Mathematics in Applied Sciences and Engineering https://ojs.lib.uwo.ca/mase
Volume **4**, Number **1**, March **2023**, pp.**61-78** https://doi.org/10.5206/mase/15536

ENERGY CRITERIA OF GLOBAL EXISTENCE FOR THE HARTREE EQUATION WITH COULOMB POTENTIAL

NA TANG, CHENGLIN WANG, AND JIAN ZHANG

Abstract. This paper studies a class of Hartree equations with Coulomb potential. Combined with the conservation of mass and energy, we analyze the variational characteristics of the corresponding nonlinear elliptic equation. According to the range of parameters, we construct the evolution invariant flows of the equation in different cases. Then the sharp energy thresholds for global existence and blow-up of solutions are discussed in detail.

1. Introduction

In this paper, we study a class of Hartree equations with Coulomb potential:

$$i\varphi_t + \Delta\varphi + \beta|x|^{-1}\varphi + (|x|^{-\gamma} * |\varphi|^2)\varphi + |\varphi|^p\varphi = 0,\ t > 0, x \in \mathbb{R}^n, \tag{1.1}$$

where

$$n \geq 3, \quad 2 < \gamma < min\{4,\ n\}, \quad 0 \leq \beta < \frac{(n-2)^2(\gamma-2)}{2(\gamma-1)}, \quad 0 < p < \frac{4}{n-4},$$

and $\varphi = \varphi(t,x)$ is a complex value wave function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Equation (1.1) is considered as the first-principle model for beam-matter interaction in X-ray free electron lasers (XFEL)[1, 4, 9]. The parameter β denotes the strength of an electron beam interaction with external Coulomb force. Recent developments using XFEL include the motion of atoms, measuring the dynamics of atomic vibrations and biomolecular imaging [3, 8, 23]. Besides, in the context of BEC, such a model equation is also known as the Gross-Pitaevskii for dipole Bose-Einstein condensation with Coulomb potential[24].

For (1.1), the local well-posedness was established in [6, 10]. Feng and Zhao [10] obtained the global well-posedness for (1.1) under some assumptions. In [15], authors proved the existence of ground states and normalized solutions for (1.1) with harmonic potential. If we remove the term $\beta|x|^{-1}$ in (1.1), this equation may occur blow up in finite time for the whole range of p, see [25, 26]. To our knowledge, the existence of blowup and the sharp criteria of global existence for (1.1) has not been studied in the literature.

We recall the Hartree equation:

$$i\varphi_t + \Delta\varphi + (|x|^{-(n-2)} * |\varphi|^\alpha)|\varphi|^{\alpha-2}\varphi = 0,\ t > 0, x \in \mathbb{R}^n. \tag{1.2}$$

When $\alpha = 2$, the equation (1.2) becomes Choquard-Pekar equation, which occurs in the modelling of quantum semiconductor devices, the electron transport and the electron-electron interaction(see [17]). There are numerous results for equation (1.2). When $n \geq 3$, $2 \leq \alpha \leq 1 + \frac{4}{n-2}$, Genev and Venkov [13] proved the local and global well-posedness and the existence of blow-up solutions. The dynamics

Received by the editors 23 November 2022; accepted 24 March 2023; published online 30 March 2023.
2010 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. Hartree equation, Coulomb potential, blow-up, global existence, sharp energy threshold.

of blow-up solutions was investigated in [5, 20, 22, 28, 29]. In [2, 12, 21], they showed the sharp criteria for blow-up and scattering in $H^1(\mathbb{R}^n)$. Huang, Zhang, Chen [16] and Tian, Yang, Zhou [25] showed the sharp criteria of global existence for the Hartree equation with subcritical perturbations. And Leng, Li, Zheng [18] showed the sharp criteria of global existence for the Hartree equation with supercritical perturbations. In [26], they detected the dynamical properties of blow-up solutions. Lieb [17] showed the uniqueness of the radial symmetric standing wave in $\mathbb{R}^3$.

The nonlinear Schrödinger equation with Coulomb potential is as follows:

$$i\varphi_t + \Delta\varphi + \beta|x|^{-1}\varphi = \lambda f(|\varphi|^2)\varphi,\ t > 0, x \in \mathbb{R}^n. \tag{1.3}$$

When $\beta > 0$, it provides a quantum mechanical description of Coulomb force between two charged particles and corresponds to having an external attractive long-range potential due to the presence of a positively charged atomic nucleus(see [19]). When $\beta \leq 0$ and $f(|\varphi|^2) = |x|^{-1} * |\varphi|^2$, Chadam, Glassey [5] obtained the existence of the unique global solution in $H^1(\mathbb{R}^3)$. Hayashi, Ozawa [14] showed the global existence and a decay rate of solutions when the initial data belongs to a weighted-L^2 space. For (1.1), we construct different invariant flows under different parameter ranges. Then we obtain the sharp energy thresholds for global existence and blow-up of solutions for (1.1). We mainly consider the following cases:

(1) $0 < p < \frac{2}{n}$, $2 < \gamma < \min\{n, 4\}$;
(2) $p = \frac{2}{n}$, $2 < \gamma < \min\{n, 4\}$;
(3) $\frac{2}{n} < p < \frac{4}{n}$, $2 < \gamma < \min\{n, 4\}$;
(4) $p = \frac{4}{n}$, $2 < \gamma < \min\{n, 4\}$;
(5) $\frac{4}{n} < p < \frac{4}{n-2}$, $2 < \gamma < \frac{np}{2}$;
(6) $\frac{4}{n} < p < \frac{4}{n-2}$, $\frac{np}{2} \leq \gamma < \min\{4, n\}$.

This paper is organized as follows: in Section 2, we establish some basic facts including local well-posedness, the conservation laws of mass and energy, and sharp inequalities. In Section 3, we give the sharp energy thresholds of blow-up and global existence for (1.1).

2. Preliminaries

We impose the initial data of (1.1) as follows

$$\varphi(0, x) = \varphi_0,\ x \in \mathbb{R}^n. \tag{2.1}$$

For the Cauchy problem (1.1) and (2.1), we define the energy space as

$$H^1(\mathbb{R}^n) := \{v : v \in L^2(\mathbb{R}^n), \nabla v \in L^2(\mathbb{R}^n)\}, \tag{2.2}$$

and introduce the inner product

$$(u, v) := \int \nabla u \cdot \nabla \overline{v} + u\overline{v}dx, \tag{2.3}$$

whose associated norm denoted by $\|\cdot\|_{H^1}$. Here and hereafter, for simplicity, we use $\int \cdot dx$ to denote $\int_{\mathbb{R}^n} \cdot dx$.

Lemma 2.1. [6, 10] *Assume $\varphi_0 \in H^1(\mathbb{R}^n)$, there exists a unique solution $\varphi(t)$ of the Cauchy problem (1.1) and (2.1) in $\mathbb{C}([0, T); H^1(\mathbb{R}^n))$ for some $T \in (0, \infty]$ (maximal existence time). We have the*

alternatives $T = \infty$ *(global existence) or else* $T < \infty$ *and* $\lim_{t \to T} \|\varphi(t)\|_{H^1} = \infty$ *(blow up). Moreover for all* $t \in [0, T)$*, the solution* $\varphi(t)$ *satisfies the following:*
(i) Conservation of mass:

$$\int |\varphi(t)|^2 dx = \int |\varphi_0|^2 dx. \tag{2.4}$$

(ii) Conservation of energy:

$$E(\varphi(t)) = \int \frac{1}{2}|\nabla\varphi(t)|^2 - \frac{\beta}{2}|x|^{-1}|\varphi(t)|^2 - \frac{1}{4}(|x|^{-\gamma} * |\varphi(t)|^2)|\varphi(t)|^2 - \frac{1}{p+2}|\varphi(t)|^{p+2} dx = E(\varphi_0). \tag{2.5}$$

By a direct calculation, we have the following result.

Lemma 2.2. *Let* $\varphi_0 \in H^1(\mathbb{R}^n)$*,* $\int |x|^2|\varphi_0|^2 dx < \infty$ *and* $\varphi(t, x)$ *be a solution of the Cauchy problem* (1.1) *and* (2.1)*. Put* $J(t) := \int |x|^2|\varphi(t, x)|^2 dx$*, then one has*

$$\begin{aligned} J''(t) &= \int 8|\nabla\varphi|^2 - 4\beta|x|^{-1}|\varphi|^2 - 2\gamma(|x|^{-\gamma} * |\varphi|^2)|\varphi|^2 - \frac{4np}{p+2}|\varphi|^{p+2} dx \\ &= 8\gamma E(\varphi_0) + \int \frac{8\gamma - 4np}{p+2}|\varphi|^{p+2} - 4(\gamma - 2)|\nabla\varphi|^2 + (4\gamma - 4)\beta|x|^{-1}|\varphi|^2 dx. \end{aligned} \tag{2.6}$$

Lemma 2.3. [27] *Let* $\varphi_0 \in H^1(\mathbb{R}^n)$ *and* $\int |x|^2|\varphi_0|^2 dx < \infty$*. Then the following estimate holds:*

$$\int |\varphi|^2 dx \le \frac{2}{n}\Big(\int |\nabla\varphi|^2 dx\Big)^{\frac{1}{2}}\Big(\int |x|^2|\varphi|^2 dx\Big)^{\frac{1}{2}}. \tag{2.7}$$

Lemma 2.4. [27] *For* $0 < p < \dfrac{4}{n-2}$ *and* $v \in H^1(\mathbb{R}^n)$*,*

$$\|v\|_{p+2}^{p+2} \le \frac{2(p+2)}{np\|\nabla R\|_2^p}\|v\|_2^{\frac{4-(n-2)p}{2}}\|\nabla v\|_2^{\frac{np}{2}}, \tag{2.8}$$

where R *is the unique positive ground state solution of equation:*

$$-\Delta R + \frac{4-(n-2)p}{np}R - |R|^p R = 0, R \in H^1(\mathbb{R}^n). \tag{2.9}$$

Lemma 2.5. [7, 29] *For* $0 < \gamma < \min\{4, n\}$ *and* $v \in H^1(\mathbb{R}^n)$*, one has*

$$\|(|x|^{-\gamma} * |v|^2)|v|^2\|_1 \le \frac{4}{\gamma\|\nabla W\|_2^2}\|v\|_2^{4-\gamma}\|\nabla v\|_2^{\gamma}, \tag{2.10}$$

where W *is a positive ground state solution of equation:*

$$-\Delta W + \frac{4-\gamma}{\gamma}W - (|x|^{-\gamma} * |W|^2)W = 0, W \in H^1(\mathbb{R}^n). \tag{2.11}$$

Lemma 2.6. [11] *Assume* $1 < \alpha < n, v \in W^{1,\alpha}(\mathbb{R}^n)$*, then*

$$\int \frac{|v|^\alpha}{|x|^\alpha} dx \le \Big(\frac{\alpha}{n-\alpha}\Big)^\alpha \int |\nabla v|^\alpha dx. \tag{2.12}$$

In the end, for simplicity, we denote

$$c_0 = \frac{1}{2} + \frac{\beta}{4} - \frac{\beta}{(n-2)^2}, \qquad \frac{1}{a_0} = \frac{1}{2} - \frac{\beta}{(n-2)^2}.$$

3. Sharp energy thresholds

In this section, we state the sharp criteria for global existence and blow up of (1.1). According to the range of parameters p and γ, we show the results in the following six cases.

Case I: $0 < p < \frac{2}{n}$, $2 < \gamma < \min\{n, 4\}$. In this case, we have three theorems. Let

$$a_1 = \frac{2}{2^{\frac{np}{2}} np \|\nabla R\|_2^p} \|\varphi_0\|_2^{\frac{4-2np+2p}{2}}, a_2 = \frac{1}{2^\gamma \gamma \|\nabla W\|_2^2} \|\varphi_0\|_2^{4-2\gamma},$$
$$D_1 = (\frac{2-np}{2\gamma-2})^{\frac{np-2}{2\gamma-np}} + (\frac{2-np}{2\gamma-2})^{\frac{2\gamma-2}{2\gamma-np}},$$
$$D_2 = \frac{np}{2}[\frac{np(2-np)}{4\gamma(\gamma-1)}]^{\frac{np-2}{2\gamma-np}} + \gamma[\frac{np(2-np)}{4\gamma(\gamma-1)}]^{\frac{2\gamma-2}{2\gamma-np}},$$
$$b_1 = [\frac{2^{2-np}(np)^{2\gamma-2}\gamma^{2-np}\|\nabla R\|_2^{2p\gamma-2p}\|\nabla W\|_2^{4-2np}}{(a_0 D_1)^{2\gamma-np}}]^{\frac{1}{4-2np+2p\gamma-2p}},$$
$$b_2 = [\frac{2^{2-np}(np)^{2\gamma-2}\gamma^{2-np}\|\nabla R\|_2^{2p\gamma-2p}\|\nabla W\|_2^{4-2np}}{(a_0 D_2)^{2\gamma-np}}]^{\frac{1}{4-2np+2p\gamma-2p}},$$
$$K_1 = \frac{np-2}{4\gamma}[\frac{(n-2)^2(2\gamma a_1 - npa_1)^{\frac{2}{np}} np}{(2\gamma-4)(n-2)^2 - 4\beta(\gamma-1)}]^{\frac{np}{2-np}}.$$

Under the constraint : $\|\varphi_0\|_2 < b_1$, we define two invariant sets:

$$G_1 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_1, \|\varphi\|_{H^1}^2 < y_1\},$$

$$B_1 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_1, \|\varphi\|_{H^1}^2 > y_1\},$$

where y_1 is the unique positive maximizer of :

$$f_1(y) := \frac{1}{a_0} y - a_1 y^{\frac{np}{2}} - a_2 y^\gamma. \tag{3.1}$$

Let $\widetilde{y_1} > 0$ be the first positive root of equation $f_1'(y) = \frac{d}{dy} f_1(y) = 0$.

Theorem 3.1. *For $0 < p < \frac{2}{n}$ and $2 < \gamma < \min\{n, 4\}$. Assume $\|\varphi_0\|_2 < b_1$, then the following facts are true:*

(i) *When $\varphi_0 \in G_1 \cup \{0\}$ and $f_1(\widetilde{y_1}) < K_1$, the solution $\varphi(t, x)$ of the Cauchy problem* (1.1) *and* (2.1) *exists globally in $t \in (0, \infty)$.*

(ii) *When $\varphi_0 \in B_1$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t, x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0, T]$,

$$\begin{aligned} E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq\; & \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + c_0\|\varphi\|_2^2 \\ & - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^\gamma - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} \\ \geq\; & [\frac{1}{2} - \frac{\beta}{(n-2)^2}]\|\varphi\|_{H^1}^2 - \frac{1}{2^\gamma\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-2\gamma}\|\varphi\|_{H^1}^{2\gamma} \\ & - \frac{2}{2^{\frac{np}{2}} np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-2np+2p}{2}}\|\varphi\|_{H^1}^{np}. \end{aligned} \tag{3.2}$$

Let $y = \|\varphi(t)\|_{H^1}^2 \geq 0$, for all $t \in (0, T]$,

$$E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq f_1(\|\varphi(t)\|_{H^1}^2) = f_1(y), \tag{3.3}$$

where f_1 is defined in (3.1).

Secondly, we claim that the maximum of $f_1(y)$ on $[0, +\infty)$ is greater than 0. Let

$$g(y) = \frac{1}{a_0} - a_1 y^{\frac{np}{2}-1} - a_2 y^{\gamma-1}.$$

It follows that $f_1(y) = yg(y)$, $\lim\limits_{y\to 0^+} g(y) = \lim\limits_{y\to+\infty} g(y) = -\infty$ and $g'(y)$ has only one zero point

$$y_0 = [\frac{(2-np)a_1}{2(\gamma-1)a_2}]^{\frac{2}{2\gamma-np}}.$$

Thus the maximum of $g(y)$ on $[0, +\infty)$ is $g(y_0)$. From $\|\varphi_0\|_2 < b_1$, we can obtain

$$a_1^{2\gamma-2} a_2^{2-np} < (a_0 D_1)^{np-2\gamma},$$

which implies

$$g(y_0) = \frac{1}{a_0} - a_1 y_0^{\frac{np}{2}-1} - a_2 y_0^{\gamma-1} > 0.$$

Note that $f_1(y) \to 0^-$ as $y \to 0^+$ and $f_1(y) \to -\infty$ as $y \to +\infty$. Therefore, $f_1(y)$ has the unique positive maximizer y_1 on $[0, \infty)$ and $f_1(y_1) \geq y_0 g(y_0) > 0$.

Thirdly, we prove the invariance of G_1 and B_1. When $f_1(\widetilde{y_1}) < K_1$, combined with the structure of $f_1(y)$, we can easily know that G_1 is a nonempty set. If $\varphi_0 \in G_1$, $f_1(\widetilde{y_1}) < K_1$ and $\varphi(t, x)$ is the corresponding solution of the Cauchy problem (1.1) and (2.1), then by Lemma 2.1, we have for all $t \in [0, T)$,

$$f_1(\|\varphi\|_{H^1}^2) \leq E(\varphi) + c_0\|\varphi\|_2^2 < K_1. \tag{3.4}$$

We only need to prove $\|\varphi\|_{H^1}^2 < y_1$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0, T)$ such that $\|\varphi(\bar{t})\|_{H^1}^2 = y_1$, and then

$$f_1(\|\varphi(\bar{t})\|_{H^1}^2) = f_1(y_1) > K_1,$$

which contradicts (3.4). Thus $\|\varphi\|_{H^1}^2 < y_1$, which implies the solution $\varphi(t, x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0, \infty)$. We can obtain B_1 is a nonempty invariant set by the same token.

Finally, we prove the statement (ii) of Theorem 3.1. From (2.6), we have

$$\begin{aligned} J''(t) &= 8\gamma E(\varphi_0) + \int \frac{8\gamma - 4np}{p+2}|\varphi|^{p+2} - 4(\gamma-2)|\nabla\varphi|^2 + (4\gamma-4)\beta|x|^{-1}|\varphi|^2 dx \\ &\leq 8\gamma E(\varphi) + \frac{16\gamma - 8np}{np}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} - 4(\gamma-2)\|\nabla\varphi\|_2^2 \\ &\quad + (4\gamma-4)\beta[\frac{2}{(n-2)^2}\|\nabla\varphi\|_2^2 + \frac{1}{2}\|\varphi\|_2^2] \\ &\leq 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_1(y), \end{aligned} \tag{3.5}$$

where

$$H_1(y) = (8\gamma - 4np)a_1 y^{\frac{np}{2}} + [\frac{8\beta(\gamma-1)}{(n-2)^2} - 4\gamma + 8]y.$$

$H_1'(y)$ has only one zero point y^* on $[0, +\infty)$,

$$y^* = [\frac{(n-2)^2(2\gamma - np)npa_1}{(n-2)^2(2\gamma-4) - 4\beta(\gamma-1)}]^{\frac{2}{2-np}}.$$

$H_1(y)$ is increasing on$(0, y^*)$ and decreasing on $(y^*, +\infty)$, so the maximum of $H_1(y)$ is

$$H_1(y^*) = (4-2np)[\frac{(n-2)^2(2\gamma a_1 - npa_1)^{\frac{2}{np}}np}{(n-2)^2(2\gamma-4)-4\beta(\gamma-1)}]^{\frac{np}{2-np}} = -8\gamma K_1. \tag{3.6}$$

By the invariance of B_1, if $\varphi_0 \in B_1$ then for all $t \in [0,T)$,

$$f_1(\|\varphi\|^2_{H^1}) \leq E(\varphi) + c_0\|\varphi\|^2_2 < K_1.$$

Inserting the results into (3.5), we obtain

$$J''(t) \leq 8\gamma[E(\varphi) + c_0\|\varphi\|^2_2] + H_1(y^*) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies that the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.1. □

Under the constraint : $b_1 \leq \|\varphi_0\|_2 < b_2$, we define two invariant sets:

$$G_2 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|^2_2 < K_1, \|\varphi\|^2_{H^1} < y_2\},$$

$$B_2 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|^2_2 < K_1, \|\varphi\|^2_{H^1} > y_2\},$$

where y_2 is the unique positive maximizer of equation (3.1).Let $\widetilde{y_2} > 0$ be the first positive root of the equation $f_1'(y) = \dfrac{d}{dy}f_1(y) = 0$ under the constraint $b_1 \leq \|\varphi_0\|_2 < b_2$.

Theorem 3.2. *For $0 < p < \dfrac{2}{n}$ and $2 < \gamma < \min\{n, 4\}$. Assume $b_1 \leq \|\varphi_0\|_2 < b_2$, then the following facts are true:*

(i) *When $\varphi_0 \in G_2 \cup \{0\}$ and $f_1(\widetilde{y_2}) < K_1$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *exists globally in $t \in (0,\infty)$.*

(ii) *When $\varphi_0 \in B_2$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, we claim that $f_1(y) \leq 0$ and $f_1(y)$ has two extrema on $[0,+\infty)$. When $b_1 \leq \|\varphi_0\|_2 < b_2$, we have

$$f_1'(y) = \frac{1}{a_0} - \frac{npa_1}{2}y^{\frac{np}{2}-1} - \gamma a_2 y^{\gamma-1},$$

and

$$f_1''(y) = \frac{np(2-np)a_1}{4}y^{\frac{np}{2}-2} - \gamma(\gamma-1)a_2y^{\gamma-2}.$$

Then $f_1'(y) \to -\infty$ as $y \to 0^+$ or $y \to +\infty$, and $f_1''(y)$ has only one zero point y_m,

$$y_m = [\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{2}{2\gamma-np}},$$

so the maximum of $f_1'(y)$ on $[0,\infty)$ is

$$f_1'(y_m) = \frac{1}{a_0} - \frac{npa_1}{2}[\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{np-2}{2\gamma-np}} - \gamma a_2[\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{2\gamma-2}{2\gamma-np}}. \tag{3.7}$$

By $b_1 \leq \|\varphi_0\|_2 < b_2$, we can get

$$(a_0D_1)^{np-2\gamma} \leq a_1^{2\gamma-2}a_2^{2-np} < (a_0D_2)^{np-2\gamma},$$

which implies that $f_1(y) \leq 0$ and $f_1'(y_m) > 0$. Note that $f_1'(y)$ is increasing on $(0, y_m)$ and decreasing on $(y_m, +\infty)$. Therefore $f_1'(y)$ has two zero points on $[0,+\infty)$, it follows that $f_1(y)$ has two extrema on $[0,+\infty)$. Let y_3 represent the minimal point and y_2 represent the maximal point. It is not hard to find

$$y_3 < y_2 \ , \ f_1(y_2) > K_1.$$

And then, the same as the proof of Theorem 3.1, we can verify that both G_2 and B_2 are nonempty invariant sets. Thus we obtain that the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0,\infty)$. Besides, we can also verify

$$J''(t) \leq 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_1(y^*) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.2. □

Under the constraint : $\|\varphi_0\|_2 \geq b_2$, we define the following invariant set:

$$B_3 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_1, \|\varphi\|_{H^1}^2 > y_k\},$$

where y_k is the unique positive solution of $f_1(y) = K_1$. Then we get a sufficient condition for blow-up of solutions.

Theorem 3.3. *Let* $0 < p < \frac{2}{n}$, $2 < \gamma < \min\{n, 4\}$ *and* $|x|\varphi_0 \in L^2(R^n)$. *When* $\|\varphi_0\|_2 \geq b_2$ *and* $\varphi_0 \in B_3$, *the solution* $\varphi(t,x)$ *of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, we claim that $f_1(y) \leq 0$ and $f_1(y)$ has no extrema on $[0,+\infty)$. When $\|\varphi_0\|_2 \geq b_2$, we have

$$f_1'(y) = \frac{1}{a_0} - \frac{npa_1}{2} y^{\frac{np}{2}-1} - \gamma a_2 y^{\gamma-1},$$

and

$$f_1''(y) = \frac{np(2-np)a_1}{4} y^{\frac{np}{2}-2} - \gamma(\gamma-1)a_2 y^{\gamma-2}.$$

Then $f_1'(y) \to -\infty$ as $y \to 0^+$ or $y \to +\infty$, and $f_1''(y)$ has only one zero point y_m,

$$y_m = [\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{2}{2\gamma-np}},$$

so the maximum of $f_1'(y)$ on $[0,\infty)$ is

$$f_1'(y_m) = \frac{1}{a_0} - \frac{npa_1}{2}[\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{np-2}{2\gamma-np}} - \gamma a_2[\frac{(2-np)npa_1}{4(\gamma-1)\gamma a_2}]^{\frac{2\gamma-2}{2\gamma-np}}. \tag{3.8}$$

By $\|\varphi_0\|_2 \geq b_2$, we can get

$$a_1^{2\gamma-2} a_2^{2-np} \geq (a_0 D_2)^{np-2\gamma},$$

it follows that $f_1(y) \leq 0$ and $f_1'(y_m) < 0$. Therefore $f_1(y)$ is decreasing on $[0,+\infty)$, which implies $f_1(y)$ has no extrema on $[0,+\infty)$. By the monotonicity of $f_1(y)$, there exists unique $y_k \in (0,+\infty)$ such that $f_1(y) = K_1$.

And then, the same as the proof of Theorem 3.1 and 3.2, we can verify that B_3 is a nonempty invariant set. Besides, we can also verify

$$J''(t) \leq 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_1(y^*) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.3. □

Case II: $p = \frac{2}{n}$, $2 < \gamma < \min\{n, 4\}$. Denote

$$y_4 = \frac{\gamma\|\nabla W\|_2^2[((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2\|\varphi\|_2^{\frac{2}{n}}]}{2^{\gamma-2}(n-2)^2\|\nabla R\|_2^{\frac{2}{n}}},$$

$$K_2 = \frac{\|\nabla W\|_2^2[((n-2)^2(\gamma-2) - 2\beta(\gamma-1))\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2(\gamma-1)\|\varphi\|_2^{\frac{2}{n}}]}{2^{\gamma-1}(n-2)^4\|\nabla R\|_2^{\frac{4}{n}}}$$
$$\times [((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2\|\varphi\|_2^{\frac{2}{n}}].$$

We define two invariant sets:

$$G_4 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_2, \|\varphi\|_{H^1}^2 < y_4, \|\varphi\|_2^{\frac{2}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}}\},$$
$$B_4 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_2, \|\varphi\|_{H^1}^2 > y_4, \|\varphi\|_2^{\frac{2}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}}\}.$$

Theorem 3.4. *For $p = \frac{2}{n}$ and $2 < \gamma < \min\{n, 4\}$, the following facts are ture:*

(i) *When $\varphi_0 \in G_4 \cup \{0\}$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *exists globally in $t \in (0,\infty)$.*

(ii) *When $\varphi_0 \in B_4$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0,T]$,

$$\begin{aligned} E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 &\geq \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + c_0\|\varphi\|_2^2 \\ &\quad - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}\|\varphi\|_{H^1}^4 - \frac{1}{2\|\nabla R\|_2^{\frac{2}{n}}}\|\varphi\|_2^{\frac{2}{n}}\|\varphi\|_{H^1}^2 \\ &= [\frac{1}{2} - \frac{\beta}{(n-2)^2}]\|\varphi\|_{H^1}^2 - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}\|\varphi\|_{H^1}^4 \\ &\quad - \frac{1}{2\|\nabla R\|_2^{\frac{2}{n}}}\|\varphi\|_2^{\frac{2}{n}}\|\varphi\|_{H^1}^2. \end{aligned} \tag{3.9}$$

Let $y = \|\varphi(t)\|_{H^1}^2 \geq 0$, for all $t \in (0,T]$,

$$E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq f_2(\|\varphi(t)\|_{H^1}^2) = f_2(y), \tag{3.10}$$

where

$$f_2(y) = [\frac{1}{2} - \frac{\beta}{(n-2)^2} - \frac{\|\varphi_0\|_2^{\frac{2}{n}}}{2\|\nabla R\|_2^{\frac{2}{n}}}]y - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}y^2,$$
$$f_2'(y) = [\frac{1}{2} - \frac{\beta}{(n-2)^2} - \frac{\|\varphi_0\|_2^{\frac{2}{n}}}{2\|\nabla R\|_2^{\frac{2}{n}}}] - \frac{1}{2^{3-\gamma}\gamma\|\nabla W\|_2^2}y.$$

By $\|\varphi\|_2^{\frac{2}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}}$, we know

$$\|\varphi\|_2^{\frac{2}{n}} < (1 - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}},$$

so $f_2'(y)$ has only one zero point y_4 on $[0,+\infty)$,

$$y_4 = \frac{\gamma\|\nabla W\|_2^2[((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2\|\varphi\|_2^{\frac{2}{n}}]}{2^{\gamma-2}(n-2)^2\|\nabla R\|_2^{\frac{2}{n}}}.$$

Then the maximum of $f_2(y)$ is

$$f_2(y_4) = \frac{\gamma\|\nabla W\|_2^2[((n-2)^2-2\beta)\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2\|\varphi\|_2^{\frac{2}{n}}]^2}{2^\gamma(n-2)^4\|\nabla R\|_2^{\frac{4}{n}}}.$$

Secondly, we prove the invariance of G_4 and B_4. Combined with the structure of $f_2(y)$, we can easily know both G_4 and B_4 are nonempty sets. If $\varphi_0 \in G_4$, by Lemma 2.1, the corresponding solution $\varphi(t,x)$ of Cauchy problem (1.1) and (2.1) satisfies: for all $t \in [0,T)$,

$$f_2(\|\varphi(t)\|_{H^1}^2) \le E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 < K_2, \tag{3.11}$$

and

$$\|\varphi\|_2^{\frac{2}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}}.$$

We only need to prove $\|\varphi\|_{H^1}^2 < y_4$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0,T)$ such that $\|\varphi(\bar{t})\|_{H^1}^2 = y_4$, then by computation we can get

$$f_2(\|\varphi(\bar{t})\|_{H^1}^2) = f_2(y_4) > K_2,$$

which contradicts (3.11). Thus $\|\varphi\|_{H^1}^2 < y_4$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0,\infty)$. We can obtain the invariance of B_4 by the same token.

Finally, we prove the statement (ii) of Theorem 3.4. From (2.6), we have

$$\begin{aligned} J''(t) &= 8\gamma E(\varphi_0) + \int \frac{8\gamma - 4np}{p+2}|\varphi|^{p+2} - 4(\gamma-2)|\nabla\varphi|^2 + (4\gamma-4)\beta|x|^{-1}|\varphi|^2 dx \\ &\le 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_2(y), \end{aligned} \tag{3.12}$$

where

$$H_2(y) = [\frac{4(\gamma-1)\|\varphi_0\|_2^{\frac{2}{n}}}{\|\nabla R\|_2^{\frac{2}{n}}} - 4(\gamma-2) + \frac{8\beta(\gamma-1)}{(n-2)^2}]y.$$

When $\|\varphi\|_2^{\frac{2}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{2}{n}}$, the maximum of $H_2(y)$ on $[y_4, +\infty)$ is :

$$\begin{aligned} H_2(y_4) &= \frac{2^{4-\gamma}\gamma\|\nabla W\|_2^2[((n-2)^2-2\beta)\|\nabla R\|_2^{\frac{2}{n}} - (n-2)^2\|\varphi\|_2^{\frac{2}{n}}]}{(n-2)^4\|\nabla R\|_2^{\frac{4}{n}}} \\ &\quad \times [(n-2)^2(\gamma-1)\|\varphi\|_2^{\frac{2}{n}} - ((n-2)^2(\gamma-2) - 2\beta(\gamma-1))\|\nabla R\|_2^{\frac{2}{n}}] \\ &= -8\gamma K_2. \end{aligned}$$

By the invariance of B_4, if $\varphi_0 \in B_4$, then for all $t \in [0,T)$,

$$f_2(\|\varphi\|_{H^1}^2) \le E(\varphi) + c_0\|\varphi\|_2^2 < K_2, \quad \|\varphi\|_{H^1}^2 > y_4.$$

Inserting the results into (3.12), we obtain

$$J''(t) \le 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_2(y_4) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.4. □

Case III: $\frac{2}{n} < p < \frac{4}{n}$, $2 < \gamma < \min\{n, 4\}$. Denote

$$K_3 = \frac{np-4}{4np\gamma}\Big[\frac{(n-2)^2(2\gamma-np)\|\varphi\|_2^{\frac{4-(n-2)p}{2}}}{[(2\gamma-4)(n-2)^2-4\beta(\gamma-1)]^{\frac{np}{4}}\|\nabla R\|_2^p}\Big]^{\frac{4}{4-np}},$$
$$D_3 = \Big(\frac{4-np}{4\gamma-4}\Big)^{\frac{np-4}{4\gamma-np}} + \Big(\frac{4-np}{4\gamma-4}\Big)^{\frac{4\gamma-4}{4\gamma-np}},$$
$$b_3 = \Big[\frac{\|\nabla R\|_2^{4p\gamma-4p}\|\nabla W\|_2^{8-2np}}{2^{np\gamma-8\gamma+4}(a_0D_3)^{4\gamma-np}}\Big]^{\frac{1}{8+4p\gamma-4p-2np}},$$
$$a_3 = \frac{1}{2\|\nabla R\|_2^P}\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}, a_4 = \frac{1}{2^\gamma\|\nabla W\|_2^2}\|\varphi_0\|_2^{4-2\gamma},$$
$$f_3(y) := \frac{1}{a_0}y - \frac{2}{np\|\nabla R\|_2^p}\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{4}} - \frac{1}{2^\gamma\gamma\|\nabla W\|_2^2}\|\varphi_0\|_2^{\frac{4-2\gamma}{2}}y^\gamma. \tag{3.13}$$

Let $\widetilde{y_3} > 0$ and y_5 be the first and second positive roots of the equation $f_3'(y) = \frac{d}{dy}f_3(y) = 0$ respectively. Then we define two invariant sets:

$$G_5 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_3, \|\varphi\|_{H^1}^2 < y_5, \|\varphi\|_2 < b_3\},$$

$$B_5 = \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_3, \|\varphi\|_{H^1}^2 > y_5, \|\varphi\|_2 < b_3\}.$$

Theorem 3.5. *For* $\frac{2}{n} < p < \frac{4}{n}$ *and* $2 < \gamma < \min\{n, 4\}$, *the following facts are ture:*

(i) *When* $\varphi_0 \in G_5 \cup \{0\}$ *and* $f_3(\widetilde{y_3}) < K_3$, *the solution* $\varphi(t, x)$ *of the Cauchy problem* (1.1) *and* (2.1) *exists globally in* $t \in (0, \infty)$.

(ii) *When* $\varphi_0 \in B_5$ *and* $|x|\varphi_0 \in L^2(R^n)$, *the solution* $\varphi(t, x)$ *of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0, T]$,

$$\begin{aligned} E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq{}& \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + c_0\|\varphi\|_2^2 \\ &- \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^\gamma - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} \\ \geq{}& \Big[\frac{1}{2} - \frac{\beta}{(n-2)^2}\Big]\|\varphi\|_{H^1}^2 - \frac{1}{2^\gamma\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-2\gamma}\|\varphi\|_{H^1}^{2\gamma} \\ &- \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\varphi\|_{H^1}^{\frac{np}{2}}. \end{aligned} \tag{3.14}$$

Let $y = \|\varphi(t)\|_{H^1}^2 \geq 0$, for all $t \in (0, T]$,

$$E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq f_3(\|\varphi(t)\|_{H^1}^2) = f_3(y), \tag{3.15}$$

where f_3 is defined in (3.13). And then

$$\begin{aligned} f_3'(y) &= \frac{1}{a_0} - \frac{1}{2\|\nabla R\|_2^p}\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{4}-1} - \frac{1}{2^\gamma\|\nabla W\|_2^2}\|\varphi_0\|_2^{4-2\gamma}y^{\gamma-1} \\ &= \frac{1}{a_0} - a_3y^{\frac{np}{4}-1} - a_4y^{\gamma-1}, \end{aligned}$$

$$f_3''(y) = -\frac{(np-4)\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}}{8\|\nabla R\|_2^p}y^{\frac{np}{4}-2} - \frac{(\gamma-1)\|\varphi_0\|_2^{4-2\gamma}}{2^\gamma\|\nabla W\|_2^2}y^{\gamma-2}.$$

We can verify that $f_3''(y)$ has only one zero point

$$\overline{y_0} = [\frac{2^{\gamma-3}(4-np)\|\nabla W\|_2^2}{(\gamma-1)\|\varphi_0\|_2^{2-2\gamma+\frac{n-2}{2}p}\|\nabla R\|_2^p}]^{\frac{4}{4\gamma-np}},$$

$f_3''(y) \to +\infty$ as $y \to 0^+$ and $f_3''(y) \to -\infty$ as $y \to +\infty$. Thus the maximum of $f_3'(y)$ on $[0,\infty)$ is $f_3'(\overline{y_0})$. By $\|\varphi_0\|_2 < b_3$, we can get

$$a_3^{4\gamma-4}a_4^{4-np} < (a_0D_3)^{np-4\gamma},$$

which implies $f_3'(\overline{y_0}) > 0$. Note that $\lim\limits_{y\to+\infty} f_3' = -\infty$, so there exists a unique $y_5 \in (\overline{y_0}, +\infty)$ such that $f_3'(y) = 0$. Thus $f_3(y)$ is increasing on $(\overline{y_0}, y_5)$ and decreasing on $(y_5, +\infty)$. So the maximum of $f_3(y)$ on $[0,+\infty)$ is $f_3(y_5)$.

Secondly, we prove the invariance of G_5 and B_5. When $f_3(\widetilde{y_3}) < K_3$, combined with the structure of $f_3(y)$, we can easily know both G_5 and B_5 are nonempty sets. If $\varphi_0 \in G_5$, by Lemma 2.1, the corresponding solution $\varphi(t,x)$ of Cauchy problem (1.1) and (2.1) satisfies: for all $t \in [0,T)$,

$$f_3(\|\varphi(t)\|_{H^1}^2) \le E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 < K_3, \quad \|\varphi\|_2 < b_3. \tag{3.16}$$

We only need to prove $\|\varphi\|_{H^1}^2 < y_5$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0,T)$ such that $\|\varphi(\bar{t})\|_{H^1}^2 = y_5$, then by computation we get

$$f_3(\|\varphi(\bar{t})\|_{H^1}^2) = f_3(y_5) > K_3,$$

which contradicts (3.16). Thus $\|\varphi\|_{H^1}^2 < y_5$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0,\infty)$. We can obtain B_5 is a nonempty invariant set by the same token.

Finally, we prove the statement (ii) of Theorem 3.5. From (2.6), we have

$$\begin{aligned} J''(t) &= 8\gamma E(\varphi_0) + \int \frac{8\gamma-4np}{p+2}|\varphi|^{p+2} - 4(\gamma-2)|\nabla\varphi|^2 + (4\gamma-4)\beta|x|^{-1}|\varphi|^2 dx \\ &\le 8\gamma E(\varphi) + \frac{16\gamma-8np}{np}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} - (4\gamma-2)\|\nabla\varphi\|_2^2 \\ &\quad + (4\gamma-4)\beta[\frac{2}{(n-2)^2}\|\nabla\varphi\|_2^2 + \frac{1}{2}\|\varphi\|_2^2] \\ &\le 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_3(y), \end{aligned} \tag{3.17}$$

where

$$H_3(y) = \frac{16\gamma-8np}{np\|\nabla R\|_2^p}\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{4}} + [-4(\gamma-2) + \frac{8\beta(\gamma-1)}{(n-2)^2}]y.$$

Then H_3' has only one zero point $\overline{y^*}$ on $[0,\infty)$,

$$\overline{y^*} = [\frac{(n-2)^2(2\gamma-np)\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}}{[(n-2)^2(2\gamma-4)-4\beta(\gamma-1)]\|\nabla R\|_2^p}]^{\frac{4}{4-np}}.$$

$H_3(y)$ is increasing on $(0,\overline{y^*})$ and decreasing on $(\overline{y^*}, +\infty)$. So the maximum of $H_3(y)$ on $[0,+\infty)$ is :

$$H_3(\overline{y^*}) = \frac{8-2np}{np}[\frac{(n-2)^{\frac{np}{2}}(2\gamma-np)\|\varphi_0\|_2^{\frac{4-(n-2)p}{2}}}{[(n-2)^2(2\gamma-4)-4\beta(\gamma-1)]^{\frac{np}{4}}\|\nabla R\|_2^p}]^{\frac{4}{4-np}} = -8\gamma K_3.$$

By the invariance of B_5, if $\varphi_0 \in B_5$, then for all $t \in [0,T)$,

$$f_3(\|\varphi\|_{H^1}^2) \le E(\varphi) + c_0\|\varphi\|_2^2 < K_3, \quad \|\varphi\|_{H^1}^2 > y_5.$$

Inserting the results into (3.17), we obtain

$$J''(t) \le 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_3(\overline{y^*}) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.5. □

Case IV: $p = \dfrac{4}{n}$, $2 < \gamma < \min\{n, 4\}$. Denote

$$\begin{aligned} y_6 &= \frac{\gamma\|\nabla W\|_2^2[((n-2)^2-2\beta)\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2\|\varphi\|_2^{\frac{4}{n}}]}{2^{\gamma-2}(n-2)^2\|\nabla R\|_2^{\frac{4}{n}}}, \\ K_4 &= \frac{\|\nabla W\|_2^2[((n-2)^2(\gamma-2)-2\beta(\gamma-1))\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2(\gamma-1)\|\varphi\|_2^{\frac{4}{n}}]}{2^{\gamma-1}(n-2)^4\|\nabla R\|_2^{\frac{8}{n}}} \\ &\quad \times [((n-2)^2-2\beta)\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2\|\varphi\|_2^{\frac{4}{n}}]. \end{aligned}$$

We define two invariant sets:

$$\begin{aligned} G_6 &= \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_4, \|\varphi\|_{H^1}^2 < y_6, \|\varphi\|_2^{\frac{4}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}}\}, \\ B_6 &= \{\varphi \in H^1 : E(\varphi) + c_0\|\varphi\|_2^2 < K_4, \|\varphi\|_{H^1}^2 > y_6, \|\varphi\|_2^{\frac{4}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}}\}. \end{aligned}$$

Theorem 3.6. *For $p = \dfrac{4}{n}$ and $2 < \gamma < \min\{n, 4\}$, the following facts are ture:*

(i) *When $\varphi_0 \in G_6 \cup \{0\}$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *exists globally in $t \in (0, \infty)$.*

(ii) *When $\varphi_0 \in B_6$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0, T]$,

$$\begin{aligned} E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 &\geq \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + c_0\|\varphi\|_2^2 \\ &\quad - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}\|\varphi\|_{H^1}^4 - \frac{1}{2\|\nabla R\|_2^{\frac{4}{n}}}\|\varphi\|_2^{\frac{4}{n}}\|\varphi\|_{H^1}^2 \\ &= [\frac{1}{2} - \frac{\beta}{(n-2)^2}]\|\varphi\|_{H^1}^2 - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}\|\varphi\|_{H^1}^4 \\ &\quad - \frac{1}{2\|\nabla R\|_2^{\frac{4}{n}}}\|\varphi\|_2^{\frac{4}{n}}\|\varphi\|_{H^1}^2. \end{aligned} \tag{3.18}$$

Let $y = \|\varphi(t)\|_{H^1}^2 \geq 0$, for all $t \in (0, T]$,

$$E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 \geq f_4(\|\varphi(t)\|_{H^1}^2) = f_4(y), \tag{3.19}$$

where

$$\begin{aligned} f_4(y) &= [\frac{1}{2} - \frac{\beta}{(n-2)^2} - \frac{\|\varphi_0\|_2^{\frac{4}{n}}}{2\|\nabla R\|_2^{\frac{4}{n}}}]y - \frac{1}{2^{4-\gamma}\gamma\|\nabla W\|_2^2}y^2, \\ f_4'(y) &= [\frac{1}{2} - \frac{\beta}{(n-2)^2} - \frac{\|\varphi_0\|_2^{\frac{4}{n}}}{2\|\nabla R\|_2^{\frac{4}{n}}}] - \frac{1}{2^{3-\gamma}\gamma\|\nabla W\|_2^2}y. \end{aligned}$$

By $\|\varphi\|_2^{\frac{4}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}}$, we know

$$\|\varphi\|_2^{\frac{4}{n}} < (1 - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}},$$

so $f_4'(y)$ has only one zero point y_6 on $[0, +\infty)$,

$$y_6 = \frac{\gamma\|\nabla W\|_2^2[((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2\|\varphi\|_2^{\frac{4}{n}}]}{2^{\gamma-2}(n-2)^2\|\nabla R\|_2^{\frac{4}{n}}}.$$

Then the maximum of $f_4(y)$ is

$$f_4(y_6) = \frac{\gamma\|\nabla W\|_2^2[((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2\|\varphi\|_2^{\frac{4}{n}}]^2}{2^{\gamma}(n-2)^4\|\nabla R\|_2^{\frac{8}{n}}}.$$

Secondly, we prove the invariance of G_6 and B_6. Combined with the structure of $f_4(y)$, we can easily know both G_6 and B_6 are nonempty sets. If $\varphi_0 \in G_6$, by Lemma 2.1, the corresponding solution $\varphi(t,x)$ of Cauchy problem (1.1) and (2.1) satisfies: for all $t \in [0,T)$,

$$f_4(\|\varphi(t)\|_{H^1}^2) \leq E(\varphi(t)) + c_0\|\varphi(t)\|_2^2 < K_4, \tag{3.20}$$
$$\|\varphi\|_2^{\frac{4}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}}.$$

We only need to prove $\|\varphi\|_{H^1}^2 < y_6$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0,T)$ such that $\|\varphi(\bar{t})\|_{H^1}^2 = y_6$. Then by computation we get

$$f_4(\|\varphi(\bar{t})\|_{H^1}^2) = f_4(y_6) > K_4,$$

which contradicts (3.20). Thus $\|\varphi\|_{H^1}^2 < y_6$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0,\infty)$. We can obtain the invariance of B_6 by the same token.

Finally, we prove the statement (ii) of Theorem 3.6. From (2.6), we have

$$\begin{aligned} J''(t) &= 8\gamma E(\varphi_0) + \int \frac{8\gamma - 4np}{p+2}|\varphi|^{p+2} - 4(\gamma-2)|\nabla\varphi|^2 + (4\gamma-4)\beta|x|^{-1}|\varphi|^2 dx \\ &\leq 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_4(y), \end{aligned} \tag{3.21}$$

where

$$H_4(y) = [\frac{4(\gamma-1)\|\varphi_0\|_2^{\frac{4}{n}}}{\|\nabla R\|_2^{\frac{4}{n}}} - 4(\gamma-2) + \frac{8\beta(\gamma-1)}{(n-2)^2}]y.$$

When $\|\varphi\|_2^{\frac{4}{n}} < (\frac{\gamma-2}{\gamma-1} - \frac{2\beta}{(n-2)^2})\|\nabla R\|_2^{\frac{4}{n}}$, the maximum of $H_4(y)$ on $[y_6, +\infty)$ is :

$$\begin{aligned} H_4(y_6) &= \frac{2^{4-\gamma}\gamma\|\nabla W\|_2^2[((n-2)^2 - 2\beta)\|\nabla R\|_2^{\frac{4}{n}} - (n-2)^2\|\varphi\|_2^{\frac{4}{n}}]}{(n-2)^4\|\nabla R\|_2^{\frac{8}{n}}} \\ &\quad \times [(n-2)^2(\gamma-1)\|\varphi\|_2^{\frac{4}{n}} - ((n-2)^2(\gamma-2) - 2\beta(\gamma-1))\|\nabla R\|_2^{\frac{4}{n}}] \\ &= -8\gamma K_4. \end{aligned}$$

By the invariance of B_6, if $\varphi_0 \in B_6$, then for all $t \in [0,T)$,

$$f_4(\|\varphi\|_{H^1}^2) \leq E(\varphi) + c_0\|\varphi\|_2^2 < K_4, \quad \|\varphi\|_{H^1}^2 > y_6.$$

Inserting the results into (3.21), we obtain

$$J''(t) \leq 8\gamma[E(\varphi) + c_0\|\varphi\|_2^2] + H_4(y_6) < 0.$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.6. □

Case V: $\frac{4}{n} < p < \frac{4}{n-2}$, $2 < \gamma < \frac{np}{2}$. Denote

$$K_5 = \frac{(n-2)^2(\gamma-2) - 2\beta(\gamma-1)}{2(n-2)^2\gamma} Y^2.$$

Then we have two invariant sets:

$$G_7 = \{\varphi \in H^1 : E(\varphi) + \frac{(\beta+1)\gamma - 1}{4\gamma}\|\varphi\|_2^2 < K_5, \|\varphi\|_2 < \frac{2}{n-2}Y, \|\nabla\varphi\|_2 < Y\},$$
$$B_7 = \{\varphi \in H^1 : E(\varphi) + \frac{(\beta+1)\gamma - 1}{4\gamma}\|\varphi\|_2^2 < K_5, \|\varphi\|_2 < \frac{2}{n-2}Y, \|\nabla\varphi\|_2 > Y\},$$

where Y is shown in the proof of the following theorem:

Theorem 3.7. *For $\frac{4}{n} < p < \frac{4}{n-2}$ and $2 < \gamma < \frac{np}{2}$, the following facts are ture:*

(i) *When $\varphi_0 \in G_7 \cup \{0\}$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *exists globally in $t \in (0,\infty)$.*

(ii) *When $\varphi_0 \in B_7$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0,T]$,

$$\begin{aligned}
E(\varphi(t)) + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi(t)\|_2^2 &\geq \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi\|_2^2 \\
&\quad - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^\gamma - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} \\
&= [\frac{1}{2} - \frac{\beta}{(n-2)^2}]\|\nabla\varphi\|_2^2 - \frac{\beta}{4\gamma}\|\varphi\|_2^2 - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^\gamma \\
&\quad - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}}. \qquad (3.22)
\end{aligned}$$

Let $y = \|\nabla\varphi(t)\|_2 \geq 0$, for all $t \in (0,T]$,

$$E(\varphi(t)) + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi(t)\|_2^2 \geq \hbar_1(\|\nabla\varphi(t)\|_2) = \hbar_1(y), \qquad (3.23)$$

where

$$\hbar_1(y) = [\frac{1}{2} - \frac{\beta}{(n-2)^2}]y^2 - \frac{\beta}{4\gamma}\|\varphi\|_2^2 - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}y^\gamma - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{2}},$$

$$\hbar_1'(y) = [1 - \frac{2\beta}{(n-2)^2} - \frac{1}{\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}y^{\gamma-2} - \frac{1}{\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{2}-2}]y = \hbar_2(y)y.$$

Thus $\hbar_2(y) = 0$ has only one positive solution,

$$\hbar_2'(y) = -(\gamma-2)\frac{1}{\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}y^{\gamma-3} - \frac{np-4}{2\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{2}-3} < 0,$$

which implies $\hbar_2$ is decreasing on $[0,+\infty)$. Note that $\hbar_2(0) = 1 - \frac{\beta}{(n-2)^2} > 0$ and

$$\hbar_2[(\frac{((n-2)^2-\beta)\|\nabla W\|_2^2}{(n-2)^2\|\varphi\|_2^{4-\gamma}})^{\frac{1}{\gamma-2}}] = -\frac{\|\varphi\|_2^{\frac{4-(n-2)p}{2}}}{\|\nabla R\|_2^p}(\frac{((n-2)^2-\beta)\|\nabla W\|_2^2}{(n-2)^2\|\varphi\|_2^{4-\gamma}})^{\frac{1}{\gamma-2}(\frac{np}{2}-2)} < 0.$$

Since $\hbar_2$ is continuous on $[0,+\infty)$, there exists a unique positive Y,

$$Y \in [0, (\frac{((n-2)^2-\beta)\|\nabla W\|_2^2}{(n-2)^2\|\varphi\|_2^{4-\gamma}})^{\frac{1}{\gamma-2}}],$$

such that $\hbar_2(Y) = 0$, thus the maximum of $\hbar_1(y)$ is $\hbar_1(Y)$.

Secondly, we prove the invariance of G_7 and B_7. Combined with the structure of $\hbar_1(y)$, we can easily know both G_7 and B_7 are nonempty sets. If $\varphi_0 \in G_7$, by Lemma 2.1 and $\|\varphi\|_2 < \frac{2}{n-2}Y$, the corresponding solution $\varphi(t,x)$ of Cauchy problem (1.1) and (2.1) satisfies: for all $t \in [0,T)$,

$$\hbar_1(\|\nabla\varphi(t)\|_2^2) \leq E(\varphi) + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi\|_2^2 < \frac{(n-2)^2(\gamma-2)-2\beta(\gamma-1)}{2(n-2)^2\gamma}Y^2 < \hbar_1(Y). \tag{3.24}$$

We only need to prove $\|\nabla\varphi\|_2 < Y$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0,T)$ such that $\|\nabla\varphi(\bar{t})\|_2 = Y$, then by computation we get

$$\hbar_1(\|\nabla\varphi(\bar{t})\|_2) = \hbar_1(Y) \leq E(\varphi) + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi\|_2^2,$$

which contradicts (3.24). Thus $\|\nabla\varphi\|_2 < Y$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and(2.1) exists globally in $t \in (0,\infty)$. We can obtain the invariance of B_7 by the same token.

Finally, we prove the statement (ii) of Theorem 3.7. From (2.6), we have

$$\begin{aligned} J''(t) &\leq 8\gamma E(\varphi_0) + \int -4(\gamma-2)|\nabla\varphi|^2 + (4\gamma-4)\beta[\frac{2}{(n-2)^2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2]dx \\ &= 8\gamma[E(\varphi) + \frac{\beta(\gamma-1)}{4\gamma}\|\varphi\|_2^2] - \frac{4(n-2)^2(\gamma-2)-8\beta(\gamma-1)}{(n-2)^2}\|\nabla\varphi\|_2^2 \\ &\leq 8\gamma\frac{(n-2)^2(\gamma-2)-2\beta(\gamma-1)}{2\gamma(n-2)^2}Y^2 - \frac{4(n-2)^2(\gamma-2)-8\beta(\gamma-1)}{(n-2)^2}Y^2 = 0. \end{aligned}$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.7. □

Case VI: $\frac{4}{n} < p < \frac{4}{n-2}$, $\frac{np}{2} \leq \gamma < \min\{4,n\}$. Denote

$$K_6 = \frac{(n-2)^2(np-4)-\beta(2np-4)}{2(n-2)^2 np}Y'^2.$$

Then we have two invariant sets:

$$G_8 = \{\varphi \in H^1 : E(\varphi) + \frac{(\beta+1)\gamma-1}{4\gamma}\|\varphi\|_2^2 < K_6, \|\varphi\|_2 < \frac{2}{(n-2)}Y', \|\nabla\varphi\|_2 < Y'\},$$

$$B_8 = \{\varphi \in H^1 : E(\varphi) + \frac{(\beta+1)\gamma-1}{4\gamma}\|\varphi\|_2^2 < K_6, \|\varphi\|_2 < \frac{2}{(n-2)}Y', \|\nabla\varphi\|_2 > Y'\},$$

where Y' is shown in the proof of the following theorem:

Theorem 3.8. *For* $\frac{4}{n} < p < \frac{4}{n-2}$ *and* $\frac{np}{2} \leq \gamma < \min\{4,n\}$*, the following facts are ture:*

(i) *When* $\varphi_0 \in G_8 \cup \{0\}$ *, the solution* $\varphi(t,x)$ *of the Cauchy problem* (1.1) *and* (2.1) *exists globally in* $t \in (0,\infty)$.

(ii) *When $\varphi_0 \in B_8$ and $|x|\varphi_0 \in L^2(R^n)$, the solution $\varphi(t,x)$ of the Cauchy problem* (1.1) *and* (2.1) *blows up in a finite time.*

Proof. Firstly, according to (2.8) , (2.10) and (2.12), we estimate the energy functional $E(\varphi)$, for all $t \in (0,T]$,

$$\begin{aligned} E(\varphi(t)) + \frac{\beta(np-2)}{4np}\|\varphi(t)\|_2^2 \geq{}& \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{\beta}{(n-2)^2}\|\nabla\varphi\|_2^2 - \frac{\beta}{4}\|\varphi\|_2^2 + \frac{\beta(np-2)}{4np}\|\varphi\|_2^2 \\ &- \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^{\gamma} - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}} \\ ={}& [\frac{1}{2} - \frac{\beta}{(n-2)^2}]\|\nabla\varphi\|_2^2 - \frac{\beta}{2np}\|\varphi\|_2^2 - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}\|\nabla\varphi\|_2^{\gamma} \\ &- \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}\|\nabla\varphi\|_2^{\frac{np}{2}}. \end{aligned} \tag{3.25}$$

Let $y = \|\nabla\varphi(t)\|_2 \geq 0$, for all $t \in (0,T]$,

$$E(\varphi(t)) + \frac{\beta(np-2)}{4np}\|\varphi(t)\|_2^2 \geq \hbar_3(\|\nabla\varphi(t)\|_2) = \hbar_3(y), \tag{3.26}$$

where

$$\begin{aligned} \hbar_3(y) &= [\frac{1}{2} - \frac{\beta}{(n-2)^2}]y^2 - \frac{\beta}{2np}\|\varphi\|_2^2 - \frac{1}{\gamma\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}y^{\gamma} - \frac{2}{np\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{2}}, \\ \hbar_3'(y) &= [1 - \frac{2\beta}{(n-2)^2} - \frac{1}{\|\nabla W\|_2^2}\|\varphi\|_2^{4-\gamma}y^{\gamma-2} - \frac{1}{\|\nabla R\|_2^p}\|\varphi\|_2^{\frac{4-(n-2)p}{2}}y^{\frac{np}{2}-2}]y = \hbar_2(y)y. \end{aligned}$$

The same as the proof of Theorem 3.7, there exists a unique positive Y' such that $\hbar_2(Y') = 0$, thus the maximum of $\hbar_3(y)$ is $\hbar_3(Y')$.

Secondly, we prove the invariance of G_8 and B_8. Combined with the structure of $\hbar_3(y)$, we can easily know both G_8 and B_8 are nonempty sets. If $\varphi_0 \in G_8$, by Lemma 2.1 and $\|\varphi\|_2^2 < \frac{8}{(n-2)^2}Y'^2$, the corresponding solution $\varphi(t,x)$ of Cauchy problem (1.1) and (2.1) satisfies: for all $t \in [0,T)$,

$$\hbar_3(\|\nabla\varphi(t)\|_2^2) \leq E(\varphi) + \frac{\beta(np-2)}{4np}\|\varphi\|_2^2 < \frac{(n-2)^2(np-4) - \beta(2np-4)}{2(n-2)^2np}Y'^2 < \hbar_3(Y'). \tag{3.27}$$

We only need to prove $\|\nabla\varphi\|_2 < Y'$. Otherwise, by the continuity of $\varphi(t)$ there exists $\bar{t} \in [0,T)$ such that $\|\nabla\varphi(\bar{t})\|_2 = Y'$, then by computation we get

$$\hbar_3(\|\nabla\varphi(\bar{t})\|_2) = \hbar_3(Y') \leq E(\varphi) + \frac{\beta(np-2)}{4np}\|\varphi\|_2^2,$$

which contradicts (3.27). Thus $\|\nabla\varphi\|_2 < Y'$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) exists globally in $t \in (0,\infty)$. We can obtain the invariance of B_8 by the same token.

Finally, we prove the statement (ii) of Theorem 3.8. From (2.6), we have

$$\begin{aligned} J''(t) &= 4npE(\varphi_0) - \int (2np-8)|\nabla\varphi|^2 - (np-2\gamma)(|x|^{-\gamma} * |\varphi|^2)|\varphi|^2 - (2np-4)\beta|x|^{-1}|\varphi|^2 dx \\ &\leq 4npE(\varphi_0) - \int (2np-8)|\nabla\varphi|^2 - (2np-4)\beta[\frac{2}{(n-2)^2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2]dx \\ &= 4np[E(\varphi) + \frac{\beta(np-2)}{4np}\|\varphi\|_2^2] - \frac{(n-2)^2(2np-8) - \beta(4np-8)}{(n-2)^2}\|\nabla\varphi\|_2^2 \\ &\leq 4np\frac{(n-2)^2(np-4) - \beta(2np-4)}{2(n-2)^2np}Y'^2 - \frac{(n-2)^2(2np-8) - \beta(4np-8)}{(n-2)^2}Y'^2 = 0. \end{aligned} \tag{3.28}$$

Therefore from Lemma 2.1 and 2.3, it must be the case $T < \infty$, which implies the solution $\varphi(t,x)$ of the Cauchy problem (1.1) and (2.1) blows up in a finite time. This completes the proof of Theorem 3.8. $\square$

References

[1] P. Antonelli, A. Athanassoulis, H. Hajaiej et al, *On the XFEL Schrödinger equation: Highly oscillatory magnetic potentials and time averaging*, Archive for Rational Mechanics and Analysis. **211** (2014), 711-732.

[2] D. G. Bhimani, *Global well-posedness for fractional Hartree equation on modulation spaces and Fourier algebra*, Journal of Differential Equations. **268** (2019), 141-159.

[3] J. D. Brock, *Watching atoms move*, Sciences. **315** (2007), 609-610.

[4] H. N. Chapman, *Femtosecond time-delay X-ray holography*, Nature. **448** (2007).

[5] J. M. Chadam, R. T. Glassey, *Global existence of solutions to the Cauchy problem for time-dependent Hartree equations*, Journal of Mathematical Physics. **16** (1975), 1122-1130.

[6] T. Cazenave, *Semilinear Schrödinger Equations*, Kluwer, New York, 2003.

[7] D. Y. Fang, Z. Han, *The nonlinear Schrödinger equation with combined nonlinearities of power-type and Hartree-type*, Chin Ann Math. **32B** (2011), 435-474.

[8] D. M. Fritz, D. A. Reis, R. A. Akre, et al, *Ultrafast bond softening in bismuth: Mapping a solid's interatomic potential with X-rays*, Science. **315** (2007), 633-636.

[9] B. Feng, D. Zhao, *Optimal bilinear control for the X-Ray free electron laser Schrödinger equation*, SIAM Journal on Control and Optimization. **57** (2019).

[10] B. Feng, D. Zhao, *On the Cauchy problem for the XFEL Schrödinger equation*, Discrete and continuous dynamical systems. **23** (2018), 4171-4186.

[11] J. P. García Azorero, I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, Journal of Differential Equations. **144** (1998).

[12] Y. Gao, Z. Wang, *Below and beyond the mass-energy threshold: scattering for the Hartree equation with radial data in* $d \geq 5$, Z Angew Math Phys. **71** (2020), 52.

[13] H. Genev, G. Venkov, *Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation*, Discrete and continuous dynamical systems. **5** (2012), 903-923.

[14] N. Hayashi, T. Ozawa, *Time decay of solutions to the Cauchy problem for time-dependent Schrödinger-Hartree equations*, Communications in Mathematical Physics. **110** (1987), 467-478.

[15] H. Han, F. H. Li, Z. P. Wang, et al, *Ground state for the X-ray free electron laser Schrödinger equation with harmonic potential*, J. Applied Mathematics and Computation. **401** (2021), 126113.

[16] J. Huang, J. Zhang, G.G. Chen, *Strong Instability of Standing Waves for Hartree Equations with Subcritical Perturbations*, J. Mathematica Applicata. **25** (2012), 527-534.

[17] E. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Applied Mathematics. **57** (1977), 93-105.

[18] L. Leng, X. Li, P. Zheng, *Sharp criteria for the nonlinear Schrödinger equation with combined nonlinearities of power-type and Hartree-type*, Applicable Analysis, **96** (2017), 2846-2851.

[19] A. Messiah, *Quantum Mechanics* , North Holland, Amsterdam, 1961.

[20] V. Moroz, J.V. Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, Journal of Functional Analysis. **265** (2013), 153-184.

[21] C. Miao, G. Xu, L. Zhao, *Global well-posedness and scattering for the mass-critical Hartree equation with radial data*, J Math Pures Appl. **91** (2009), 49-79.

[22] C. Miao, G. Xu, L. Zhao, *On the blow-up phenomenon for the mass-critical focusing Hartree equation in* $\mathbb{R}^4$, Colloq Math. **119** (2010), 23-50.

[23] R. Neutze, R. Wouts, D. Van Der Spoel, et al, *Potential for biomolecular imaging with femtosecond X-ray pulses*, Nature. **406** (2000), 752-757.

[24] S. Ronen, D. Bortolotti, D. Blume, et al, *Dipolar Bose-Einstein condensates with dipole-dependent scattering length*, Physical Review A. **74** (2006): 033611.

[25] S. Tian, Y, Yang, R. Zhou, S. H. Zhu, *Energy thresholds of blow-up for the Hartree equation with a focusing subcritical perturbation*, Studies in Applied Mathematics, **146** (2021), 658-676.

[26] S. Tian, S. H. Zhu, *Dynamics of the nonlinear Hartree equation with a focusing and defocusing perturbation*, J. Nonlinear Analysis, **222** (2022): 112980.

[27] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Communications in Mathematical Physics. **87** (1983), 567-576.

[28] S. H. Zhu, *On the blow-up solutions for the nonlinear fractional Schrödinger equation*, J. Differ Equ. **261** (2016), 1506-1531.

[29] J. Zhang, S. H. Zhu, *Stability of standing waves for the nonlinear fractional Schrödinger equation*, J. Dyn Differ Equ. **29** (2017), 1017-1030.

N. Tang, School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 711731, P. R. China
Email address: tangna514@163.com

C. Wang, School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 711731, P. R. China
Email address: wangchenglinedu@163.com

J. Zhang, corresponding author, School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 711731, P. R. China
Email address: zhangjian@uestc.edu.cn