CAUCHY –Jurnal Matematika Murni dan Aplikasi
Volume 7(1) (2021), Pages 22-27
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Inclusion Properties of Herz-Morrey Spaces With Variable Exponent

Hairur Rahman

Departement of Mathematics, Islamic State University of Maulana Malik Ibrahim Malang

Email: hairur@mat.uin-malang.ac.id

ABSTRACT

The inclusion properties in Herz-Morrey spaces has proved by Rahman in 2020. This paper aims to discuss the inclusion of the homogeneous Herz-Morrey spaces and homogeneous weak Herz-Morrey spaces with variable exponent. We also investigated the inclusion between both spaces. This result will be useful to prove fractional integral on the homogeneous Herz-Morrey spaces with variable exponent.

Keywords: Herz-Morrey spaces; inclusion properties; variable exponent.

INTRODUCTION

Inclusion properties or inclusion relation between spaces has received a lot of attention from researchers. It seems that many authors have studied this issue in some spaces (see [1]-[5]). Thus, this lead the author for discussing the inclusion properties especially in Herz-Morrey spaces.

Herz spaces can be traced back to the work of Beurling. Beurling [6] introduced a space $\mathcal{A}_p$, which is the original version of non homogeneous Herz spaces. Lu *et al* [7] has given the inclusion properties in homogeneous Herz spaces, as a proposition below.

Proposition 1.1. *Let $\alpha \in \mathbb{R}$, $p > 0$, and $q \leq \infty$. The following inclusions are valid.*

a. If $p_1 \leq p_2$, then $K_q^{\alpha,p_1}(\mathbb{R}^n) \subset K_q^{\alpha,p_2}(\mathbb{R}^n)$

b. If $q_2 \leq q_1$, then $K_{q_1}^{\alpha,p}(\mathbb{R}^n) \subset K_{q_2}^{\alpha-n\left(\frac{1}{q_1}-\frac{1}{q_2}\right),p}(\mathbb{R}^n)$.

This proposition can be proved by simply computation. In fact, if $0 < r < 1$, (a) is a consequence of the inequality

$$\left(\sum_{k=1}^{\infty}|a_k|\right)^r \leq \sum_{k=1}^{\infty}|a_k|^r.$$

While, (b) can be deduced directly from the Hölder inequality.

In 2016, Gunawan *et al.* (see [1] [2]) have proved the inclusion of Morrey spaces and generalized Morrey spaces. Recently, Rahman [8] also has proved the inclusion properties in Herz-Morrey spaces. These result have been motivated the author to study more about inclusion in homogenous Herz-Morrey spaces, but in this case the author uses variable exponent.

Since 1991, the research of Kovacik and Rakosnik [9] motivated many researchers to study about function spaces with variable exponent in several discussion. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, $p(\cdot)\colon \Omega \to [1,\infty)$ is a measurable

Submitted: May 02, 2021 Reviewed: August 24, 2021 Accepted: October 06, 2021
DOI: https://doi.org/10.18860/ca.v7i1.12141

function and $L^{p(\cdot)}(\Omega)$ is denoted the set of measurable functions f on Ω, such that for some positive λ satisfied

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

If $L^{p(\cdot)}(\Omega)$ equipped by the Luxemburg-Nakano norm

$$\| f \|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\},$$

then $L^{p(\cdot)}(\Omega)$ becomes a Banach function spaces. Since these spaces generalize the standard L^p spaces, they are also referred to as variable L^p spaces. $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$, when $p(x) = p$ is a constant.

In 2010, the boundedness of sublinear operators on Herz-Morrey space with variable exponent $\mathcal{M}\dot{K}^{\alpha,q}_{p(\cdot)}$ and $\mathcal{M}\dot{K}^{\overline{\alpha},q}_{p(\cdot)}$ was proved by Izuki [10]. Then, Xu and Yang [11] developed the definition of Herz-Morrey spaces with variabel exponents. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, $0 \le \lambda < \infty$, and $\alpha(\cdot)$ is a bounded real-valued measurable function on $\mathbb{R}^n$, the homogeneous Herz-Morrey spaces with variable exponent $\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)$ consists all functions $f \in L^q_{loc}(\mathbb{R}^n/\{0\})$ such that

$$\|f\|_{\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p} \| f\chi_k \|^p_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{p(\cdot)}} < \infty,$$

where $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k/B_{k-1}$ and $\chi_k = \chi_{A_k}$ is the characteristic function of the set A_k for $k \in \mathbb{Z}$.

As another spaces which have their weak type spaces, Herz-Morrey spaces also have their weak type spaces. For $\alpha(\cdot) \in \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \le \lambda \le \infty$ and $0 < q \le \infty$, the homogeneous weak Herz-Morrey spaces with variabel exponent $\left(W\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)\right)$ is a set of measurable $f \in L^q_{loc}(\mathbb{R}^n/\{0\})$ which is equipped with norm such that

$$\| f \|_{W\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}(\mathbb{R}^n)} = \sup_{\gamma>0} \gamma \sup_{L\in\mathbb{Z}} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)} m_k(\gamma, f)^{\frac{p(\cdot)}{q}} \right)^{\frac{1}{p(\cdot)}} < \infty,$$

where $m_k(\gamma, f) = |\{x \in A_k : |f(x)| > \gamma\}|$.

Some authors have investigated those spaces in various terms of discussion (see [12] - [15]). Meanwhile, this article aims to discuss in terms inclusion properties and inclusion relation of the homogeneous Herz-Morrey spaces and homogeneous weak Herz-Morrey spaces with variable exponent.

RESULT AND DISCUSSION

Our main results are the following:

Theorem 2.1. *Let* $1 \le p_1(\cdot) \le p_2(\cdot) < q < \infty$, *and* $\alpha(\cdot)$ *is a bounded real-valued measurable fuction on* $\mathbb{R}^n$. *Then, the inclusion*

$$\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n),$$

Is valid.

Proof. We may take any $f \in \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n)$. Then, by using Hölder inequality and $p_1 \le p_2$ we have

$$\|f\|_{\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n)} = \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_1(\cdot)} \|f\chi_k\|^{p_1(\cdot)}_{L^q(\mathbb{R}^n)} \right)^{\frac{1}{p_1(\cdot)}}$$

$$\le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\left(\sum_{k=-\infty}^{L} \left(2^{k\alpha(\cdot)p_1(\cdot)}\right)^{\frac{p_2(\cdot)}{p_1(\cdot)}} \right)^{\frac{p_1(\cdot)}{p_2(\cdot)}} \left(\sum_{k=-\infty}^{L} \left(\|f\chi_k\|^{p_1(\cdot)}_{L^q(\mathbb{R}^n)} \right)^{\frac{p_2(\cdot)}{p_2(\cdot)-p_1(\cdot)}} \right)^{1-\frac{p_1(\cdot)}{p_2(\cdot)}} \right)^{\frac{1}{p_1(\cdot)}}$$

$$\le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_2(\cdot)} \right)^{\frac{p_1(\cdot)}{p_2(\cdot)}} \left(\sum_{k=-\infty}^{L} \|f\chi_k\|^{\frac{p_1(\cdot)p_2(\cdot)}{p_2(\cdot)-p_1(\cdot)}}_{L^q(\mathbb{R}^n)} \right)^{1-\frac{p_1(\cdot)}{p_2(\cdot)}} \right)^{\frac{1}{p_1(\cdot)}}$$

$$\le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_2(\cdot)} \left(\sum_{k=-\infty}^{L} \|f\chi_k\|^{\frac{p_1(\cdot)p_2(\cdot)}{p_2(\cdot)-p_1(\cdot)}}_{L^q(\mathbb{R}^n)} \right)^{\frac{p_2(\cdot)-p_1(\cdot)}{p_1(\cdot)}} \right)^{\frac{1}{p_2(\cdot)}}$$

$$\le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_2(\cdot)} \|f\chi_k\|^{p_2(\cdot)}_{L^q(\mathbb{R}^n)} \right)^{\frac{1}{p_2(\cdot)}}$$

$$\le \|f\|_{\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n)}.$$

It is easy to know that $f \in \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n)$, where $\alpha(\cdot) \in (\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, we have $\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n)$.

By the previous theorem, the author established the following inclusions.

Theorem 2.2. *Let* $1 \le p_1(\cdot) \le p_2(\cdot) < q < \infty$, *and* $\alpha(\cdot)$ *is a bounded real-valued measurable fuction on* $\mathbb{R}^n$, *then the following inclusion is valid.*

$$L^q(R^n) = \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{q,q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n).$$

Proof. Theorem 2.1 has stated that $\mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_1(\cdot),q}(\mathbb{R}^n)$. Then, we only prove that $L^q(\mathbb{R}^n) = \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{q,q}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}^{\alpha(\cdot),\lambda}_{p_2(\cdot),q}(\mathbb{R}^n)$. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,q}(\mathbb{R}^n)$, by using similar method as before, we get

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,q}(\mathbb{R}^n)} \le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)q} \left(\left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}} \left(\int_{B(0,2^k)} |\chi_k|^q dy \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}}$$

$$\le \sup_{L\in Z} \frac{1}{2^{L\lambda}} \sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)} \left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}} (2^{kd})^{\frac{1}{q}}$$

$$\le C \left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}}$$

$\leq \| f \|_{L^q(\mathbb{R}^n)}$.

Hence, $f \in L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$. In the other hand, for any $f \in L^q(\mathbb{R}^n)$, there exist any constant C such that $C = \sup_{L\in Z} \frac{1}{2^{L\lambda}} \sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)+\frac{kd}{q}}$. Consequently, we have $f \in \mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ and $\mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$. It gives conclusion that $L^q(\mathbb{R}^n) = \mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, where $\alpha(\cdot) \in (\mathbb{R}^n)$.

Furthermore, we will prove that $\mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq \mathcal{M}\dot{K}_{p_2(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$. By using similar method as the proof of Theorem 2.1, we have $\| f \|_{\mathcal{M}\dot{K}_{p_2(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \leq \| f \|_{\mathcal{M}\dot{K}_{q,q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}$, where $\alpha(\cdot) \in (\mathbb{R}^n)$.

The author also added the inclusion of the homogeneous weak Herz-Morrey spaces with variable exponent by the following theorem.

Theorem 2.3. *Let $1 \leq p_1(\cdot) \leq p_2(\cdot) \leq q < \infty$, and $\alpha(\cdot)$ is a bounded real-valued measurable fuction on $\mathbb{R}^n$, the following inclusion holds:*

$$W\mathcal{M}\dot{K}_{p_2(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}_{p_1(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n).$$

Proof. Let $f \in \|f\|_{W\mathcal{M}\dot{K}_{p_1(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}$, we have

$$\| f \|_{W\mathcal{M}\dot{K}_{p_1(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{\gamma>0} \gamma \sup_{L\in\mathbb{Z}} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_1(\cdot)} m_k(\gamma, f)^{\frac{p_1(\cdot)}{q}} \right)^{\frac{1}{p_1(\cdot)}}$$

$$\leq \sup_{\gamma>0} \gamma \sup_{L\in\mathbb{Z}} \frac{1}{2^{L\lambda}} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p_2(\cdot)} m_k(\gamma, f)^{\frac{p_2(\cdot)}{q}} \right)^{\frac{1}{p_2(\cdot)}}$$

$$\leq \|f\|_{W\mathcal{M}\dot{K}_{p_2(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}.$$

The above inequality has shown that $W\mathcal{M}\dot{K}_{p_2(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}_{p_1(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$.

Now, we state the inclusion relation between both spaces.

Theorem 2.4. *Let $1 \leq p(\cdot) \leq q$, and $\alpha(\cdot)$ is a bounded real-valued measurable function on $\mathbb{R}^n$. Then, the inclusion*

$$\mathcal{M}\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$$

is proper.

Proof. We use similar idea as before to prove this theorem. Let $f \in \mathcal{M}\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, $a(\cdot) \in \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\gamma > 0$. We have observed that

$$|\{x \in A_k : |f(x)| > \gamma\}|^{\frac{p(\cdot)}{q}} \leq \left(\int_{B(0,2^k)} |f(x)\chi_k|^q \, dx \right)^{\frac{p(\cdot)}{q}} = \| f\chi_k \|_{L^q(\mathbb{R}^n)}^{p(\cdot)}.$$

Multiplying both sides by $\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)}$, we get

$$\sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)}|\{\, x \in A_k : |f(x)| > \gamma \,\}|^{\frac{p(\cdot)}{q}} \leq \sum_{k=-\infty}^{L} 2^{k\alpha(\cdot)p(\cdot)} \parallel f\chi_k \parallel_{L^q(\mathbb{R}^n)}^{p(\cdot)}.$$

Clearly, we see that $\parallel f \parallel_{W\mathcal{M}\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \leq \parallel f \parallel_{\mathcal{M}\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}$ and $f \in W\,\mathcal{M}\,\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$, which implies that $\mathcal{M}\,\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n) \subseteq W\,\mathcal{M}\,\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$.

CONCLUSION

By this result, the author can conclude that the homogeneous Herz-Morrey spaces with variable exponent have inclusion properties ... . This result will be useful to be used in proving fractional integral on the homogeneous Herz-Morrey spaces with variable exponent.

ACKNOWLEDGMENT

This paper is partially supported by UIN Maulana Malik Ibrahim Malang Research and Innovation Program 2020.

REFERENCES

[1] H. Gunawan, D. I. Hakim, K. M. Limanta and A. A. Masta, "Inclusion property of generalized Morrey spaces," *Math. Nachr.,* pp. 1-9, 2016.

[2] H. Gunawan, D. I. Hakim and M. Idris, "Proper inclusions of Morrey spaces," *Glasnik Matematicki,* vol. 53, no. 1, 2017.

[3] H. Gunawan, D. I. Hakim, E. Nakai and Y. Sawano, "On inclusion relation between weak Morrey spaces and Morrey spaces," *Nonlineae Analysis,* vol. 168, pp. 27-31, 2018.

[4] H. Gunawan, E. Kikianty and C. Schwanke, "Discrete Morrey spaces and their inclusion properties," *Math. Nachr.,* pp. 1-14, 2017.

[5] A. A. Masta, H. Gunawan and W. Setya-Budhi, "An Inclusion Property of Orlicz-Morrey Spaces," *J. Phys.: Conf. Ser,* vol. 893, pp. 1-7, 2017.

[6] A. Beurling, "Construction and analysis of some convolution algebras," *Annales de L'Institut Fourier Grenoble,* vol. 14, pp. 1-32, 1964.

[7] S. Lu, D. Yang and H. Guoen, Herz Type Spaces and Their Applications, Beijing: Science Press, 2008.

[8] H. Rahman, "Inclusion properties of the homogeneous Herz-Morrey," *Cauchy,* vol. 6, no. 3, pp. 117-121, 2020.

[9] O. Kovacik and J. Rakosnik, "On space and," *Czchoslovak Math. J.,* vol. 41, pp. 592-618, 1991.

[10] M. Izuki, "Boundedness of Sublinear Operators on Herz Spaces with Variable Exponent and Application to Wavelet Characterization," vol. 36 (1), no. Analysis Mathematics, pp. 33-50, 2010.

[11] J. Yang and J. Xu, "Herz-Morrey-Hardy Spaces with Variable Exponents and Their Applications," no. Journal of Function Spaces, pp. 1-19, 2015.

[12] S. Lu and L. Xu, "Boundedness of Rough Singular Integral Operators on The Homogeneous Morrey-Herz Spaces," *Hokkaido Math. Journal,* vol. 34, pp. 299-314,

2005.

[13] M. Izuki, "Fractional Integral on Herz-Morrey spaces with variable exponent," *Hiroshima Math. J.,* vol. 40, pp. 343-355, 2010.

[14] Y. Mizuta and T. Ohno, "Herz-Morrey spaces of variable exponent, Riesz potential operator and duality," *Complex Variable and Elliptic Equations,* vol. 60, no. 2, pp. 211-240, 2015.

[15] Y. Shi, X. Tao and T. Zheng, "Multilinier Riesz potential on Morrey-Herz spaces with non-doubling measure," *Journal of Inequality and Applications,* vol. 10, 2010.