)}\, x^{-\alpha-3\beta+1}}{\sqrt{\pi}} \int_0^\infty \cos(y\tau) \left[\hat{\mathcal{K}}^{\alpha-\beta+1/2}\, g_2\right](\tau) d\tau \end{aligned} \tag{72}$$

where $\hat{\mathcal{K}}^{\alpha-\beta+1/2}$ is the left hand sided Erdelyi-Kober fractional integral operator, represented by

$$\left[\hat{\mathcal{K}}^b g\right](y) = \frac{1}{\Gamma(b)} \int_y^\infty (y^2 - x^2)^{b-1}\, x\, g(x)\, dx, \quad R(b) > 0,\ y \in \mathbb{R},$$

and the functions $g_1(x)$ and $g_2(x)$ on which it acts in (72) involve f as $g_1(x) = x^{\alpha+3\beta-1} f(x)$, $g_2(x) = x^{-3\alpha-\beta} f(x)$.

Relations (72) holds under the assumption that both $xg_1(x)$ and $xg_2(x)$ are inegrable. Note that according to Sonine's first integral for Bessel functions, we may say that the right hand sided Erdelyi-Kober operator of order b acts on the function $x^{\alpha-\beta}\, J_{\alpha-\beta}(x)$ as a rising operator turning it into $x^{\alpha-\beta+b}\, J_{\alpha-\beta+b}(x)$.

Expressions similar to (72) can be deduced for the fractional order transforms, of course.

In addition (72) as paralleled by similar expressions involving the Barut-Girardello transform of the first and second type. As an example, we deduce here the relation involving the conventional transform (49). On account of the integral representation of the modified Bessel function of the first kind, $I_{\alpha-\beta}$, i.e.

$$I_{\alpha-\beta}(xy) = \frac{2^{1-(\alpha-\beta)}\, y^{\alpha-\beta}\, x^{-(\alpha-\beta)}}{\sqrt{\pi}\,\Gamma(\alpha-\beta+1/2)} \int_0^x (x^2 - \tau^2)^{\alpha-\beta-1/2} \cosh(y\tau) d\tau, \quad R(\alpha-\beta+1/2) > 0,$$

It is easy to rewrite (49) in the form

$$\left[\hat{\mathcal{G}}_{\alpha-\beta} f\right](y) = \sqrt{\frac{2}{\pi}}\, 2^{1+\left(\frac{\alpha-\beta}{2}\right)}\, e^{-(1/2)y^2} \int_0^\infty \cosh(\sqrt{2}y\tau)\, [\hat{\mathcal{K}}^{\alpha-\beta+1/2}\, h](\tau) d\tau,$$

with the function $h(x)$ being

$$h(x) = x^{-1-\left(\frac{\alpha-\beta}{2}\right)}\, e^{-(1/2)y^2}\, f(x).$$

The above relation holds under the assumption that $xh(x)$ is integrable.

We finally note that, since several forms of the Erdelyi-Kober operator exist in the literature, a rather wide set of relation linking the Hankel transform to such forms can be deduced.

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