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# On the Relation of the Total Graph of a Ring and a Product of Graphs

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**Kata Kunci:**
*Grup, Total graf, Isomorfisma, Perkalian Kartesius*

**Abstrak.** Graf total atas suatu ring $R$, dinotasikan dengan $T(\Gamma(R))$, didefinisikan sebagai suatu graf dengan himpunan titik $V\left(T(\Gamma(R))\right) = R$ dan dua titik berbeda $u, v \in V\left(T(\Gamma(R))\right)$ bertetangga jika dan hanya jika $u + v \in Z(R)$, di mana $Z(R)$ merupakan pembagi nol dari $R$. Perkalian Kartesius dari dua graf $G$ dan $H$ merupakan suatu graf yang dinotasikan dengan $G \times H$ di mana himpunan titiknya adalah $V(G \times H) = V(G) \times V(H)$ dan dua titik berbeda $(u_1, v_1)$ dan $(u_2, v_2)$ di $V(G \times H)$ bertetangga jika dan hanya jika: 1) $u_1 = u_2$ dan $v_1 v_2 \in H$; atau 2) $v_1 = v_2$ dan $u_1 u_2 \in E(G)$. Isomorfisma dari graf $G$ dan $H$ adalah suatu fungsi bijektif $\phi: V(G) \to V(H)$ sedemikian sehingga $u, v \in V(G)$ bertetangga jika dan hanya jika $f(u), f(v) \in V(H)$ bertetangga. Akan dibuktikan bahwa graf $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ isomorf dengan graf $P_2 \times K_p$ untuk setiap bilangan prima $p$.

**Keywords:**
*Group, Total graph, Isomorphism, Cartesian product*

**Abstract.** The total graph of a ring $R$, denoted as $T(\Gamma(R))$, is defined to be a graph with vertex set $V\left(T(\Gamma(R))\right) = R$ and two distinct vertices $u, v \in V\left(T(\Gamma(R))\right)$ are adjacent if and only if $u + v \in Z(R)$, where $Z(R)$ is the zero divisor of $R$. The Cartesian product of two graphs $G$ and $H$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if: 1) $u_1 = u_2$ and $v_1 v_2 \in H$; or 2) $v_1 = v_2$ and $u_1 u_2 \in E(G)$. An isomorphism of graphs $G$ dan $H$ is a bijection $\phi: V(G) \to V(H)$ such that $u, v \in V(G)$ are adjacent if and only if $f(u), f(v) \in V(H)$ are adjacent. This paper proved that $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ and $P_2 \times K_p$ are isomorphic for every odd prime $p$.

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## 1. Introduction

Investigating group or ring properties and its structures from their graph representation become a new trend in graph theoretic research. Many authors proved that there are tight bonds between the rings and graphs. Aalipour in [1] investigated the chromatic number and clique number of a commutative ring. [2] gave a novel application of a central-vertex complete graph to a commutive ring. In 2008, [3] investigated the commutive graph of rings generated from matrices over a finite filed. Three years after the graph of a ring was introduced in [4], [5] proposed useful applications of semirings in mathematics and theoretical computer science. One interest in applying graph invariant on a group also showed in [6] in the properties of zero-divisor graphs. Another useful graph generated from group or ring structure is Cayley graphs which has many useful applications in solving and understanding a variety problem in several scientific interests [7].

The graph isomorphism itself has many applications in real life and many scientific fields [8]. [9] stated briefly about its application in the atomic structures and [10] showed how it can be applied in biochemical data. To prove the isomorphism of two graphs is an NP-problem in which there is no specific algorithm or certain way that works for all graphs in consideration [11]. In 1996, [12] proposed a good graph isomorphism algorithm but still troublesome for a large graphs.

Considering those applications of ring generated graphs, the applications of the graph isomorphisms, and the isomorphism-related algorithm complexity, finding an isomorphism of ring-structured graphs and the graph obtained from certain operation is a challenging task and a potential new interest in graph theory research. This paper considers the relation between the total graph of $\mathbb{Z}_p$ and $P_2 \times K_p$ for all odd prime $p$.

## 2. Preliminaries

A graph $G$ is a pair $G = (V, E)$ for non-empty set $V$ and $E \subseteq [V]^2$ (the elements of $E$ are 2-element subsets of $V$). For terminologies and notations concerning to a graph and its invariants, please consider [13]. This preliminary covers the definitions related to ring and the total graph of a ring. It also provides some definitions related to graph isomorphism and a graph operation.

**Definition 1. Ring** [14]
A ring $R$ is a set with two binary operations, addition and multiplication, such that for all $a, b, c \in R$:
1. $a + b = b + a$,
2. $(a + b) + c = a + (b + c)$,
3. There is an additive identity 0,
4. There is an element $-a \in R$ such that $a + (-a) = 0$,
5. $a(bc) = (ab)c$, and
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

With this definition, $\mathbb{Z}_{2p}$, an integer modulo $2p$ set, equipped with addition and multiplication modulo $2p$ operation is a ring.

**Definition 2. Zero Divisor** [14]
A zero-divisor is a nonzero element $a$ of a commutative ring $R$ such that there is a nonzero element $b \in R$ with $ab = 0$.

**Definition 3. Total Graph of a Ring** [15]
Let $R$ be a ring and $Z(R)$ denotes the zero divisor of $R$. The total graph of $R$, denoted by $T\big(\Gamma(R)\big)$ is an undirected graph with elements of $R$ as its vertices, and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$.

From those definitions of zero divisor and total graph, we will construct a total graph of $\mathbb{Z}_{2p}$ for an odd prime $p$.

**Definition 4. Graph Homomorphism and Isomorphism** [13]
Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. A map $\varphi: V_G \to V_H$ is a homomorphism from $G$ to $H$ if it preserves the adjacency of the vertices. In another word, $\{x, y\} \in E_G \Rightarrow \{\varphi(x), \varphi(y)\} \in E_H$. If $\varphi$ is bijective and $\varphi^{-1}$ is also a homomorphism, then $\varphi$ is an isomorphism and $G$ is said to be isomorphic to $H$.

**Definition 5. Cartesian Product** [13]
The Cartesian product of two graphs $G$ and $H$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if: 1) $u_1 = u_2$ and $v_1 v_2 \in H$; or 2) $v_1 = v_2$ and $u_1 u_2 \in E(G)$. An isomorphism of graphs $G$ dan $H$ is a bijection $\phi: V(G) \to V(H)$ such that $u, v \in V(G)$ are adjacent if and only if $f(u), f(v) \in V(H)$ are adjacent.

## 3. Main Results

In this section we will prove the isomorphism of the total graph of $\mathbb{Z}_{2p}$ and $P_2 \times K_p$. We will investigate several properties of $T\left(\Gamma\big(\mathbb{Z}_{2p}\big)\right)$ before we proof the isomorphism. Those investigations will be provided as lemmas and theorems equipped with their proofs. To characterize $T\left(\Gamma\big(\mathbb{Z}_{2p}\big)\right)$, we consider its vertex set, the degree of each vertex, and the clique it has as subgraphs, since $P_2 \times K_p$ can easily be considered and seen from those properties.

**Lemma 1.** The zero divisor of $\mathbb{Z}_{2p}$ is
$$Z\big(\mathbb{Z}_{2p}\big) = \{p\} \cup \{2n : n = 1, 2, \dots, n - 1\}$$
for every odd prime $p$.
**Proof.**
For each $x \in \mathbb{Z}_{2p}$, the exactly one of the following holds: $x = p$, $x$ is even, and $x \neq p$ is odd.
**Case 1**, $x = p$
Since $2p = 0$ and $2 \in \mathbb{Z}_{2p}$, we conclude that $p \in Z\big(\mathbb{Z}_{2p}\big)$.

**Case 2**, $x$ is even
Let $x = 2m$ for some $m \in \mathbb{Z}$. Since $xp = 2mp = m \cdot 2p = m \cdot 0 = 0$ and $p \in \mathbb{Z}_{2p}$, we conclude that $x \in Z\big(\mathbb{Z}_{2p}\big)$ for all even $x \in \mathbb{Z}_{2p}$.

**Case 3**, $x \neq p$ is odd
If $x = 1$, then $xy \neq 0$ for all $0 \neq y \in \mathbb{Z}_{2p}$.
We will show that $1 \neq x \notin Z\big(\mathbb{Z}_{2p}\big)$ by using a contradiction. Suppose on the contrary, that $x \in Z\big(\mathbb{Z}_{2p}\big)$. Consequently, there exists $0 \neq y \in \mathbb{Z}_{2p}$ such that $xy = 0$. It follows that $\gcd(x, 2p) > 1$. Since the factor of $2p$ is 2 and $p$, we obtain that $x$ divides $p$. It is a contradiction since $p$ is a prime number.

**Lemma 2**. Let $p$ be an odd prime. Let $A \subseteq \mathbb{Z}_{2p}$ be the set of all odd elements of $\mathbb{Z}_{2p}$ and $B \subseteq \mathbb{Z}_{2p}$ be the set of all even elements of $\mathbb{Z}_{2p}$. $\{u, v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$ for all $u, v \in A$ and $\{x, y\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$ for all $x, y \in B$. In another word, the vertices in $A$ dan $B$ form cliques in $T\left(\Gamma(\mathbb{Z}_{2p})\right)$.

**Proof.**
Let $u, v \in A$ and let $u = 2s + 1$ and $v = 2t + 1$ for some $s, t \in \mathbb{Z}$. We obtain

$$\begin{aligned} u + v &= 2s + 1 + 2t + 1 \\ &= 2(s + t + 1) \in Z(\mathbb{Z}_{2p}). \end{aligned}$$

Therefore $\{u, v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$ for all $u, v \in A$.
Let $x, y \in A$ and let $x = 2s$ and $y = 2t$ for some $s, t \in \mathbb{Z}$. We obtain

$$\begin{aligned} x + y &= 2s + 2t \\ &= 2(s + t) \in Z(\mathbb{Z}_{2p}). \end{aligned}$$

Therefore $\{x, y\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$ for all $x, y \in B$.
It proves that $A$ and $B$ form cliques in $T\left(\Gamma(\mathbb{Z}_{2p})\right)$.

**Lemma 3**. Let $A$ and $B$ be sets defined in Lemma 2 and $p$ be an odd prime number. For each $v \in A$ there is a unique $x \in B$ such that

$$\{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right).$$

**Proof.**
For each $v \in A$, choose $x = p - v$. It can be easily verified that $x \in B$ since $p$ and $v$ are both odd numbers. On the other hand, let $x \in B$ and $x \neq p - v$. Suppose that $\{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$, that is $v + x \in Z(\mathbb{Z}_{2p})$. Since $v$ is odd and $x$ is even, it follows that $v + x$ is an odd number and $v + x = p \Leftrightarrow x = p - v$, a contradiction. This proves that $\{v, p - v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right), \forall x \in A$.
Analogous to this proof, we can easily prove that for each $x \in B$ there is a unique $v \in A$ such that

$$\{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right).$$

Before we discuss the main problem, consider Figure 1 that represents the graph $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ for several $p$.

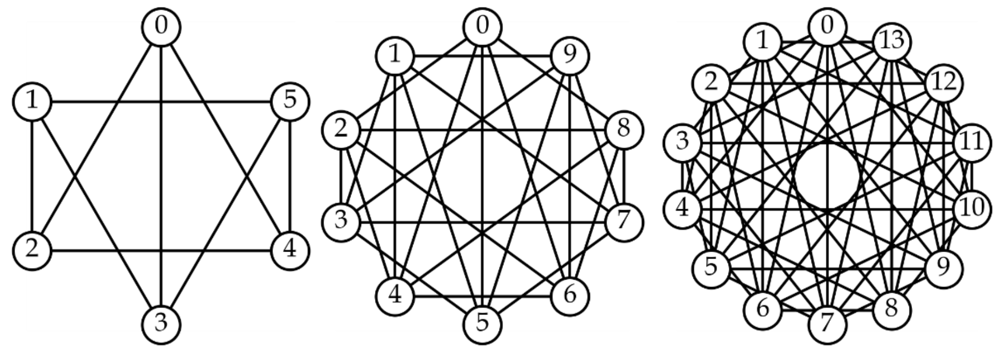


**Figure 1.** $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ for $p \in \{3,5,7\}$.

**Theorem 1.** For any odd prime $p$, $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ is isomorph to $P_2 \times K_p$.

**Proof.** Let $V(P_2)$ and $V(K_p)$ be labeled as $\{p_1, p_2\}$ and $\{k_0, k_1, \ldots, k_{p-1}\}$ respectively. The vertices of the resulting graph obtained from the Cartesian product, $P_2 \times K_p$, is therefore labeled

$$\{(p_1, k_1), (p_1, k_2), \ldots, (p_1, k_p), (p_2, k_1), (p_2, k_2), \ldots, (p_2, k_p)\}$$

in which

$$\{(p_s, k_i), (p_s, k_j)\} \in E(P_2 \times K_p), \forall i, j \in \{1,2, \ldots, p\}$$

and $i \neq j$, for $s \in \{1,2\}$. Other edges to consider is $\{(p_1, k_i), (p_2, k_{(p-i \bmod p)+1})\} \in E(P_2 \times K_p), \forall i \in \{1,2, \ldots, p\}$. Here, the "mod" in "$p - i \bmod p$" is a modulus operator, not a modulus relation.

Consider the function $\varphi: V\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \to V(P_2 \times K_p)$ defined as follows:

$$\varphi(x) = \begin{cases} \left(P_1, \left(\frac{p-x}{2} \bmod p\right) + 1\right), \text{if } x \text{ is odd} \\ \left(P_2, \frac{x}{2} + 1\right), \text{if } x \text{ is even.} \end{cases}$$

Since $\varphi$ is a bijective function that preserves adjacency of the vertices of $V\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right)$ and $V(P_2 \times K_p)$, we conclude that $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ and $P_2 \times K_p$ are isomorphic.

Figure 1 and Figure 2 show some examples of the mapping result of $\varphi$.

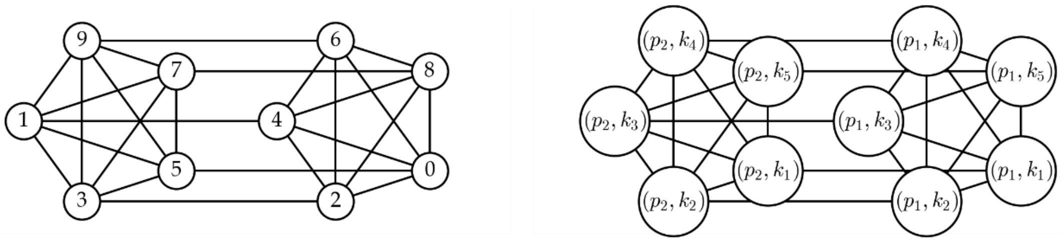


**Figure 2.** the mapping result of $\varphi$ from $T(\Gamma(\mathbb{Z}_{2\cdot 5}))$ to $P_2 \times K_5$

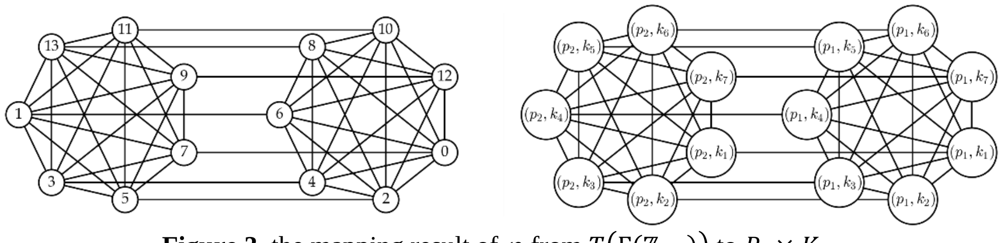


**Figure 3.** the mapping result of $\varphi$ from $T(\Gamma(\mathbb{Z}_{2\cdot 7}))$ to $P_2 \times K_7$

## 4. Conclusion

From the discussion, we conclude that $T\left(\Gamma(\mathbb{Z}_{2p})\right)$ and $P_2 \times K_p$ are isomorphic.

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