Eur. J. Math. Anal. 1 (2021) 1-18
doi: 10.28924/ada/ma.1.1

Existence and Stability Results for Second-Order Neutral Stochastic Differential Equations With Random Impulses and Poisson Jumps

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Abstract. The objective of this paper is to investigate the existence and stability results of second-order neutral stochastic functional differential equations (NSFDEs) in Hilbert space. Initially, we establish the existence results of mild solutions of the aforementioned system using the Banach contraction principle. The results are formulated using stochastic analysis techniques. In the later part, we investigate the stability results through the continuous dependence of solutions on initial conditions.

1. Introduction

Stochastic differential equations (SDEs) captures disturbances from random factors. Mathematical models obtained by integrating stochastic process provide a better understanding of the real-world system [12]. For elementary study of stochastic differential equations, the reader may refer to [7, 12, 14, 24].

Impulsive differential equations also attracted the attention of researchers (see [4, 11, 13, 21, 22] etc.). Impulse in general occurs as deterministic or random models. Nevertheless by natural phenomena, the impulses often occur at random time points. Many researches have been undergone solving various differential equations with fixed time impulses [1, 9, 16, 23]. Random impulsive differential equations involving fractional derivative are also studied see [20, 25].

It is known that impulsive stochastic differential equations play a vital role in modelling practical processes. Not only from Guassian white noise there are certain other factors that results in the rise of random effects. Random impulsive stochastic differential equations (ISDEs) are widely used

Received: 20 Aug 2021.
Key words and phrases. existence; stability; Banach contraction principle; second-order neutral stochastic functional differential equations; random impulse; stochastic differential system.

in the fields of medicine, biology, economy, finance and so on. For example, the classical stock price model [28].

$$\begin{aligned} d[\mathfrak{S}(t)] &= \mathfrak{f}\mathfrak{S}(t)dt + \sigma\mathfrak{S}(t)dw(t),\ t \geq 0,\ t \neq \tau_k, \\ \mathfrak{S}(\tau_k) &= a_k\mathfrak{S}(\tau_k^-),\ k = 1, 2, ..., \\ \mathfrak{S}(0) &= \mathfrak{S}_0, \end{aligned}$$

is described using an ISDEs. Here w_t is a Brownian motion or Wiener process, $\mathfrak{S}(t)$ represents the price of the stock at time t, and $\{\tau_k\}$ represents the release time of the important information relating to the stock. $\mathfrak{S}(\tau_k^-) = \lim_{t\to\tau_k-0}\mathfrak{S}(t)$ and $\mathfrak{S}_0 \in \mathbb{R}$. In reality, $\{\tau_k\}$ is a sequence of random variables, which satisfies $0 < \tau_2 < \tau_3 < \cdots$. Recently, in [10] the authors have contributed the existence and Hyers-Ulam stability of mild solutions for random impulsive stochastic functional ordinary differential equations which are studied using Krasnoselskii's fixed point theorem.

Solving second-order differential equations has been observed by many scholars. Many authors solved second-order stochastic differential equations see [5,6,8,19]. However, there are not many papers considering the existence and stability results on stochastic differential equations with random impulse. Anguraj et.al [3], considered the SDEs with random impulses and Poisson jumps of the form

$$\begin{aligned} d[\mathfrak{x}(t)] &= \mathfrak{f}(t, \mathfrak{x}_t) + \mathfrak{g}(t, \mathfrak{x}_t)dw(t) + \int_{\mathfrak{U}} \mathfrak{h}(t, \mathfrak{x}_t, u)\widetilde{N}(dt, du),\ t \geq t_0,\ t \neq \tau_k, \\ \mathfrak{x}(\zeta_k) &= b_k(\tau_k)\mathfrak{x}(\zeta_k^-),\ k = 1, 2, ..., \\ \mathfrak{x}_{t_0} &= \zeta = \{\zeta(\theta) : -\tau \leq \theta \leq 0\}. \end{aligned}$$

The authors studied the existence, uniqueness, and stability through continuous dependence on initial conditions for SDEs with random impulses and Poisson jumps by using Banach fixed point theorem. Very recently, Anguraj et.al [2] investigated the Existence and Hyers Ulam stability of random impulsive stochastic functional integrodifferential equations with finite delays.

Motivated by the above discussion, here we consider the following second-order NSFDEs with random impulses and Poisson jumps.

$$\begin{aligned} d\left[x'(t) - \mathfrak{h}(t, x_t)\right] &= \left[\mathfrak{A}x(t) + \mathfrak{f}(t, x_t)\right]dt + \mathfrak{g}(t, x_t)d\omega(t) + \int_{\mathfrak{U}} \sigma(t, x_t, u)\widetilde{N}(dt, du),,\ t \geq t_0,\ t \neq \xi_k, \\ x(\xi_k) &= \mathfrak{b}_k(\delta_k)x(\xi_k^-),\ x'(\xi_k) = \mathfrak{b}_k(\delta_k)x'(\xi_k^-),\ k = 1, 2, ..., \\ x_{t_0} &= \phi,\ x'(t_0) = \varphi, \end{aligned} \tag{1.1}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family $\{\mathcal{C}(t), t \geq 0\}$. $W(t)$ is a given Q-Wiener process with a finite trace nuclear covariance operator $Q > 0$. δ_k is a random variable defined from Ω to $D \equiv (0, d_k)$ for $k = 1, 2\cdots$. Suppose that δ_i and δ_j are independent of each other as $i \neq j$, $(i, j = 1, 2, \cdots)$. The impulsive moments ξ_k are random variables and satisfy $\xi_k = \xi_{k-1} + \delta_k$, $k = 1, 2, \cdots$. Obviously, $\{\xi_k\}$ is a process with independent increments. $0 < t_0 = \xi_0 < \xi_2 < \xi_3 < \cdots < \lim_{k\to\infty}\xi_k = \infty$, and $x(\xi_k^-) = \lim_{t\to\xi_{k-0}} x(t)$. $\mathfrak{b}_k : D_k \to H$,

for each $k = 1, 2, \cdots$. The time history $x_t(\theta) = \{x(t+\theta) : -\delta \leq \theta \leq 0\}$ with some given $\delta > 0$. Moreover, $\mathfrak{h}, \mathfrak{f}, \mathfrak{g}, \sigma$, and ϕ, φ will be specified later.

To the best of authors knowledge, up to now, no work has been reported to derive the second-order NSFDEs with random impulses and Poisson jumps. The main contributions are summarized as follows:

(1) second-order NSFDEs with random impulses and Poisson jumps is formulated.

(2) Initially, we establish the existence results of mild solutions of the aforementioned system using Banach contraction principle.

(3) Next, we investigate the stability results through continuous dependence of solutions on initial conditions.

(4) An example is provided to illustrate the obtained theoretical results.

The rest of the paper is organised as follows. Section 2 is devoted to basic definitions, notions and lemma. In section 3, existence of mild solutions of the aforementioned system (1.1) is investigated using Banach contraction principle. Eventually in section 4, the stability of mild solution is obtained through continuous dependence of solutions on initial conditions.

2. Preliminaries

Let $(\Omega, \Im, \mathcal{P})$ be a complete probability space equipped with the normal filtration $\{\Im\}_{t \geq t_0}$. $\Im_{t_0}$ containing all $\mathcal{P}$-null sets. H and K be two real Hilbert spaces. $\mathcal{L}(H, K)$ denotes the space of all bounded linear operators from K to H.

We may assume that, $\{\mathcal{N}(t), t \geq t_0\}$ be a counting process generated by $\{\xi_k, k \geq 0\}$. $\Im_t^{(1)}$ denote the minimal σ algebra denoted by $\{\mathcal{N}(r), r \leq t\}$ and denote $\Im_t^{(2)}$ the σ-algebra generated by $\{\omega(s), s \leq t\}$. We assume that $\Im_\infty^{(1)}, \Im_\infty^{(2)}$ and ξ are mutually independent and $\Im_t = \Im_t^{(1)} \vee \Im_t^{(2)}$.

We assume that there exist a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in $\mathcal{K}$, a bounded sequence of non-negative real numbers λ_n such that, $Qe_n = \lambda_n e_n$, $n = 1, 2, \cdots$. Let $\{\beta_n(t)\}(n = 1, 2, 3...)$ be a sequence of real valued one dimensional standard Brownian motion mutually independent over $(\Omega, \Im, \mathcal{P})$. A Q-Wiener process can be defined by $\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n$, $(t \geq 0)$. Set $\Phi \in \mathcal{L}(K, H)$ we define,

$$\|\Phi\|_Q^2 = Tr(\Phi Q \Phi^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\lambda_n}\Phi e_n\right\|^2$$

If $\|\Phi\|_Q^2 < \infty$, then Φ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operator $\Phi : K \to H$. The completion $\mathcal{L}_Q(K, H)$ of $\mathcal{L}(K, H)$ with respect to the topology induced by the norm $\|.\|_Q$, where $\|\Phi\|_Q^2 = \langle \Phi, \Phi \rangle$ is a Hilbert space.

Let $T \in (t_0, +\infty), J := [t_0, T], J_k = [\xi_k, \xi_{k+1}), k = 0, 1, \cdots, \tilde{J} = \{t : t \in J, t \neq \xi_k, k = 1, 2, \cdots\}$. $\mathcal{L}_2(\Omega, H)$ be the collection of square integrable $\Im_t$-measurable, H-valued random variables defined by the norm $\|x\|_{\mathcal{L}_2} = \left(\mathbb{E}\|x\|^2\right)^{\frac{1}{2}}$, the expectation being expressed by the form $\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 \, d\mathcal{P}$. Let $PC(J, \mathcal{L}_2(\Omega, H)) = \{x : J \to \mathcal{L}_2(\Omega, H)\}$, x is continuous on every J_k, and the left limits $x(\xi_k^-), x'(\xi_k^-)$ exist $k = 1, 2, \cdots$ be a piecewise continuous space.

We may define the space $\mathrm{C} = \mathrm{C}([-\delta, 0], \mathrm{H})$ which contains all piecewise continuous functions mapping from $[-\delta, 0]$ to H with the norm $\|x\|_t = \sup_{t-\delta \le s \le t} \|x(s)\|$ for each $t \ge t_0$. $\mathcal{B}$ be the Banach space, $\mathcal{B}([t_0 - \delta, T], \mathcal{L}_2(\Omega, \mathrm{H}))$ consists of continuous, $\Im_t$-measurable, C-valued processes. The norm is defined by

$$\|x\|_{\mathcal{B}} = \left(\sup_{t \in \mathrm{J}} \mathbb{E} \|x\|_t^2 \right)^{\frac{1}{2}}.$$

In (1.1), $\widetilde{N}(dt, du) = N(dt, du) - dt\nu(du)$ denotes the compensated Poisson measure independent of $\omega(t)$ and $N(dt, du)$ represents the Poisson counting measure associated with a characteristic measure ν. For a basic study on the Poisson jumps we refer to the book by [27].

Subsequently, we introduce certain definitions of sine and cosine operators.

A bounded linear operators family $\{\mathcal{C}(t), t \in \mathbb{R}\}$ is called a strongly continuous cosine family if and only if

(i) $\mathcal{C}(0) = I$ (I is the identity operator in H);

(ii) $\mathcal{C}(t)x$ is continuous in t, for all $x \in \mathrm{H}$;

(iii) $\mathcal{C}(t+s) + \mathcal{C}(t-s) = 2\mathcal{C}(t)\mathcal{C}(s)$ for all $t, s \in \mathbb{R}$.

The corresponding strongly continuous sine family $\{\mathcal{S}(t), t \in \mathbb{R}\}$ is defined by

$$\mathcal{S}(t)x = \int_0^t \mathcal{C}(s)x\,ds, \ \ x \in \mathrm{H}, \ \ t \in \mathbb{R}$$

Then the following property holds:

$$\mathfrak{A} \int_{t_0}^t \mathcal{S}(s)x\,ds = [\mathcal{C}(t) - \mathcal{C}(t_0)]\,x$$

Lemma 2.1. *[18] Let $\{\mathcal{C}(t), t \in \mathbb{R}\}$ be a strongly continuous cosine family in* H*, then for all $s, t \in \mathbb{R}$, the following results are true:*

(i) $\mathcal{C}(t) = \mathcal{C}(-t)$;

(ii) $\mathcal{S}(s+t) + \mathcal{S}(s-t) = 2\mathcal{S}(s)\mathcal{C}(t)$;

(iii) $\mathcal{S}(s+t) = \mathcal{S}(s)\mathcal{C}(t) + \mathcal{S}(t)\mathcal{C}(s)$;

(iv) $\mathcal{S}(t) = -\mathcal{S}(-t)$;

(v) $\mathcal{C}(t+s) + \mathcal{C}(s-t) = 2\mathcal{C}(s)\mathcal{C}(t)$;

(vi) $\mathcal{C}(t+s) - \mathcal{C}(t-s) = 2\mathfrak{A}\mathcal{S}(t)\mathcal{S}(s)$.

Before investigating mild solution (1.1), we consider the second-order neutral functional differential equation, which is given by

$$\begin{cases} d[u'(t) - g(t, u_{(t)})] = \mathfrak{A}u_t\,dt + \mathfrak{f}(t, u_t)dt, \ \ t \ge 0, \\ u_0 = \phi \in \mathrm{C}, u'(0) = \varphi \in \mathrm{H}, \ \ t \in (-r, 0], \end{cases} \tag{2.1}$$

where $\mathfrak{A}$ is the infinitesimal generator of a strongly continuous cosine family $\{\mathcal{C}(t), t \in \mathbb{R}^+\}$ and the functions $\mathfrak{g}, \mathfrak{f} \in \mathcal{L}^1(0, T; \mathrm{H})$.

Lemma 2.2. *[15] A continuously differentiable function $u(t) : [0, T] \to H$ is called the mild solution for the Cauchy problem (2.1), if it satisfies,*

$$u(t) = \mathcal{C}(t)\phi(0) + \mathcal{S}(t)[\varphi - \mathfrak{g}(0, \phi)] + \int_0^t \mathcal{C}(t-s)\mathfrak{g}(s, u_s)ds + \int_0^t \mathcal{S}(t-s)\mathfrak{f}(s, x_s)ds, \ t \geq 0,$$

where

$$\mathcal{S}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^2; \mathfrak{A})d\lambda;$$

$$\mathcal{C}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda R(\lambda^2; \mathfrak{A})d\lambda,$$

and Γ is a suitable path.

Consider the linear second-order linear differential equation with impulse conditions,

$$\begin{cases} u''(t) = \mathfrak{A}u(t) + \mathfrak{f}(t), \ t \geq 0, t \neq t_k, \\ u(0) = u_0, u'(0) = v_0, \\ u(t_k) = \mathfrak{b}_k u(t_k^-), u'(t_k) = \mathfrak{b}_k u'(t_k^-), \ k = 1, 2, \cdots, \end{cases} \tag{2.2}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \{t_k, k \geq 1\}$ is a sequence of fixed impulsive points, $\mathfrak{f}(t) : [0, T) \to H$ is an integrable function.

Lemma 2.3. *The piecewise continuous differentiable function $u(t) : [0, T] \to H$ is a mild solution of (2.2), if and only if $x(t)$ satisfies the integral equation*

$$\begin{aligned} u(t) = &\prod_{i=1}^{k}\mathfrak{b}_i\mathcal{C}(t)u_0 + \prod_{i=1}^{k}\mathfrak{b}_i\mathcal{S}(t)v_0 + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j\int_{t_{i-1}}^{t_i}\mathcal{S}(t-s)\mathfrak{f}(s)ds \\ &\times \int_{t_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds, \ t \in [t_k, t_{k+1}), \ k = 0, 1, \cdots. \end{aligned} \tag{2.3}$$

Proof. (i)For $t \in [0, t_1)$, the mild solution is studied in [17],

$$u(t) = \mathcal{C}(t)u_0 + \mathcal{S}(t)v_0 + \int_0^t \mathcal{S}(t-s)\mathfrak{f}(s)ds, \ t \in [0, t_1).$$

(ii) For $t \in [t_1, t_2)$, we set

$$u(t) = \mathcal{C}(t-t_1)u(t_1) + \mathcal{S}(t-t_1)u'(t_1) + \int_{t_1}^t \mathcal{S}(t-s)\mathfrak{f}(s)ds, \ t \in [t_1, t_2). \tag{2.4}$$

Since,

$$u(t_1) = \mathfrak{b}_1 u(t_1^-), \ u'(t_1) = \mathfrak{b}_1 u'(t_1^-),$$

and from (i) we know

$$u(t_1^-) = \mathcal{C}(t_1)u_0 + \mathcal{S}(t_1)v_0 + \int_0^{t_1}\mathcal{S}(t_1-s)\mathfrak{f}(s)ds; \tag{2.5}$$

$$u'(t_1^-) = \mathfrak{A}\mathcal{S}(t_1)u_0 + \mathcal{C}(t_1)v_0 + \int_0^{t_1}\mathcal{C}(t_1-s)\mathfrak{f}(s)ds. \tag{2.6}$$

Thus,

$$\begin{aligned} u(t) &= \mathfrak{b}_1\mathcal{C}(t-t_1)\mathcal{C}(t_1)u_0 + \mathfrak{b}_1\mathcal{S}(t-t_1)\mathfrak{AS}(t_1)u_0 + \mathfrak{b}_1\mathcal{C}(t-t_1)\mathcal{S}(t_1)v_0 + \mathfrak{b}_1\mathcal{S}(t-t_1)\mathcal{C}(t_1)v_0 \\ &+ \mathfrak{b}_1\mathcal{C}(t-t_1)\int_0^{t_1}\mathcal{S}(t_1-s)\mathfrak{f}(s)ds + \mathfrak{b}_1\mathcal{S}(t-t_1)\int_0^{t_1}\mathcal{S}(t_1-s)\mathfrak{f}(s)ds \\ &+ \int_{t_1}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds,\ \ t\in[t_1,t_2). \end{aligned}$$

Applying Lemma 2.1, we get

$$u(t) = \mathfrak{b}_1\mathcal{C}(t)u_0 + \mathfrak{b}_1\mathcal{S}(t)v_0 + \mathfrak{b}_1\int_0^{t_1}\mathcal{S}(t_1-s)\mathfrak{f}(s)ds + \int_{t_1}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds,\ \ t\in[t_1,t_2).$$

(iii) For $t\in[t_2,t_3)$,

$$\begin{aligned} u(t) &= \mathcal{C}(t-t_2)u(t_2) + \mathcal{S}(t-t_2)u'(t_2) + \int_{t_2}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds \\ &= \mathcal{C}(t-t_2)\mathfrak{b}_2u(t_2^-) + \mathcal{S}(t-t_2)\mathfrak{b}_2u'(t_2^-) + \int_{t_2}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds. \end{aligned} \tag{2.7}$$

From the conclusions of (ii), it is known that,

$$u(t_2^-) = \mathfrak{b}_1\mathcal{C}(t_2)u_0 + \mathfrak{b}_1\mathcal{S}(t_2)v_0 + \mathfrak{b}_1\int_0^{t_2}\mathcal{S}(t_2-s)\mathfrak{f}(s)ds + \int_{t_1}^{t_2}\mathcal{S}(t_2-s)\mathfrak{f}(s)ds; \tag{2.8}$$

$$u'(t_2^-) = \mathfrak{b}_1\mathfrak{AS}(t_2)u_0 + \mathfrak{b}_1\mathcal{C}(t_2)v_0 + \mathfrak{b}_1\int_0^{t_2}\mathcal{C}(t_2-s)\mathfrak{f}(s)ds + \int_{t_1}^{t_2}\mathcal{C}(t_2-s)\mathfrak{f}(s)ds \tag{2.9}$$

Along with (2.7) and using Lemma 2.1, we have

$$\begin{aligned} u(t) &= \mathfrak{b}_2\mathfrak{b}_1\mathcal{C}(t)u_0 + \mathfrak{b}_2\mathfrak{b}_1\mathcal{S}(t)v_0 + \mathfrak{b}_2\mathfrak{b}_1\int_0^{t_1}\mathcal{S}(t-s)\mathfrak{f}(s)ds \\ &+ \mathfrak{b}_2\int_{t_1}^{t_2}\mathcal{S}(t-s)\mathfrak{f}(s)ds + \int_{t_1}^{t_2}\mathcal{S}(t-s)\mathfrak{f}(s)ds,\ \ t\in[t_2,t_3) \end{aligned}$$

Similarly, for all $t\in[t_k,t_{k-1})$.

$$x(t) = \prod_{i=1}^{k}\mathfrak{b}_i\mathcal{C}(t)u_0 + \prod_{i=1}^{k}\mathfrak{b}_i\mathcal{S}(t)v_0 + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j\int_{t_{i-1}}^{t_i}\mathcal{S}(t-s)\mathfrak{f}(s)ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s)ds.$$

□

By Lemma 2.2, Lemma 2.3 the mild solution of the system (1.1) applying index function for $t\in J$.

Definition 2.1. *For a given $T\in(t_0,+\infty)$, a $\mathfrak{F}$-adapted process function $\{x\in\mathcal{B}, t_0-\delta\le t\le T\}$ is called a mild solution of system (1.1), if (i) $x_{t_0}(s)=\phi(s)\in\mathcal{L}_2^0(\Omega,\mathcal{B})$ for $\delta\le s\le 0$;*
(ii) $x'_{(}t_0)=\varphi(t)\in\mathcal{L}_2^0(\Omega,H)$ for $t\in J$;
(iii) The functions $\mathfrak{f}(s,x_t),\mathfrak{g}(s,x_t),\mathfrak{h}(s,x_t)$ and $\sigma(s,x_s,u)$ are integrable, and for a.e. $t\in J$, the

following integral equation is satisfied.

$$
\begin{aligned}
x(t) \quad = \quad & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{C}(t-t_0)\phi(0)+\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{S}(t-t_0)[\varphi-\mathfrak{h}(0,\phi)]+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
\times \quad & \int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds+\int_{\xi_k}^{t}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\\
\times \quad & \mathfrak{f}(s,x_s)ds+\int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s)\\
+ \quad & \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s)+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du)\\
+ \quad & \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du)\Bigg]I_{[\xi_k,\xi_{k+1})}(t),\ \ t\in[t_0,T]. \qquad (2.10)
\end{aligned}
$$

where,

$$\prod_{j=i}^{k}b_j(\delta_j)=b_k(\delta_k)b_{k-1}(\delta_{k-1})\cdots b_i(\delta_i),$$

and $I_{(A)}(.)$ is the index function, i.e.,

$$I_{\mathrm{A}}(t)=\begin{cases}1,\ \ if\ \ t\in \mathrm{A},\\ 0,\ \ if\ \ t\notin \mathrm{A}.\end{cases}$$

Lemma 2.4. *For any $p\geq 1$, and for $\mathcal{L}_Q(\mathrm{K},\mathrm{H})$-valued predictable process $u(.)$ such that,*

$$\sup_{s\in[0,T]}\mathbb{E}\left\|\int_0^s u(\eta)d\omega(\eta)\right\|^{2p}\leq(p(2p-1))^p\left(\int_0^t\left(\mathbb{E}\left\|u(s)\right\|_Q^{2p}\right)^{1/p}ds\right)^p,\ \ t\in \mathrm{J}.$$

3. Existence Results of Mild Solution

To prove the existence of mild solutions of random impulsive stochastic differential equations, the following assumptions are to be made.

(H1) $\mathcal{C}(t)$, $\mathcal{S}(t)(t\in \mathrm{J})$ are equicontinuous and there exist positive constants $\mathcal{M}$, $\tilde{\mathcal{M}}$ such that

$$\sup_{t\in \mathrm{J}}\left\|\mathcal{C}(t)\right\|\leq\mathcal{M},\ \ \sup_{t\in \mathrm{J}}\left\|\mathcal{S}(t)\right\|\leq\tilde{\mathcal{M}}. \qquad (3.1)$$

(H2) The functions $\mathfrak{f}:\mathrm{J}\times\mathrm{C}\to\mathrm{H}$; $\mathfrak{h}:\mathrm{J}\times\mathrm{C}\to\mathrm{H}$; $\mathfrak{g}:\mathrm{J}\times\mathrm{C}\to\mathcal{L}_Q(\mathrm{K},\mathrm{H})$ and $\sigma:\mathrm{J}\times\mathrm{C}\times\mathfrak{U}\to\mathrm{H}$

$$\mathbb{E}\left\|\mathfrak{f}(t,x_t)-\mathfrak{f}(t,y_t)\right\|^2\leq\mathcal{L}_{\mathfrak{f}}\left\|x-y\right\|_t^2,$$
$$\mathbb{E}\left\|\mathfrak{g}(t,x_t)-\mathfrak{g}(t,y_t)\right\|^2\leq\mathcal{L}_{\mathfrak{g}}\left\|x-y\right\|_t^2,$$
$$\mathbb{E}\left\|\mathfrak{h}(t,x_t)-\mathfrak{h}(t,y_t)\right\|^2\leq\mathcal{L}_{\mathfrak{h}}\left\|x-y\right\|_t^2,$$
$$\int_{\mathfrak{U}}\mathbb{E}\left\|\sigma(t,x_t,u)-\sigma(t,y_t,u)\right\|^2\nu(du)ds\vee\int_{\mathfrak{U}}\left(\mathbb{E}\left\|\sigma(t,x_t,u)-\sigma(t,y_t,u)\right\|^4\nu(du)ds\right)^{\frac{1}{2}}\leq\mathcal{L}_\sigma\left\|x-y\right\|_t^2,$$
$$\int_{\mathfrak{U}}\left(\mathbb{E}\left\|\sigma(t,x_t,u)-\sigma(t,y_t,u)\right\|^4\nu(du)ds\right)^{\frac{1}{2}}\leq\mathcal{L}_\sigma\left\|x\right\|_t^2.$$

(H3) For all $t \in J$, there exist constants $\kappa_{\mathfrak{f}}, \kappa_{\mathfrak{g}}, \kappa_{\mathfrak{h}}, \kappa_{\sigma} \in \mathcal{L}'(J, \Re^+)$ such that,

$$\mathbb{E}\left\|\mathfrak{f}(t,0)\right\|^2 \leq \kappa_{\mathfrak{f}},\ \mathbb{E}\left\|\mathfrak{g}(t,0)\right\|^2 \leq \kappa_{\mathfrak{g}},$$
$$\mathbb{E}\left\|\mathfrak{h}(t,0)\right\|^2 \leq \kappa_{\mathfrak{h}},\ \mathbb{E}\left\|\sigma(t,0,u)\right\|^2 \leq \kappa_{\sigma}.$$

(H4) $\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}$ is uniformly bounded then there exist constant $\mathcal{N}$ for all $\delta_j \in D_j$ such that

$$\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\} \leq \mathcal{N}.$$

Theorem 3.1. *If assumptions (H1)-(H4) gets satisfied then there exist a unique continuous mild solution of the system (1.1).*

Proof. We define an operator $\phi : \mathcal{B} \to \mathcal{B}$ by ϕx such that,

$$(\phi x)(t) = \begin{cases} \phi(t),\ \ t \in [t_0 - \delta, t_0], \\ \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{C}(t-t_0)\phi(0) + \prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{S}(t-t_0)[\varphi - \mathfrak{h}(0,\phi)] + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j) \\ \times \int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s) \\ \times\ \mathfrak{f}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) \\ + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du) \\ + \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du)\Big]I_{[\xi_k,\xi_{k+1})}(t),\ \ t \in [t_0,T]. \end{cases}$$

We need to prove that ϕ maps $\mathcal{B}$ into itself.

$$\begin{aligned}\mathbb{E}\left\|(\phi x)(t)\right\|^2 \quad &\leq \quad \mathbb{E}\Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{C}(t-t_0)\phi(0) + \prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{S}(t-t_0)[\varphi - \mathfrak{h}(0,\phi)] \\ &+ \quad \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du) \\
&+ \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\mathcal{N}(ds,du)\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^2 \\
\leq\ & 6\mathbb{E}\left[\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|\left\|\mathcal{C}(t-t_0)\right\|\left\|\phi(0)\right\|\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2 + 6\mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|\left\|\mathcal{S}(t-t_0)\right\| \\
&\times \left\|\varphi-\mathfrak{h}(0,\phi)\right\|\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2 + 6\mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{C}(t-s)\right\|\left\|\mathfrak{h}(s,x_s)\right\|ds \\
&+ \int_{\xi_k}^{t}\left\|\mathcal{C}(t-s)\right\|\left\|\mathfrak{h}(s,x_s)\right\|ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2 + 6\mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{S}(t-s)\right\| \\
&\times \left\|\mathfrak{f}(s,x_s)\right\|ds + \int_{\xi_k}^{t}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{f}(s,x_s)\right\|ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2 + 6\mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\| \\
&\times \int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{g}(s,x_s)\right\|d\omega(s) + \int_{\xi_k}^{t}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{g}(s,x_s)\right\|d\omega(s)\Bigg]I_{[\xi_k,\xi_{k+1})}(t) \\
&+ 6\mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\| \times \int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\left\|\mathcal{S}(t-s)\right\|\left\|\sigma(s,x_s,u)\right\|\widetilde{\mathcal{N}}(ds,du) \\
&+ \int_{\xi_k}^{t}\int_{\mathfrak{U}}\left\|\mathcal{S}(t-s)\right\|\left\|\sigma(s,x_s,u)\right\|\widetilde{\mathcal{N}}(ds,du)\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2, \\
=\ & 6\sum_{i=1}^{6}\mathfrak{G}_i.
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathfrak{G}_1 &\leq \mathbb{E}\left[\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|\left\|\mathcal{C}(t-t_0)\right\|\left\|\phi(0)\right\|\right]I_{[\xi_k,\xi_{k+1})}(t)\right]^2 \\
&\leq \mathcal{M}^2\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}^2\mathbb{E}\left\|\phi(0)\right\|^2 \\
&\leq \mathcal{M}^2\mathcal{N}^2\mathbb{E}\left\|\phi(0)\right\|^2, \\
\mathfrak{G}_2 &\leq \mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|\left\|\mathcal{S}(t-t_0)\right\|\left\|\varphi-\mathfrak{h}(0,\phi)\right\|\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2
\end{aligned}
$$

$$\begin{aligned}
&\leq \tilde{\mathcal{M}}^2\mathbb{E}\left\{\max_{i,k}\{\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2\\
&\leq \tilde{\mathcal{M}}^2\mathcal{N}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_3 &\leq \mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{C}(t-s)\right\|\left\|\mathfrak{h}(s,x_s)\right\|ds+\int_{\xi_k}^{t}\left\|\mathcal{C}(t-s)\right\|\left\|\mathfrak{h}(s,x_s)\right\|ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2\\
&\leq \mathcal{M}^2\mathbb{E}\left\{\max_{i,k}\{1,\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}^2(T-t_0)\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{h}(s,x_s)\right\|^2ds\\
&\leq 2\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)\Bigg[\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{h}(s,x_s)-\mathfrak{h}(s,0)\right\|^2ds+\int_{t_0}^{t}\left\|\mathfrak{h}(s,0)\right\|^2ds\Bigg]\\
&\leq 2\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)\int_{t_0}^{t}\left[\mathcal{L}_{\mathfrak{h}}\mathbb{E}\left\|x\right\|_s^2+\kappa_{\mathfrak{h}}\right]ds\\
&\leq 2\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)\int_{t_0}^{t}\mathcal{L}_{\mathfrak{h}}\mathbb{E}\left\|x\right\|_s^2ds+2\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_{\mathfrak{h}},\\
\mathfrak{G}_4 &\leq \mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{f}(s,x_s)\right\|ds+\int_{\xi_k}^{t}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{f}(s,x_s)\right\|ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2\\
&\leq \tilde{\mathcal{M}}^2\mathbb{E}\left\{\max_{i,k}\{1,\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}^2(T-t_0)\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{f}(s,x_s)\right\|^2ds\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\Bigg[\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{f}(s,x_s)-\mathfrak{f}(s,0)\right\|^2ds+\int_{t_0}^{t}\left\|\mathfrak{f}(s,0)\right\|^2ds\Bigg]\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\int_{t_0}^{t}\left[\mathcal{L}_{\mathfrak{f}}\mathbb{E}\left\|x\right\|_s^2+\kappa_{\mathfrak{f}}\right]ds\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\int_{t_0}^{t}\mathcal{L}_{\mathfrak{f}}\mathbb{E}\left\|x\right\|_s^2ds+2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_{\mathfrak{f}},\\
\mathfrak{G}_5 &\leq \mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\int_{\xi_{i-1}}^{\xi_i}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{g}(s,x_s)\right\|d\omega(s)+\int_{\xi_k}^{t}\left\|\mathcal{S}(t-s)\right\|\left\|\mathfrak{g}(s,x_s)\right\|d\omega(s)\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2\\
&\leq \tilde{\mathcal{M}}^2\mathbb{E}\left\{\max_{i,k}\{1,\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\|\}\right\}^2\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{g}(s,x_s)\right\|^2ds\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q)\Bigg[\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{g}(s,x_s)-\mathfrak{g}(s,0)\right\|^2ds+\int_{t_0}^{t}\left\|\mathfrak{g}(s,0)\right\|^2ds\Bigg]\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q)\int_{t_0}^{t}\left[\mathcal{L}_{\mathfrak{g}}\mathbb{E}\left\|x\right\|_s^2+\kappa_{\mathfrak{g}}\right]ds\\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q)\int_{t_0}^{t}\mathcal{L}_{\mathfrak{g}}\mathbb{E}\left\|x\right\|_s^2ds+2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)Tr(Q)\kappa_{\mathfrak{g}}.
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{G}_6 &\leq \mathbb{E}\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{j=i}^{k}\left\|\mathfrak{b}_j(\delta_j)\right\| \times \int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\left\|\mathcal{S}(t-s)\right\|\left\|\sigma(s,x_s,u)\right\|\widetilde{N}(ds,du) \\
&+ \int_{\xi_k}^{t}\int_{\mathfrak{U}}\left\|\mathcal{S}(t-s)\right\|\left\|\sigma(s,x_s,u)\right\|\widetilde{N}(ds,du)\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2 \\
&\leq 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\int_{\mathfrak{U}}\left[\mathbb{E}\left\|\sigma(s,x_s,u)-\sigma(s,0,u)\right\|^2+\left\|\sigma(s,0,u)\right\|^2\right]ds \\
&+ 2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\left(\int_{t_0}^{t}\int_{\mathfrak{U}}\mathbb{E}\left\|\sigma(s,x_s,u)\right\|^4\nu(du)ds\right)^{\frac{1}{2}} \\
&\leq 4\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_\sigma\mathbb{E}\left\|x\right\|_s^2\,ds+2\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\kappa_\sigma,
\end{aligned}
$$

Thus we would obtain,

$$
\begin{aligned}
\mathbb{E}\left\|(\phi x)(t)\right\|_t^2 &\leq 6\mathcal{M}^2\mathcal{N}^2\mathbb{E}\left\|\phi(0)\right\|^2+6\tilde{\mathcal{M}}^2\mathcal{N}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2+12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0) \\
&\times \int_{t_0}^{t}\mathcal{L}_\mathfrak{h}\mathbb{E}\left\|x\right\|_s^2\,ds+12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_\mathfrak{h}+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0) \\
&\times \int_{t_0}^{t}\mathcal{L}_\mathfrak{f}\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_\mathfrak{f}+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q) \\
&\times \int_{t_0}^{t}\mathcal{L}_\mathfrak{g}\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)Tr(Q)\kappa_\mathfrak{g} \\
&+ 24\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_\sigma\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\kappa_\sigma.
\end{aligned}
$$

Taking supremum over t,

$$
\begin{aligned}
&\sup_{t_0\leq t\leq T}\mathbb{E}\left\|(\phi x)(t)\right\|_t^2 \\
\leq\ & 6\mathcal{M}^2\mathcal{N}^2\mathbb{E}\left\|\phi(0)\right\|^2+6\tilde{\mathcal{M}}^2\mathcal{N}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2+12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0) \\
\times\ & \int_{t_0}^{t}\mathcal{L}_\mathfrak{h}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_s^2\,ds+12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_\mathfrak{h}+12\tilde{\mathcal{M}}^2max\{1,\mathcal{N}^2\} \\
\times\ & (T-t_0)\int_{t_0}^{t}\mathcal{L}_\mathfrak{f}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_\mathfrak{f}+12\tilde{\mathcal{M}}^2 \\
\times\ & \max\{1,\mathcal{N}^2\}Tr(Q)\int_{t_0}^{t}\mathcal{L}_\mathfrak{g}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2max\{1,\mathcal{N}^2\}(T-t_0)Tr(Q)\kappa_\mathfrak{g} \\
+\ & 24\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_\sigma\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_s^2\,ds+12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\kappa_\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\leq 6\mathcal{M}^2\mathcal{N}^2\mathbb{E}\left\|\phi(0)\right\|^2 + 6\tilde{\mathcal{M}}^2\mathcal{N}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2 + 12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\\
&\times \mathcal{L}_{\mathfrak{h}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_t^2 + 12\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_{\mathfrak{h}} + 12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\\
&\times (T-t_0)^2\mathcal{L}_{\mathfrak{f}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_t^2 + 12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)^2\kappa_{\mathfrak{f}}\\
&+ 12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q)(T-t_0)\mathcal{L}_{\mathfrak{g}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_t^2\\
&+ 12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)Tr(Q)\kappa_{\mathfrak{g}}\\
&+ 24\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\mathcal{L}_{\sigma}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x\right\|_s^2 ds + 12\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\kappa_{\sigma}\\
&\leq 6\Big[\mathcal{N}^2\Big[\mathcal{M}^2\mathbb{E}\left\|\phi(0)\right\|^2 + \tilde{\mathcal{M}}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2\Big]\Big] + 12\max\{1,\mathcal{N}^2\}(T-t_0)\\
&\times \Big[\mathcal{M}^2(T-t_0)\kappa_{\mathfrak{h}} + \tilde{\mathcal{M}}^2(T-t_0)\kappa_{\mathfrak{f}} + \tilde{\mathcal{M}}^2Tr(Q)\kappa_{\mathfrak{g}} + \tilde{\mathcal{M}}^2\kappa_{\sigma}\Big] + 12\max\{1,\mathcal{N}^2\}\\
&\times (T-t_0)\Big[\mathcal{M}^2(T-t_0)\mathcal{L}_{\mathfrak{h}} + \tilde{\mathcal{M}}^2(T-t_0)\mathcal{L}_{\mathfrak{f}} + \tilde{\mathcal{M}}^2Tr(Q)\mathcal{L}_{\mathfrak{g}} + 2\tilde{\mathcal{M}}^2\mathcal{L}_{\sigma}\Big]\left\|x\right\|_t^2\\
\left\|\phi x\right\|_{\mathcal{B}}^2 &\leq \mathfrak{c}_1 + \mathfrak{c}_2\left\|x\right\|_{\mathcal{B}}^2.
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathfrak{c}_1 &= 6\Big[\mathcal{N}^2\Big[\mathcal{M}^2\mathbb{E}\left\|\phi(0)\right\|^2 + \tilde{\mathcal{M}}^2\mathbb{E}\left\|\varphi-\mathfrak{h}(0,\phi)\right\|^2\Big]\Big] + 12\max\{1,\mathcal{N}^2\}(T-t_0)\\
&\times \Big[\mathcal{M}^2(T-t_0)\kappa_{\mathfrak{h}} + \tilde{\mathcal{M}}^2(T-t_0)\kappa_{\mathfrak{f}} + \tilde{\mathcal{M}}^2Tr(Q)\kappa_{\mathfrak{g}} + \tilde{\mathcal{M}}^2\kappa_{\sigma}\Big],\\
\mathfrak{c}_2 &= 12\max\{1,\mathcal{N}^2\}(T-t_0)\Big[\mathcal{M}^2(T-t_0)\mathcal{L}_{\mathfrak{h}} + \tilde{\mathcal{M}}^2(T-t_0)\mathcal{L}_{\mathfrak{f}} + \tilde{\mathcal{M}}^2Tr(Q)\mathcal{L}_{\mathfrak{g}} + 2\tilde{\mathcal{M}}^2\mathcal{L}_{\sigma}\Big].
\end{aligned}
$$

where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are constants.

Hence ϕ is bounded.

Now we need to prove that ϕ is a contraction mapping. For any $x, y \in \mathcal{B}$ we have,

$$
\begin{aligned}
&\left\|(\phi x)(t)-(\phi y)(t)\right\|^2\\
\leq\ &\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{C}(t-s)\mathfrak{h}(s,x_s)ds\\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s,x_s)ds + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
&\times \int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{g}(s,x_s)d\omega(s) + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
&\times \int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\widetilde{N}(ds,dt) + \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,x_s,u)\widetilde{N}(ds,dt)\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg\|^2\\
-\ &\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\mathfrak{h}(s,y_s)ds + \int_{\xi_k}^{t}\mathcal{C}(t-s)\mathfrak{h}(s,y_s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{f}(s,y_s)ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{f}(s,y_s)ds + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
&\times \int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\mathfrak{g}(s,y_s)d\omega(s) + \int_{\xi_k}^{t}\mathcal{S}(t-s)\mathfrak{g}(s,y_s)d\omega(s) + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\\
&\times \int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,y_s,u)\widetilde{\mathcal{N}}(ds,dt) + \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\sigma(s,y_s,u)\widetilde{\mathcal{N}}(ds,dt)\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^2\\
\leq\ & 4\max\{1,\mathcal{N}^2\}\mathcal{M}^2(T-t_0)\int_{t_0}^{t}\left\|\mathfrak{h}(t,x_s)-\mathfrak{h}(t,y_s)\right\|^2 ds + 4\max\{1,\mathcal{N}^2\}\\
&\times \tilde{\mathcal{M}}^2(T-t_0)\int_{t_0}^{t}\left\|\mathfrak{f}(t,x_s)-\mathfrak{f}(t,y_s)\right\|^2 ds + 4\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2\\
&\times \int_{t_0}^{t}\left\|\mathfrak{g}(t,x_s)-\mathfrak{g}(t,y_s)\right\|^2 ds + 4\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2\\
&\times \Bigg[\int_{t_0}^{t}\int_{\mathfrak{U}}\left\|\sigma(t,x_s,u)-\sigma(t,y_s,u)\right\|^2\nu(du)ds\\
&+ \left(\int_{t_0}^{t}\int_{\mathfrak{U}}\left\|\sigma(t,x_s,u)-\sigma(t,y_s,u)\right\|^4\nu(du)ds\right)^{\frac{1}{2}}\Bigg]
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\sup_{t_0\leq t\leq T}\mathbb{E}\left\|(\phi x)(t)-(\phi y)(t)\right\|^2\\
\leq\ & 4\max\{1,\mathcal{N}^2\}\mathcal{M}^2(T-t_0)^2\mathcal{L}_h\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-y\right\|_s^2 ds + 4\max\{1,\mathcal{N}^2\}\\
&\times \tilde{\mathcal{M}}^2(T-t_0)^2\mathcal{L}_f\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-y\right\|_s^2 ds + 4\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2 Tr(Q)\\
&\times (T-t_0)\mathcal{L}_g\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-y\right\|_s^2 ds + 4\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2\\
&\times (T-t_0)\mathcal{L}_\sigma\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-y\right\|_s^2 ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \Big[4\max\{1,\mathcal{N}^2\}\mathcal{M}^2(T-t_0)^2\mathcal{L}_h + 4\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2(T-t_0)\\
&\times [(T-t_0)\mathcal{L}_f + Tr(Q)\mathcal{L}_\mathfrak{g} + \mathcal{L}_\sigma]\Big]\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-y\right\|_t^2
\end{aligned}
$$

Hence,

$$\left\|(\phi x)-(\phi y)\right\|_{\mathcal{B}}^2 \leq \gamma(T)\left\|x-y\right\|_{\mathcal{B}}^2.$$

where,

$$\gamma(T) = 3\max\{1,\mathcal{N}^2\}\mathcal{M}^2(T-t_0)^2\mathcal{L}_h + 3\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2(T-t_0)[(T-t_0)\mathcal{L}_f + Tr(Q)\mathcal{L}_\mathfrak{g} + \mathcal{L}_\sigma].$$

By taking suitable $0 < T_1 < T$ sufficiently small such that, $\gamma(T_1) < 1$.
Hence ϕ is a contraction on $\mathcal{B}$. By Banach contraction principle, a unique fixed point $x \in \mathcal{B}$ is obtained for the operator ϕ and therefore $\phi x = x$ is a mild solution of the system.
The solution can be extended to the entire interval $(-\delta, T]$ in finitely many steps. Thus the existence and uniqueness of the mild solution on $(-\delta, T]$ is proved.

□

4. Stability

The stability through continuous dependence of solutions on initial conditions are established.

Definition 4.1. *A mild solution $x^{\xi,x}(t)$ of the system (1.1) with the initial value (ξ, x) is said to be stable in mean square if for all $\epsilon > 0$ such that*

$$\mathbb{E}\left(\sup_{0\le s\le T}\left\|x^{\xi,x}(s) - y^{\xi,x}(t)\right\|^2\right) \le \epsilon, \text{ when } \mathbb{E}\left\|\xi - \zeta\right\|^2 + \mathbb{E}\left\|x - y\right\|^2 < \delta,$$

where $x^{\zeta,y}(t)$ is another solution of the system (1.1) with initial value (ζ, y).

Theorem 4.1. *Let $x(t)$ and $\overline{x}(t)$ be mild solution of the system (1.1) with the initial condition ϕ_1 and ϕ_2 respectively. If the assumptions of Theorem 3.1 gets satisfied, the mean solution of the system (1.1) is stable in the mean square.*

Proof. We may assume that $x(t)$ and $\overline{x}(t)$ be the mild solutions of the system (1.1) with initial values ϕ_1 and ϕ_2 respectively.

$$\begin{aligned}
x(t) - \overline{x}(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{C}(t-t_0)[\phi_1-\phi_2] + \prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathcal{S}(t-t_0)[(\varphi_1-\varphi_2) - [(\mathfrak{h}(0,\phi_1) - (\mathfrak{h}(0,\phi_2))]] \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{C}(t-s)\left[\mathfrak{h}(s,x_s) - \mathfrak{h}(s,\overline{x}(s))\right]ds + \int_{\xi_k}^{t}\mathcal{C}(t-s)\left[\mathfrak{h}(s,x_s) - \mathfrak{h}(s,\overline{x}_s)\right]ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\left[\mathfrak{f}(s,x_s) - \mathfrak{f}(s,\overline{x}_s)\right]ds + \int_{\xi_k}^{t}\mathcal{S}(t-s)\left[\mathfrak{f}(s,x_s) - \mathfrak{f}(s,\overline{x}_s)\right]ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\mathcal{S}(t-s)\left[\mathfrak{g}(s,x_s) - \mathfrak{g}(s,\overline{x}_s)\right]d\omega(s) + \int_{\xi_k}^{t}\mathcal{S}(t-s)\left[\mathfrak{g}(s,x_s) - \mathfrak{g}(s,\overline{x}_s)\right]d\omega(s) \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\xi_{i-1}}^{\xi_i}\int_{\mathfrak{U}}\mathcal{S}(t-s)\left[\sigma(s,x_s,u) - \sigma(s,\overline{x}_s,u)\right]\widetilde{N}(ds,du) \\
&+ \int_{\xi_k}^{t}\int_{\mathfrak{U}}\mathcal{S}(t-s)\left[\sigma(s,x_s,u) - \sigma(s,\overline{x}_s,u)\right]\widetilde{N}(ds,du)\Bigg]I_{[\xi_k,\xi_{k+1})}(t)
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left\|x(t)-\overline{x}(t)\right\|^2\\
\leq\ & 6\mathcal{N}^2\mathcal{M}^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+12\mathcal{N}^2\tilde{\mathcal{M}}^2\mathbb{E}\left\|\varphi_1-\varphi_2\right\|^2+12\mathcal{N}^2\tilde{\mathcal{M}}^2\mathbb{E}\left\|\mathfrak{h}(0,\phi_1)-\mathfrak{h}(0,\phi_2)\right\|^2\\
+\ & 6\mathcal{M}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{h}(s,x_s)-\mathfrak{h}(s,\overline{x}_s)\right\|^2ds+6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\\
\times\ & \int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{f}(s,x_s)-\mathfrak{f}(s,\overline{x}_s)\right\|^2ds+6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathbb{E}\left\|\mathfrak{g}(s,x_s)-\mathfrak{g}(s,\overline{x}_s)\right\|^2ds\\
+\ & 6\max\{1,\mathcal{N}^2\}\tilde{\mathcal{M}}^2\times\Bigg[\int_{t_0}^{t}\int_{\mathfrak{U}}\left\|\sigma(t,x_s,u)-\sigma(t,\overline{x}_s,u)\right\|^2\nu(du)ds\\
+\ & \left(\int_{t_0}^{t}\int_{\mathfrak{U}}\left\|\sigma(t,x_s,u)-\sigma(t,\overline{x}_s,u)\right\|^4\nu(du)ds\right)^{\frac{1}{2}}\Bigg]\\
\leq\ & 6\mathcal{N}^2\mathcal{M}^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+10\mathcal{N}^2\tilde{\mathcal{M}}^2\mathbb{E}\left\|\varphi_1-\varphi_2\right\|^2+12\mathcal{N}^2\tilde{\mathcal{M}}^2\mathcal{L}_{\mathfrak{h}}\mathbb{E}\left\|\phi_1-\phi_2\right\|^2\\
+\ & 6\mathcal{M}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_{\mathfrak{h}}\mathbb{E}\left\|x-\overline{x}\right\|_s^2ds+6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_{\mathfrak{f}}\mathbb{E}\left\|x-\overline{x}\right\|_s^2ds\\
+\ & 6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}Tr(Q)\int_{t_0}^{t}\mathcal{L}_{\mathfrak{g}}\mathbb{E}\left\|x-\overline{x}\right\|_s^2ds\\
+\ & 6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}\int_{t_0}^{t}\mathcal{L}_{\sigma}\mathbb{E}\left\|x-\overline{x}\right\|_s^2ds
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2 \leq\ & 6\mathcal{N}^2\mathcal{M}^2\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+12\mathcal{N}^2\tilde{\mathcal{M}}^2\mathbb{E}\left\|\varphi_1-\varphi_2\right\|^2+12\mathcal{N}^2\tilde{\mathcal{M}}^2\mathcal{L}_{\mathfrak{h}}\mathbb{E}\left\|\phi_1-\phi_2\right\|^2\\
+\ & 6\mathcal{M}^2\max\{1,\mathcal{N}^2\}(T-t_0)\mathcal{L}_{\mathfrak{h}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2+6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\\
\times\ & \mathcal{L}_{\mathfrak{f}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2+6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)Tr(Q)\mathcal{L}_{\mathfrak{g}}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2\\
+\ & 6\tilde{\mathcal{M}}^2\max\{1,\mathcal{N}^2\}(T-t_0)\mathcal{L}_{\sigma}\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2\\
\sup_{t_0\leq t\leq T}\mathbb{E}\left\|x-\overline{x}\right\|_t^2 \leq\ & \frac{6\mathcal{N}^2\left[\mathcal{M}^2+\tilde{\mathcal{M}}^2\mathcal{L}_{\mathfrak{h}}\right]}{1-6\max\{1,\mathcal{N}^2\}(T-t_0)\left[\mathcal{M}^2\mathcal{L}_{\mathfrak{h}}+\tilde{\mathcal{M}}^2[\mathcal{L}_{\mathfrak{f}}+Tr(Q)\mathcal{L}_{\mathfrak{g}}+\mathcal{L}_{\sigma}]\right]}\mathbb{E}\left\|\phi_1-\phi_2\right\|^2\\
+\ & \frac{12\mathcal{N}^2\tilde{\mathcal{M}}^2}{1-6\max\{1,\mathcal{N}^2\}(T-t_0)\left[\mathcal{M}^2\mathcal{L}_{\mathfrak{h}}+\tilde{\mathcal{M}}^2[\mathcal{L}_{\mathfrak{f}}+Tr(Q)\mathcal{L}_{\mathfrak{g}}+\mathcal{L}_{\sigma}]\right]}\mathbb{E}\left\|\varphi_1-\varphi_2\right\|^2\\
\leq\ & \rho\mathbb{E}\left\|\phi_1-\phi_2\right\|^2+\Upsilon\mathbb{E}\left\|\varphi_1-\varphi_2\right\|^2
\end{aligned}
$$

where,

$$
\rho=\frac{5\mathcal{N}^2\left[\mathcal{M}^2+\tilde{\mathcal{M}}^2\mathcal{L}_{\mathfrak{h}}\right]}{1-5\max\{1,\mathcal{N}^2\}(T-t_0)\left[\mathcal{M}^2\mathcal{L}_{\mathfrak{h}}+\tilde{\mathcal{M}}^2[\mathcal{L}_{\mathfrak{f}}+Tr(Q)\mathcal{L}_{\mathfrak{g}}+\mathcal{L}_{\sigma}]\right]}
$$

$$
\Upsilon=\frac{10\mathcal{N}^2\tilde{\mathcal{M}}^2}{1-5\max\{1,\mathcal{N}^2\}(T-t_0)\left[\mathcal{M}^2\mathcal{L}_{\mathfrak{h}}+\tilde{\mathcal{M}}^2[\mathcal{L}_{\mathfrak{f}}+Tr(Q)\mathcal{L}_{\mathfrak{g}}+\mathcal{L}_{\sigma}]\right]}
$$

Given $\epsilon > 0, \mu > 0$ choose, $\lambda = \frac{\epsilon}{\rho}, \mu = \frac{\sigma}{\Upsilon}$ such that,

$$\mathbb{E}\left\|\phi_1 - \phi_2\right\|^2 \leq \lambda \text{ and } \mathbb{E}\left\|\varphi_1 - \varphi_2\right\|^2 \leq \mu$$

Therefore,

$$\left\|x - y\right\|_{\mathcal{B}}^2 \leq \epsilon.$$

Thus the proof is complete.

5. Illustration

In this section, the results obtained are applied to a stochastic partial differential equations with random impulses. Let us consider a space $\mathrm{H} = \mathcal{L}^2([0,\pi])$. The infinitesimal generator $\mathfrak{A}$ is defined to be $\mathfrak{A} : \mathrm{D}(\mathfrak{A}) \subset \mathrm{H} \to \mathrm{H}$ by $\mathfrak{A} = \frac{\partial^2}{\partial x^2}$, with the domain,

$$\mathrm{D}(\mathfrak{A}) = \{z \in \mathrm{H} \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 z}{\partial x^2} \in \mathrm{H}, z(0) = z(\pi) = 0\}$$

. For $z \in \mathrm{D}(\mathfrak{A}), \mathfrak{A}z = -\sum_{n=1}^{\infty} n^2 < z, z_n > z_n$, where $\{z_n : n \in \mathbb{Z}\}$ is an orthonormal basis of H, $z_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}^+, x \in [0,\pi]$. It is known that $\mathfrak{A}$ generates strongly continuous operators $\mathcal{C}(t)$ and $\mathcal{S}(t)$ in a Hilbert space H, such that $\mathcal{C}(t)z = \sum_{n=1}^{\infty} cos(nt) < z, z_n > z_n$, and $\mathcal{S}(t)z = \sum_{n=1}^{\infty} sin(nt)/n < z, z_n > z_n$, for $t \in \mathbb{R}$. And we assume that $\mathcal{S}(t)$ is not a compact semigroup and $\vartheta(\mathcal{S}(t)\mathrm{D}) \leq \vartheta(\mathrm{D})$, where $\mathrm{D} \in \mathrm{H}$ denotes a bounded set, ϑ is the Hausdroff measure of non-compactness.

In the sequel, we may consider second-order neutral stochastic functional differential equation of the form,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} z(t,x) - \frac{\mathfrak{m}_1}{5}\int_{-r}^{0} \varepsilon_1(s) z(t+s,x) ds\right] \qquad (5.1)\\
&= \left[\frac{\partial^2}{\partial x^2} z(t,x) + \frac{\mathfrak{m}_2}{5}\int_{-r}^{0} \varepsilon_1(s) z(t+s) ds\right] dt\\
&+ \frac{m_3}{5}\int_{-r}^{0} \varepsilon_3(s) z(t+s) d\omega(t)\\
&+ \frac{m_4}{5}\int_{\mathfrak{U}}\int_{-r}^{0} \varepsilon_4(s) z(t+s)\widetilde{N}(dt,du),\ t \geq t_0,\ t \neq \xi_{\mathrm{k}},\ x \in [0,\pi],\\
z(\xi_{\mathrm{k}}, x) &= \varrho(\mathrm{k})\delta_{\mathrm{k}} z(\xi_k^-, x), \mathrm{k} = 1,2,3..., \qquad (5.2)\\
\frac{\partial}{\partial t} z(\xi_{\mathrm{k}}, x) &= \varrho(\mathrm{k})\delta_{\mathrm{k}} \frac{\partial}{\partial t} z(\xi_{\mathrm{k}}^-, x),\\
z(t_0, x) &= \phi(\theta, x), \theta \in [-r, 0],\ x \in [0,\pi], r > 0,\\
\frac{\partial}{\partial t} z(t_0, x) &= \varphi(x),\ x \in [0,\pi],\\
z(t,0) &= z(t,\pi) = 0.
\end{aligned}$$

Let δ_k be a random variable defined on $D_k \equiv (0, d_k)$ where, $0 < d_k < +\infty$, for $k = 1, 2, \cdots$. $\xi_0 = t_0 > 0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1, 2, \cdots$. $\omega(t)$ denotes a standard cylindrical Weiner process in H. Furthermore, let ϱ be a function of k. $\varepsilon_i : [-r, 0] \to \mathbb{R}$ are positive functions and $\mathfrak{m}_i > 0$ for $i = 1, 2, 3, 4$. $\left\|\mathcal{C}(t)\right\|, \left\|\mathcal{S}(t)\right\|$ are bounded on $\mathbb{R}$. $\left\|\mathcal{C}(t)\right\| \leq e^{-\pi^2 t}$ and $\left\|\mathcal{S}(t)\right\| \leq e^{-\pi^2 t} (t \geq 0)$.
We may assume that,

(i)The function $\varepsilon(\theta) \geq 0$ is continuous on $[-r, 0]$, $\int_{-r}^{0} \varepsilon_i^2(\theta) d\theta < \infty (i = 1, 2, 3, 4.)$

(ii)$\max_{i,k} = \{\prod_{j=i}^{k} \mathbb{E}[\left\|\varrho(j)\delta_j\right\|^2]\} < \mathcal{N}$.

Using above assumptions and functions $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varrho$ we can show that $\mathcal{L}_{\mathfrak{g}} = \frac{r\mathfrak{m}_1}{25} \int_{-r}^{0} \varepsilon_1^2(\theta) d\theta$,$\mathcal{L}_{\mathfrak{f}} = \frac{r\mathfrak{m}_2}{25} \int_{-r}^{0} \varepsilon_1^2(\theta) d\theta$, $\mathcal{L}_{\mathfrak{h}} = \frac{r\mathfrak{m}_3}{25} \int_{-r}^{0} \varepsilon_1^2(\theta) d\theta$ and $\mathcal{L}_{\sigma} = \frac{r\mathfrak{m}_4}{25} \int_{-r}^{0} \varepsilon_1^2(\theta) d\theta$. Hence stability in mean square of mild solution (5.1) is obtained.

□

6. Conclusion

In this paper, the existence and stability results of second-order neutral stochastic functional systems with random impulse is presented. The existence results of aforementioned system is established using Banach contraction principle. Then the stability of mild solutions through continuous dependence of solutions on initial conditions are calculated.

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