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on Graceful Chromatic Number of Vertex amalgamation of Tree Graph Family

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ABSTRACT

Definition graceful k-coloring of graph $G = (V, E)$ is proper vertex coloring $c: V(G) \rightarrow \{1,2, \ldots, k\}; k \geq 2$, which induces a proper edge coloring $c' : E(G) \rightarrow \{1,2, \ldots, k-1\}$ defined $c'(uv) = |c(u) - c(v)|$. The minimum vertex coloring from graph G can be colored with graceful coloring called a graceful chromatic number with notation $\chi_g(G)$. The method used in this research is the axiomatic deductive method and pattern recognition. The detection method finds coloring patterns and graceful chromatic numbers based on the amalgamation operation of point family tree graphs. In this paper, we have investigated the graceful chromatic number of vertex amalgamation of tree graph family: path graph, centipede graph, broom, and E graph. The results of this study are expected to be used as a basis for studies in developing science and applications related to graceful chromatic numbers on the results of point amalgamation operations in tree family graphs.

Keywords: graceful coloring; tree graph family; graceful chromatic number; vertex amalgamation.

INTRODUCTION

Graceful coloring is the topic of vertex coloring used in this research. Vertex coloring is the assignment of color to a vertex where each neighboring vertex is assigned a different color [1][2][3]. Graceful coloring is the coloring of each point that induces the coloring of each sedge by calculating the difference between two neighboring points [4]. The minimum number of colors used in the vertex coloring of graph G is referred to as the chromatic numbers in the graceful coloring of G, which is denoted by $\chi_g(G)$.

Research on the chromatic number on graceful coloring has been done before. In 2016, Zhang [4] investigated the graceful coloring of path, wheel, complete bipartite, and circle graphs. In 2017, English et. al [5] studied graceful chromatic numbers in tree graphs. Furthermore, in 2019, Mincu et al [6] investigated the graceful coloring of 13 graphs goldner-harary, fritsch, friendship, fan, wagner, Petersen, house prism, octahedron, jellyfish, umbrella, spindle, and diamond graphs. In the same year, Alfarisi et al [7] studied graceful coloring on pan, tadpole, and sun graphs. In 2020, Sania et al [8] studied graceful staining on a family of uncyclic graphs consisting of bull, cricket, caveman, peach, and flowerpot graphs. Furthermore, in 2021, Khoirunnisa et al [9] continued their research with the same topic, but the graph used was the graph of the results of the comb operation.

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The graph that results from the comb operation is a ladder graph with a first-degree path and a ladder graph with a circle. Based on this description, research was conducted to find a particular graph's graceful chromatic number. The graph that will be studied is the result of vertex amalgamation operations in the tree graph family. The family of tree graphs used is path graphs, centipedes, brooms, and E.

This research will use a point amalgamation operation. The vertex amalgamation operation is the acquisition of a new graph from several graphs shown by attaching it to the point that has been selected, which is called a terminal point [10][11][12][13]. The reason for choosing the vertex amalgamation operation is that the basic graph arrangement of the vertex amalgamation cannot change when it is operated compared to the edge amalgamation operation, so the colouring pattern used is no different from the base graph [14].

This article aims to obtain the of a vertex amalgamation of a tree graph family. A tree graph family is a graph that has the same properties as a tree graph. For example, a connected graph with no circuits is called a tree graph [15]. If graph G has the same starting and ending points, then the graph is a circuit graph. Path graph, sweep graph, centipede graph, and graph E are some examples of tree graphs [16][17][18].

In order to achieve the objectives of the research in this article, here are some definitions, lemmas, and propositions related to graceful coloring.

Definition 1. [4] Graceful k-coloring of graph G is proper vertex coloring $c: V(G) \rightarrow \{1,2,\dots,k\}$, where $k \geq 2$ induces proper edge coloring $c': E(G) \rightarrow \{1,2,\dots,k-1$ defined $c'(uv) = |c(u) - c(v)|$. Proper vertex coloring c of graph G is graceful coloring if c is graceful k-coloring for $k \in N$.

Definition 2. [4] The graceful chromatic number of a graph G, denoted $\chi_g(G)$, is the minimum m where G has graceful m-coloring.

Definition 3. [10] The amalgamation operation $amal(G_i, v_{0i}, t)$ is a graph operation used to obtain a new graph by gluing all graphs G_i as much as t at the terminal point (v_{0i}).

Lemma 1. [4] If H is a subgraph of a graph G, then $\chi_g(G) \geq \chi_g(H)$.

Lemma 2. [4] it follows that if G is a nontrivial connected graph, then $\chi_g(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is maximum degree in G.

Proposition 1. [4] For each integer $n \geq 5$, $\chi_g(P_n) = 4$

METHODS

The method used in this research is the axiomatic deductive method and pattern recognition. The detection method is used to find coloring patterns and graceful chromatic numbers on the result of the amalgamation operation of point family tree graphs.

RESULTS AND DISCUSSION

In this article, we determine the graceful chromatic numbers in graphs resulting from the amalgamation vertex of family tree graphs. The following are the results of the research presented in four theorems.

Theorem 1. Based on Definition 3, given a graph of $amal(P_n, v, m)$ for $n \geq 3$, then the graceful coloring chromatic number is

$$\chi_g amal(P_n, v, m) = \begin{cases} 4, & \text{for } m = 2 \\ m+1, & \text{for } m \geq 3\,. \end{cases}$$

Proof. The graph $amal(P_n, v, m)$ has a vertex set $V(amal(p_n, v, m)) = \{x_j^i \mid 1 \le i \le m, 1 \le j \le n-1\} \cup \{v\}$ and an edge set $E(amal(p_n, v, m)) = \{vx_1^i, x_j x_{j+1} \mid 1 \le i \le m, 1 \le j \le n-2\}$. Proving graceful chromatic numbers on a graph $amal(P_n, v, m)$ will be divided into two cases as follows.

Case 1. $m = 2$

We know that graph P_n is a subgraph of graph $amal(P_n, v, m)$, such that based on Lemma 1 and Proposition 1 we get $\chi_g amal(P_n, v, m) \ge \chi_g(P_n) = 4$ or $\chi_g amal(P_n, v, m) \ge 4$. Furthermore, we prove that $\chi_g amal(P_n, v, m) \le 4$, define a proper vertex coloring $c : V(amal(P_n, v, m)) \to \{1,2, \dots ,4\}$ as follows

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0 \ (mod\ 4); 1 \le i \le n-1; 1 \le z \le 2\} \\ 2, & \text{for } v \in \{x_i^1, i \equiv 1 \ (mod\ 4); 1 \le i \le n-1\}, \{x_i^2, i \equiv 3 \ (mod\ 4); 1 \le i \le n-1\} \\ 3, & \text{for } v \in \{x_i^1, i \equiv 3 \ (mod\ 4); 1 \le i \le n-1\}, \{x_i^2, i \equiv 1 \ (mod\ 4); 1 \le i \le n-1\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2 \ (mod\ 4); 1 \le i \le n-1; 1 \le z \le 2\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(P_n, v, m)) \to \{1,2,3\}$coloring as follows.

$$f(e) = \begin{cases} 1, & \text{for } e \in \{Vx_1^1\}, \{x_i^1 x_{i+1}^1, i \equiv 0 \ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^2 x_{i+1}^2, i \equiv 1 \ (mod\ 2); 1 \le i \le n-2\} \\ 2, & \text{for } e \in \{vx_1^2\}, \{x_i^1 x_{i+1}^1, i \equiv 1 \ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^2 x_{i+1}^2, i \equiv 0 \ (mod\ 2); 1 \le i \le n-2\} \end{cases}$$

f is a graceful 4−coloring of $amal(P_n, v, m)$. Therefore, it obtained that $\chi_g amal(P_n, v, m) \le 4$, hence $\chi_g amal(P_n, v, m) = 4$.

Case 2. $m \ge 3$

We know that $\Delta(amal(P_n, v, m)) = m$, such that based on Lemma 2 we get $\chi_g amal(P_n, v, m) \ge \Delta\ (amal(P_n, v, m)) + 1 = m + 1$ or $\chi_g amal(P_n, v, m) \ge \mathrm{m} + 1$. Furthermore, we prove that $\chi_g amal(P_n, v, m) \le \mathrm{m} + 1$, define a proper vertex coloring $c : V(amal(P_n, v, m)) \to \{1,2, \dots, \mathrm{m} + 1\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0 \ (mod\ 4); 1 < i \le n-1; 1 \le z \le 2\}, \\ & \{x_i^z, i \equiv 3 \ (mod\ 4); 1 < i \le n-1; 3 \le z \le m\} \quad , \\ 2, & \text{for } v \in \{x_i^1, i \equiv 1 \ (mod\ 4); 1 < i \le n-1\}, \{x_i^2, i \equiv 3 \ (mod\ 4); 1 < i \le n-1\}, \\ & \{x_i^z, i \equiv 2 \ (mod\ 4); 1 < i \le n-1; 3 \le z \le m\} \\ 3, & \text{for } v \in \{x_i^1, i \equiv 3 \ (mod\ 4); 1 < i \le n-1\}, \{x_i^2, i \equiv 1 \ (mod\ 4); 1 < i \le n-1\}, \\ & \{x_i^z, i \equiv 0 \ (mod\ 4); 1 < i \le n-1; 3 \le z \le m\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2 \ (mod\ 4); 1 < i \le n-1; 1 \le z \le 2\}, \\ & \{x_i^z, i \equiv 1 \ (mod\ 4); 1 < i \le n-1; 3 \le z \le m\} \\ z+1 & \text{for } v \in \{x_i^z, 1 \le z \le m\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(P_n, v, m)) \to \{1,2, \dots, \mathrm{m}\}$ coloring as follows.

$$f(e)=\begin{cases} 1, & \text{for } e \in \{x_i^1x_{i+1}^1, i \equiv 0\ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^2x_{i+1}^2, i \equiv 1\ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^zx_{i+1}^z, i \equiv 0\ (mod\ 2); 1 \le i \le n-2; 3 \le z \le m\} \\ 2, & \text{for } e \in \{x_i^1x_{i+1}^1, i \equiv 1\ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^2x_{i+1}^2, i \equiv 0\ (mod\ 2); 1 \le i \le n-2\}, \\ & \{x_i^zx_{i+1}^z, i \equiv 1\ (mod\ 2); 1 \le i \le n-2; 3 \le z \le m\} \\ z, & \text{for } e \in \{Vx_1^z, 1 \le z \le m\} \\ z-1, & \text{for } e \in \{x_1^zx_2^z, 3 \le z \le m\} \end{cases}$$

f is a graceful $m+1$–coloring of $amal(P_n, v, m)$. Therefore, it obtained that $\chi_g amal(P_n, v, m) \le \mathrm{m}+1$, hence $\chi_g amal(P_n, v, m) = \mathrm{m}+1$.

Theorem 2. Based on Definition 3, given a graph of $amal(Cp_n, v, m)$ for $n \ge 5$, then the graceful coloring chromatic number is

$$\chi_g amal(P_n, v, m) = \begin{cases} 5, & \text{for } m \le 3 \\ m+1, & \text{for } m \ge 4. \end{cases}$$

Proof. The graph $amal(\mathrm{C}p_n, v, m)$ has a vertex set $V(amal(\mathrm{C}p_n, v, m)) = \{x_i^z | 1 \le i \le n, 1 \le z \le m\} \cup \{y_i^z | 1 \le i \le n-1;\ 1 \le z \le m\} \cup \{v\}$ and an edge set $E\big(amal(Cp_n, v, m)\big) = \{vx_1^z | 1 \le z \le m\} \cup \{x_i^zx_{i+1}^z | 1 \le z \le m; 1 \le i \le n-1\} \cup \{x_{i+1}^zy_i^z | 1 \le i \le n-1;\ 1 \le z \le m\}$. Proving graceful chromatic numbers on a graph $amal(Cp_n, v, m)$ will be divided into two cases as follows.

Case 1. $m \le 3$

We know that graph P_n is a subgraph of graph $amal(Cp_n, v, m)$, such that based on Lemma 1 and Proposition 1 we get $\chi_g amal(Cp_n, v, m) \ge \chi_g(P_n) = 4$ or $\chi_g amal(Cp_n, v, m) \ge 4$. The proper vertex coloring will induce supposed $\chi_g amal(Cp_n, v, m) = 4$, and it is known that the centipede is a subgraph of the $amal(Cp_n, v, m)$. Then graceful coloring can be done on the centipede graph first. The proper side coloring is as follows. Given a graph Cp_n, according to the definition of graceful coloring of edges $c'\ (x_2x_3) \ne c'\ (x_3x_4) \ne c'\ (x_3y_2)$. Based on the several possibilities that have been obtained, we can see that the color of edge 3 that is formed can only be formed using a combination colors of vertex 1 and 4. While coloring the color of the next edge 3, we cannot do it. So it can be concluded if $\chi_g amal(Cp_n, v, m) \ge 5$. Furthermore, we prove that $\chi_g amal(Cp_n, v, m) \le 5$, define a proper vertex coloring $c : V(amal(Cp_n, v, m)) \to \{1,2, \ldots, 5\}$ as follows

$$f(v)=\begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 3\ (mod\ 4); 1 < i \le n; 1 \le z \le 2\}, \\ & \{x_i^3, i \equiv 1\ (mod\ 4); 1 < i \le n\} \quad , \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 4); 1 < i \le n; 1 \le z \le 2\}, \\ & \{x_i^3, i \equiv 3\ (mod\ 4); 1 < i \le n\}, \{y_1^2\} \\ 3, & \text{for } v \in \{y_i^1, 1 \le i \le n-1\}, \{y_i^2, 1 < i \le n-1\}, \{y_i^3, 1 \le i \le n-1\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 4); 1 < i \le n; 1 \le z \le 2\}, \\ & \{x_i^3, i \equiv 0\ (mod\ 4); 1 < i \le n\} \\ 5, & \text{for } v \in \{x_i^z, i \equiv 0\ (mod\ 4); 1 < i \le n; 1 \le z \le 2\}, \\ & \{x_i^3, i \equiv 2\ (mod\ 4); 1 < i \le n\} \\ z+1, & \text{for } v \in \{x_i^z, 1 \le z \le 3\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(Cp_n, v, m)) \rightarrow \{1,2, \dots ,4\}$ coloring as follows.

$$f(e) = \begin{cases} 1, & \text{for } e \in \{x_{i+1}^z y_i^z, i \equiv 1\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_{i+1}^z y_i^z, i \equiv 0\ (mod\ 4); 1 \leq i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_{i+1}^3 y_i^3, i \equiv 2\ (mod\ 4); 1 \leq i \leq n-1\}, \\ & \{x_{i+1}^3 y_i^3, i \equiv 3\ (mod\ 4); 1 \leq i \leq n-1\}, \{x_1^2 x_2^2\}, \{x_1^3 x_2^3\}, \{x_2^1 y_1^1\} \\ 2, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_i^3 x_{i+1}^3, i \equiv 3\ (mod\ 4); 1 < i \leq n-1\}, \\ & \{x_{i+1}^z y_i^z, i \equiv 2\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_{i+1}^z y_i^z, i \equiv 3\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_{i+1}^3 y_i^3, i \equiv 1\ (mod\ 4); 1 < i \leq n-1\}, \\ & \{x_{i+1}^3 y_i^3, i \equiv 0\ (mod\ 4); 1 < i \leq n-1\}, \{x_2^2 y_1^2\}, \{x_1^1 x_2^1\} \\ 3, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 3\}, \\ & \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\} \\ 4, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 3\ (mod\ 4); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_i^3 x_{i+1}^3, i \equiv 1\ (mod\ 4); 1 < i \leq n-1\} \\ z, & \text{for } e \in \{V x_1^z, 1 \leq z \leq 3\} \end{cases}$$

There is graceful m+1−coloring of $amal(Cp_n, v, m)$. Therefore, it obtained that $\chi_g amal(Cp_n, v, m) \leq \text{m} + 1$, hence $\chi_g amal(Cp_n, v, m) = \text{m} + 1$.

Case 2. For $m \geq 4$

We know that $\Delta(amal(Cp_n, v, m)) = \text{m}$, such that based on Lemma 2 we get $\chi_g amal(Cp_n, v, m) \geq \Delta\ (amal(Cp_n, v, m)) + 1 = m + 1$ or $\chi_g amal(Cp_n, v, m) \geq \text{m} + 1$. Furthermore, we prove that $\chi_g amal(Cp_n, v, m) \leq \text{m} + 1$, define a proper vertex coloring $c : V(amal(P_n, v, m)) \rightarrow \{1,2, \dots , \text{m} + 1\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 3\ (mod\ 4); 1 < i \leq n; 1 \leq z \leq 2\}, \\ & \{x_i^3, i \equiv 1\ (mod\ 4); 1 < i \leq n\}, \{x_i^z, i \equiv 0\ (mod\ 4); 1 < i \leq n; 4 \leq z \leq m\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 4); 1 < i \leq n; 1 \leq z \leq 2\},, \\ & \{x_i^3, i \equiv 3\ (mod\ 4); 1 < i \leq n\}, \{y_1^2\} \\ 3, & \text{for } v \in \{y_i^1, 1 \leq i \leq n-1\}, \{y_i^2, 1 < i \leq n-1\}, \\ & \{y_i^z, 1 \leq i \leq n-1; 3 \leq z \leq m\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 4); 1 < i \leq n; 1 \leq z \leq 2\}, \\ & \{x_i^3, i \equiv 0\ (mod\ 4); 1 < i \leq n\}, \\ & \{x_i^z, i \equiv 3\ (mod\ 4); 1 < i \leq n; 4 \leq z \leq m\} \\ 5, & \text{for } v \in \{x_i^z, i \equiv 0\ (mod\ 4); 1 < i \leq n; 1 \leq z \leq 2\}, \\ & \{x_i^3, i \equiv 2\ (mod\ 4); 1 < i \leq n\}, \\ & \{x_i^z, i \equiv 1\ (mod\ 4); 1 < i \leq n; 4 \leq z \leq m\} \\ z+1, & \text{for } v \in \{x_i^z, 1 \leq z \leq m\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(Cp_n, v, m)) \rightarrow \{1,2, \dots , \text{m}\}$ coloring as follows.

$$f(e)=\begin{cases} 1, & \text{for } e\in\{x_{i+1}^{z}y_i^{z}, i\equiv 1\ (mod\ 4); 1< i\le n-1; 1\le z\le 2\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 0\ (mod\ 4); 1\le i\le n-1; 1\le z\le 2\}, \\ & \{x_{i+1}^{3}y_i^{3}, i\equiv 2\ (mod\ 4); 1\le i\le n-1\}, \\ & \{x_{i+1}^{3}y_i^{3}, i\equiv 3\ (mod\ 4); 1\le i\le n-1\}, \{x_1^2x_2^2\}, \{x_1^3x_2^3\}, \{x_2^1y_1^1\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 1\ (mod\ 4); 1\le i\le n-1; 4\le z\le m\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 2\ (mod\ 4); 1\le i\le n-1; 4\le z\le m\} \\ 2, & \text{for } e\in\{x_i^{z}x_{i+1}^{z}, i\equiv 1\ (mod\ 4); 1< i\le n-1; 1\le z\le 2\}, \\ & \{x_i^{3}x_{i+1}^{3}, i\equiv 3\ (mod\ 4); 1< i\le n-1\}, \\ & \{x_i^{z}x_{i+1}^{z}, i\equiv 2\ (mod\ 4); 1< i\le n-1; 4\le z\le m\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 2\ (mod\ 4); 1\le i\le n-1; 1\le z\le 2\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 3\ (mod\ 4); 1\le i\le n-1; 1\le z\le 2\}, \\ & \{x_{i+1}^{3}y_i^{3}, i\equiv 1\ (mod\ 4); 1\le i\le n-1\}, \\ & \{x_{i+1}^{3}y_i^{3}, i\equiv 0\ (mod\ 4); 1< i\le n-1\}, \{x_2^2y_1^2\}, \{x_1^1x_2^1\} \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 3\ (mod\ 4); 1\le i\le n-1; 4\le z\le m\}, \\ & \{x_{i+1}^{z}y_i^{z}, i\equiv 0\ (mod\ 4); 1\le i\le n-1; 4\le z\le m\} \\ 3, & \text{for } e\in\{x_i^{z}x_{i+1}^{z}, i\equiv 2\ (mod\ 4); 1< i\le n-1; 1\le z\le 3\}, \\ & \{x_i^{z}x_{i+1}^{z}, i\equiv 0\ (mod\ 4); 1< i\le n-1; 1\le z\le 3\}, \\ & \{x_i^{z}x_{i+1}^{z}, i\equiv 3\ (mod\ 4); 1< i\le n-1; 4\le z\le m\}, \\ & \{x_i^{z}x_{i+1}^{z}, i\equiv 1\ (mod\ 4); 1< i\le n-1; 1\le z\le 2\} \\ 4, & \text{for } e\in\{x_i^{z}x_{i+1}^{z}, i\equiv 3\ (mod\ 4); 1< i\le n-1; 1\le z\le 2\}, \\ & \{x_i^{3}x_{i+1}^{3}, i\equiv 1\ (mod\ 4); 1< i\le n-1\} \\ & \{x_i^{z}x_{i+1}^{z}, i\equiv 0\ (mod\ 4); 1< i\le n-1; 4\le z\le m\}, \\ z-1, & \text{for } e\in\{x_1^{z}x_2^{z}, 4\le z\le m\} \\ z, & \text{for } e\in\{Vx_1^{z}, 1\le z\le m\} \end{cases}$$

There is graceful $m+1$–coloring of $amal(Cp_n, v, m)$. Therefore, it obtained that $\chi_g amal(Cp_n, v, m) \le \mathrm{m}+1$, hence $\chi_g amal(Cp_n, v, m) = \mathrm{m}+1$.

Theorem 3. Based on Definition 3, given a graph of $amal(B_{k,n}, v, m)$ for $n\ge 4$ and $k-n\ge 3$, then the graceful coloring chromatic number is

$$\chi_g amal(B_{k,n}, v, m) = \begin{cases} k-n+2, & \text{for } m+n< k+1 \\ m+1, & \text{for } m+n\ge k+1. \end{cases}$$

Proof. The graph $amal(B_{k,n}, v, m)$ has a vertex set $V(amal(B_{k,n}, v, m)) = \{x_i^z, y_j^z | 1\le i\le n-1, 1\le j\le k-n, 1\le z\le m\} \cup \{v\}$ and an edge set $E\left(amal(B_{k,n}, v, m)\right) = \{vx_1^z, x_i^z x_{i+1}^z, x_{i+1}^z y_j^z | 1\le i\le n-2, 1\le j\le k-n, 1\le z\le m\}$. Proving graceful chromatic numbers on a graph $amal(B_{k,n}, v, m)$ will be divided into two cases as follows.

Case 1. For $m+n< k+1$

We know that $\Delta(amal(B_{k,n}, v, m)) = k-n+1$, such that based on Lemma 2 we get $\chi_g amal(B_{k,n}, v, m) \ge \Delta\,(amal(B_{k,n}, v, m)) + 1 = k-n+2$ or $\chi_g amal(B_{k,n}, v, m) \ge k-\mathrm{n}+2$. Furthermore, we prove that $\chi_g amal(B_{k,n}, v, m) \le \mathrm{k}-\mathrm{n}+2$, define a proper vertex coloring $c: V(amal(B_{k,n}, v, m)) \to \{1,2,\dots,\mathrm{k}-\mathrm{n}+2\}$ as follows.

Subcase 1. $n \equiv 1$ (mod 3)

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq m\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-1; 3 \leq z \leq m\} \\ 3, & \text{untuk } v \in \{y_2^z, 1 \leq z \leq 2\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq 2\}, \\ & \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq n-1; 3 \leq z \leq m\}, \\ z+1, & \text{untuk } v \in \{x_i^z, 1 \leq z \leq m\} \\ j+2, & \text{for } v \in \left\{y_j^z, 3 \leq j \leq k-n; 1 \leq z \leq 2\right\}, \\ & \left\{y_j^z, 1 \leq j \leq k-n; 3 \leq z \leq m\right\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(B_{k,n}, v, m)) \rightarrow \{1,2,\dots,\mathrm{k}-\mathrm{n}+1\}$ coloring as follows.

$$f(e) = \begin{cases} 1, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq 2\}, \\ & \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-2; 3 \leq z \leq m\}, \\ & \{x_{n-1}^z y_1^z, 1 \leq z \leq 2\} \\ 2, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq m\}, \\ & \{x_{n-1}^2 y_2^z, 1 \leq z \leq 2\}, \{x_1^1 x_2^1\} \\ 3, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq 2\}, \\ & \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \leq n-2; 3 \leq z \leq m\} \\ z, & \text{for } e \in \{V x_1^z, 1 \leq z \leq m\} \\ z-1, & \text{for } e \in \{x_1^z x_2^z, 2 \leq z \leq m\} \\ j+1, & \text{for } e \in \left\{x_{n-1}^z y_j^z, 3 \leq j \leq k-n; 1 \leq z \leq 2\right\} \\ & \left\{x_{n-1}^z y_j^z, 1 \leq j \leq k-n; 3 \leq z \leq m\right\} \end{cases}$$

Subcase 2. For $n \equiv 2$ (mod 3)

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq m\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq m\} \\ 3, & \text{for } v \in \{x_2^3\}, \{x_2^z, 5 \leq z \leq m\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-2; 1 \leq z \leq 2 \text{ and } z = 4\}, \\ & \{x_i^z, i \equiv 2\ (mod\ 3); 2 < i \leq n-2; z = 3 \text{ dan } 5 \leq z \leq m\}, \\ z+1, & \text{for } v \in \{x_i^z, 1 \leq z \leq m\} \\ j+1, & \text{for } v \in \left\{y_j^z, 1 \leq j \leq k-n; 1 \leq z \leq m\right\} \\ k-n+2, & \text{for } v \in \{x_{n-1}^z, 1 \leq z \leq m\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(B_{k,n}, v, m)) \rightarrow \{1,2,\dots,\mathrm{k}-\mathrm{n}+1\}$ coloring as follows.

$$f(e)=\begin{cases} 1, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 0\ (mod\ 3); 1<i\le n-3; 1\le z\le m\}, \\ & \{x_1^z x_2^z, 2\le z\le 4\} \\ 2, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\le n-3; 1\le z\le m\}, \\ & \{x_2^z x_3^z, 5\le z\le m\}, \{x_2^3 x_3^3\}, \{x_1^1 x_2^1\} \\ 3, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 2\ (mod\ 3); 2<i\le n-3; 1\le z\le m\}, \\ & \{x_2^1 x_3^1\}, \{x_2^2 x_3^2\}, \{x_2^4 x_3^4\} \\ z, & \text{for } e\in\{Vx_1^z, 1\le z\le m\} \\ z-2 & \text{for } e\in\{x_1^z x_2^z, 5\le z\le m\} \\ k-n+1, & \text{for } e\in\{x_{n-2}^z x_{n-1}^z, 1\le z\le m\} \\ k-n-j+1, & \text{for } e\in\left\{x_{n-1}^z y_j^z, 1\le j\le k-n; 1\le z\le m\right\} \end{cases}$$

Subcase 3. For $n\equiv 0$ (mod 3)

$$f(v)=\begin{cases} 1, & \text{for } v\in\{V\}, \{x_i^z, i\equiv 0\ (mod\ 3); 1<i\le n-2; 1\le z\le m\}, \\ & \{y_1^z, 1\le z\le m\} \\ 2, & \text{for } v\in\{x_i^z, i\equiv 1\ (mod\ 3); 1<i\le n-2; 1\le z\le m\} \\ 3, & \text{for } v\in\{x_2^3\}, \{x_2^z, 5\le z\le m\} \\ 4, & \text{for } v\in\{x_i^z, i\equiv 2\ (mod\ 3); 1<i\le n-2; 1\le z\le 2 \text{ dan } z=4\}, \\ & \{x_i^z, i\equiv 2\ (mod\ 3); 2<i\le n-2; z=3 \text{ dan } 5\le z\le m\} \\ z+1, & \text{for } v\in\{x_i^z, 1\le z\le m\} \\ j+1, & \text{for } v\in\left\{y_j^z, 2\le j\le k-n; 1\le z\le m\right\} \\ k-n+2, & \text{for } v\in\{x_{n-1}^z, 1\le z\le m\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(B_{k,n}, v, m)) \rightarrow \{1,2,\dots,k-n+1\}$ coloring as follows.

$$f(e)=\begin{cases} 1, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 0\ (mod\ 3); 1<i\le n-3; 1\le z\le m\}, \\ & \{x_1^z x_2^z, 2\le z\le 4\} \\ 2, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\le n-3; 1\le z\le m\}, \\ & \{x_2^z x_3^z, 5\le z\le m\}, \{x_2^3 x_3^3\}, \{x_1^1 x_2^1\} \\ 3, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 2\ (mod\ 3); 2<i\le n-3; 1\le z\le m\}, \\ & \{x_2^1 x_3^1\}, \{x_2^2 x_3^2\}, \{x_2^4 x_3^4\} \\ z, & \text{for } e\in\{Vx_1^z, 1\le z\le m\} \\ z-2 & \text{for } e\in\{x_1^z x_2^z, 5\le z\le m\} \\ k-n & \text{for } e\in\{x_{n-2}^z x_{n-1}^z, 1\le z\le m\} \\ k-n+1, & \text{for } e\in\{x_{n-1}^z y_1^z, 1\le z\le m\} \\ k-n-j+1, & \text{for } e\in\left\{x_{n-1}^z y_j^z, 2\le j\le k-n; 1\le z\le m\right\} \end{cases}$$

There is graceful $k-n-2$–coloring of $amal(B_{k,n}, v, m)$. Therefore, it obtained that $\chi_g amal(B_{k,n}, v, m)\le k-n+2$, hence $\chi_g amal(B_{k,n}, v, m) = k-n+2$.

Case 2. For $m+n\ge\ k+1$

We know that $\Delta(amal(B_{k,n}, v, m)) = m$, such that based on Lemma 2 we get $\chi_g amal(B_{k,n}, v, m)\ge\Delta\ (amal(B_{k,n}, v, m))+1 = \mathrm{m}+1$ or $\chi_g amal(B_{k,n}, v, m)\ge m+1$. Furthermore, we prove that $\chi_g amal(B_{k,n}, v, m)\le \mathrm{m}+1$ as same Case 1.

Theorem 4. Based on Definition 3, given a graph of $amal(E_{3,n}, v, m)$ for $n \geq 2$, then the graceful coloring chromatic number is

$$\chi_g amal(E_{3,n}, v, m) = \begin{cases} 4, & \text{for } m = 2 \\ m+1, & \text{for } m \geq 3 \end{cases}$$

Proof. The graph $amal(E_{3,n}, v, m)$ has a vertex set $V(amal(E_{3,n}, v, m)) = \{x_i^z | 1 \leq i \leq 2n+2, 1 \leq z \leq m\} \cup \{y_i^z \ / 1 \leq i \leq n, 1 \ \leq z \leq m\} \cup \{v\}$ and an edge set $E\left(amal\left(E_{3,n}, v, m\right)\right) = \{vx_1^z | 1 \leq z \leq m\} \cup \{x_i^z x_{i+1}^z | 1 \leq i \leq 2n \ + \ 1, 1 \leq z \leq m\} \cup \{y_i^z y_{i+1}^z | 1 \leq i \leq n, 1 \leq z \leq m\} \cup \{x_{n+1}^z y_1^z | 1 \leq z \leq m\}$. Proving graceful chromatic numbers on a graph $amal\left(E_{3,n}, v, m\right)$ will be divided into two cases as follows.

Case 1. for $m = 2$

We know that graph P_n is a subgraph of graph $amal(E_{3,n}, v, m)$, such that based on Lemma 1 and Proposition 1 we get $\chi_g amal(E_{3,n}, v, m) \geq \chi_g(P_n) = 4$ or $\chi_g amal(E_{3,n}, v, m) \geq 4$. Furthermore, we prove that $\chi_g amal(B_{k,n}, v, m) \leq \mathrm{k} - \mathrm{n} + 2$, define a proper vertex coloring $c : V(amal(E_{3,n}, v, m)) \rightarrow \{1,2, \dots, 4\}$ as follows.

Subcase 1. For $n \equiv 1 \pmod 3$

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq 2n+2; 1 \leq z \leq 2\}, \\ & \{y_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq n; 1 \leq z \leq 2\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 0\ (mod\ 3); 3 < i \leq 2n+2; 1 \leq z \leq 2\}, \\ & \{x_3^2\}, \{y_i^z, i \equiv 0\ (mod\ 3); 1 < i \leq n; 1 < z \leq 2\}, \{x_1^1\} \\ 3, & \text{for } v \in \{x_3^1\}, \{y_1^z, 1 \leq z \leq 2\}, \{x_1^2\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq 2n+2; 1 \leq z \leq 2\ \}, \\ & \{y_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq n;\ 1 \leq z \leq 2\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(E_{3,n}, v, m)) \rightarrow \{1,2,3\}$ coloring as follows.

$$f(e) = \begin{cases} 1, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 3 < i \leq 2n+1; 1 \leq z \leq 2\}, \\ & \{x_{n+1}^z y_1^z, 1 \leq z \leq 2\}, \{x_2^1 x_3^1\}, \{x_1^2 x_2^2\}, \{x_3^2 x_4^2\}, \{vx_1^1\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq 2\} \\ 2, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 2 < i \leq 2n+1; 1 \leq z \leq 2\}, \\ & \{y_1^z y_2^z, 1 \leq z \leq 2\}, \{x_2^2 x_3^2\}, \{x_1^1 x_2^1\}, \{x_3^1 x_4^1\}, \{vx_1^2\} \\ & \{y_i^z y_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq 2\} \\ 3, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \leq 2n+1; 1 \leq z \leq 2\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \leq n-1; 1 \leq z \leq 2\} \end{cases}$$

Subcase 2. For $n \equiv 2 \pmod 3$

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0\ (mod\ 3); 1 < i \leq 2n+2; 1 \leq z \leq 2\}, \\ & \{y_i^z, i \equiv 0\ (mod\ 3); 1 < i \leq n; 1 \leq z \leq 2\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq 2n+2; 1 \leq z \leq 2\}, \\ & \{x_1^1\}, \{y_i^z, i \equiv 1\ (mod\ 3); 1 < i \leq n; 1 < z \leq 2\} \\ 3, & \text{for } v \in \{x_1^2\}, \{y_1^z, 1 \leq z \leq 2\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq 2n+2; 1 \leq z \leq 2\ \}, \\ & \{y_i^z, i \equiv 2\ (mod\ 3); 1 < i \leq n;\ 1 \leq z \leq 2\} \end{cases}$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(E_{3,n}, v, m)) \rightarrow \{1,2,3\}$ coloring as follows.

$$f(e)=\begin{cases}1, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 0\ (mod\ 3); 1<i\leq 2n+1; 1\leq z\leq 2\},\\ & \{y_1^z y_2^z, 1\leq z\leq 2\},\{x_1^2x_2^2\},\{vx_1^1\},\\ & \{y_i^z y_{i+1}^z, i\equiv 0(mod\ 3); 1<i\leq n; 1\leq z\leq 2\}\\ 2, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\leq 2n+1; 1\leq z\leq 2\},\\ & \{x_{n+1}^z y_1^z, 1\leq z\leq 2\},\{x_1^1x_2^1\},\{vx_1^2\}\\ & \{y_i^z y_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\leq n-1; 1\leq z\leq 2\}\\ 3, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 2\ (mod\ 3); 1<i\leq 2n+1; 1\leq z\leq 2\},\\ & \{y_i^z y_{i+1}^z, i\equiv 2\ (mod\ 3); 1<i\leq n; 1\leq z\leq 2\}\end{cases}$$

Subcase 3. $n\equiv 0$ (mod 3)

$$f(v)=\begin{cases}1, & \text{for } v\in\{V\},\{x_i^z, i\equiv 1\ (mod\ 3); 1<i\leq 2n+2; 1\leq z\leq 2\},\\ & \{y_i^z, i\equiv 0\ (mod\ 3); 1<i\leq n; 1\leq z\leq 2\}\\ 2, & \text{for } v\in\{x_i^2, i\equiv 0\ (mod\ 3); 1<i\leq 2n+2\},\\ & \{x_1^1\},\{y_1^1\},\{y_i^z, i\equiv 1\ (mod\ 3); 1<i\leq n; 1<z\leq 2\}\\ 3, & \text{for } v\in\{x_i^1, i\equiv 0\ (mod\ 3); 1<i\leq 2n+2\},\{y_1^2\},\{x_1^2\}\\ 4, & \text{for } v\in\{x_i^z, i\equiv 2\ (mod\ 3); 1<i\leq 2n+2; 1\leq z\leq 2\},\\ & \{y_i^z, i\equiv 2\ (mod\ 3); 1<i\leq n;\ 1\leq z\leq 2\}\end{cases}$$

The proper vertex coloring will induce the proper edge of $c'\ : E(amal(E_{3,n},v,m))\rightarrow\{1,2,3\}$ coloring as follows.

$$(e)=\begin{cases}1, & \text{for } e\in\{x_i^1 x_{i+1}^1, i\equiv 2\ (mod\ 3); 1<i\leq 2n+1\},\\ & \{x_1^2x_2^2\},\{y_1^2y_1^2\},\{vx_1^1\},\{x_i^2x_{i+1}^2, i\equiv 0\ (mod\ 3); 1<i\leq 2n+1\},\\ & \{y_i^z y_{i+1}^z, i\equiv 0(mod\ 3); 1<i\leq n; 1\leq z\leq 2\}\\ 2, & \text{for } e\in\{x_i^1 x_{i+1}^1, i\equiv 0\ (mod\ 3); 1<i\leq 2n+1\},\\ & \{x_i^2x_{i+1}^2, i\equiv 2\ (mod\ 3); 1<i\leq 2n+1\},\{x_{n+1}^2y_1^2\},\{vx_1^2\},\\ & \{y_i^z y_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\leq n-1; 1\leq z\leq 2\},\{y_2^1y_2^2\}\\ 3, & \text{for } e\in\{x_i^z x_{i+1}^z, i\equiv 1\ (mod\ 3); 1<i\leq 2n+1; 1\leq z\leq 2\},\\ & \{y_i^z y_{i+1}^z, i\equiv 2\ (mod\ 3); 1<i\leq n-1; 1\leq z\leq 2\}\end{cases}$$

There is graceful 4–coloring of $amal(E_{3,n},v,m)$. Therefore, it obtained that $\chi_g amal(E_{3,n},v,m)\leq 4$, hence $\chi_g amal(B_{k,n},v,m)=4$.

Case 2. For $m\geq 3$

We know that $\Delta(amal(E_{3,n},v,m))=\mathrm{m}$, such that based on Lemma 2 we get $\chi_g amal(E_{3,n},v,m)\geq\Delta\ (amal(E_{3,n},v,m))+1=m+1$ or $\chi_g amal(E_{3,n},v,m)\geq \mathrm{m}+1$. Furthermore, we prove that $\chi_g amal(E_{3,n},v,m)\leq \mathrm{m}+1$, define a proper vertex coloring $c: V(amal(E_{3,n},v,m))\rightarrow\{1,2,\ldots,\mathrm{m}+1\}$ as follows.

Subcase 1. For $n\equiv 1$ (mod 3)

$$f(v)=\begin{cases}1, & \text{for } v\in\{V\},\{x_i^z, i\equiv 1\ (mod\ 3); 1<i\leq 2n+2; 1\leq z\leq m\},\\ & \{y_i^z, i\equiv 2\ (mod\ 3); 1<i\leq n; 1\leq z\leq m\}\\ 2, & \text{for } v\in\{x_i^z, i\equiv 0\ (mod\ 3); 3<i\leq 2n+2; 1\leq z\leq m\},\\ & \{x_3^2\},\{y_i^z, i\equiv 0\ (mod\ 3); 1<i\leq n; 1<z\leq m\},\{x_2^z; 3\leq z\leq m\}\\ 3, & \text{for } v\in\{x_3^1\},\{x_3^z; 3\leq z\leq m\},\{y_1^z, 1\leq z\leq m\}\\ 4, & \text{for } v\in\{x_i^z, i\equiv 2\ (mod\ 3); 2<i\leq 2n+2; 1\leq z\leq m\},\\ & \{y_i^z, i\equiv 1\ (mod\ 3); 1<i\leq n;\ 1\leq z\leq m\},\{x_2^1\},x_2^2\}\\ z+1, & \text{for } v\in\{x_1^z, 1\leq z\leq m\}\end{cases}$$

The proper vertex coloring will induce the proper edge of c' $: E(amal(E_{3,n}, v, m)) \rightarrow \{1,2,3, \dots, \mathrm{m}\}$ coloring as follows.

$$f(e) = \begin{cases} 1, & \text{untuk } e \in \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 3 < i \le 2n+1; 1 \le z \le m\}, \\ & \{x_{n+1}^z y_1^z, 1 \le z \le m\}, \{x_2^1 x_3^1\}, \{x_3^2 x_4^2\}, \{x_2^z x_3^z, 3 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le n-1; 1 \le z \le m\} \\ 2, & \text{untuk } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 2 < i \le 2n+1; 1 \le z \le m\}, \\ & \{y_1^z y_2^z, 1 \le z \le m\}, \{x_2^2 x_3^2\}, \{x_3^z x_4^z, z = 1 \text{ dan } 3 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \le n-1; 1 \le z \le M\}, \{x_1^1 x_2^1\} \\ 3, & \text{untuk } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+1; 1 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le n-1; 1 \le z \le m\} \\ z, & \text{untuk } e \in \{v x_1^z, 1 \le z \le m\} \\ z-1, & \text{untuk } e \in \{x_1^z x_2^z, 2 \le z \le m\} \end{cases}$$

Subcase 2. For $n \equiv 2$ (mod 3)

$$f(v) = \begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 0\ (mod\ 3); 1 < i \le 2n+2; 1 \le z \le m\}, \\ & \{y_i^z, i \equiv 0\ (mod\ 3); 1 < i \le n; 1 \le z \le m\} \\ 2, & \text{for } v \in \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+2; 1 \le z \le 2\} \\ & \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+2; 3 \le z \le m\}\ , \\ & \{y_i^z, i \equiv 1\ (mod\ 3); 1 < i \le n; 1 < z \le m\} \\ 3, & \text{for } v \in \{y_1^z, 1 \le z \le m\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+2; 1 \le z \le 2\}, \\ & \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+2; 3 \le z \le m\}, \\ & \{y_i^z, i \equiv 2\ (mod\ 3); 1 < i \le n;\ 1 \le z \le m\} \\ z+1, & \text{for } v \in \{x_1^z, 1 \le z \le m\} \end{cases}$$

The proper vertex coloring will induce the proper edge of c' $: E(amal(E_{3,n}, v, m)) \rightarrow \{1,2,3, \dots, \mathrm{m}\}$ coloring as follows.

$$f(e) = \begin{cases} 1, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \le 2n+1; 1 \le z \le 2\} \\ & \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+1; 3 \le z \le m\}\ , \\ & \{y_1^z y_2^z, 1 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 0 (mod\ 3); 1 < i \le n; 1 \le z \le m\} \\ 2, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+1; 1 \le z \le m\}, \\ & \{x_{n+1}^z y_1^z, 1 \le z \le m\}, \{x_1^1 x_2^1\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le n-1; 1 \le z \le m\} \\ 3, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+1; 1 \le z \le 2\}, \\ & \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \le 2n+1; 3 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le n; 1 \le z \le m\} \\ z, & \text{for } e \in \{v x_1^z, 1 \le z \le m\} \\ z-1, & \text{for } e \in \{x_1^z x_2^z, 2 \le z \le m\} \end{cases}$$

Subcase 3. For $n \equiv 0$ (mod 3)

$$f(v)=\begin{cases} 1, & \text{for } v \in \{V\}, \{x_i^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+2; 1 \le z \le m\}, \\ & \{y_i^z, i \equiv 0\ (mod\ 3); 1 < i \le n; 1 \le z \le m\} \\ 2, & \text{for } v \in \{x_i^2, i \equiv 0\ (mod\ 3); 1 < i \le 2n+2\}, \\ & \{x_2^z, 3 \le z \le m\}, \{y_i^z, i \equiv 1\ (mod\ 3); 1 < i \le n; 1 < z \le m\}, \\ & \{y_1^z, z = 1 \text{ dan } 3 \le z \le m\} \\ 3, & \text{for } v \in \{x_i^z, i \equiv 0\ (mod\ 3); 1 < i \le 2n+2; z = 1 \text{ and } 3 \le z \le m\} \\ & \{y_1^2\}, \{x_1^2\} \\ 4, & \text{for } v \in \{x_i^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+2; 1 \le z \le m\}, \\ & \{y_i^z, i \equiv 2\ (mod\ 3); 1 < i \le n;\ 1 \le z \le m\}, \{x_2^1\}, \{x_2^2\} \\ z+1, & \text{for } v \in \{x_1^z, 1 \le z \le m\} \end{cases},$$

The proper vertex coloring will induce the proper edge of $c' : E(amal(E_{3,n}, v, m)) \to \{1,2,3,\dots,\mathrm{m}\}$ coloring as follows.

$$f(e)=\begin{cases} 1, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le 2n+1, z = 1, \\ & \text{and } 3 \le z \le m, \{y_1^2 y_1^2\}, \{x_i^2 x_{i+1}^2, i \equiv 0\ (mod\ 3); 1 < i \le 2n+1\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 0 (mod\ 3); 1 < i \le n; 1 \le z \le m\}, \\ & \{x_{n+1}^z y_1^z, z = 1 \text{ dan } 3 \le z \le m\} \\ 2, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 0\ (mod\ 3); 1 < i \le 2n+1; z = 1 \\ & \text{and } 3 \le z \le m\}, \{x_i^2 x_{i+1}^2, i \equiv 2\ (mod\ 3); 1 < i \le 2n+1\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le n-1; 1 \le z \le m\}, \\ & \{y_2^z y_2^z, z = 1 \text{ and } 3 \le z \le m\}, \{x_1^1 x_2^1\}, \{x_{n+1}^2 y_1^2\} \\ 3, & \text{for } e \in \{x_i^z x_{i+1}^z, i \equiv 1\ (mod\ 3); 1 < i \le 2n+1; 1 \le z \le m\}, \\ & \{y_i^z y_{i+1}^z, i \equiv 2\ (mod\ 3); 1 < i \le n-1; 1 \le z \le m\} \\ z, & \text{for } e \in \{v x_1^z, 1 \le z \le m\} \\ z-1, & \text{for } e \in \{x_1^z x_2^z, 2 \le z \le m\} \end{cases},$$

There is graceful $m+1$–coloring of $amal(E_{3,n}, v, m)$. Therefore, it obtained that $\chi_g amal(E_{3,n}, v, m) \le \mathrm{m}+1$, hence $\chi_g amal(B_{k,n}, v, m) = m+1$.

CONCLUSIONS

Based on the discussion described in chapter four, four new theorems of graceful coloring are obtained from the amalgamation operation of vertex family tree graphs. The graceful chromatic number obtained is as follows.

$$\chi_g amal(P_n, v, m) = \begin{cases} 4, & \text{for } m = 2 \\ m+1, & \text{for } m \ge 3. \end{cases}$$

$$\chi_g amal(P_n, v, m) = \begin{cases} 5, & \text{for } m \le 3 \\ m+1, & \text{for } m \ge 4. \end{cases}$$

$$\chi_g amal(B_{k,n}, v, m) = \begin{cases} k-n+2, & \text{for } m+n < k+1 \\ m+1, & \text{for } m+n \ge k+1. \end{cases}$$

$$\chi_g amal(E_{3,n}, v, m) = \begin{cases} 4, & \text{for } m = 2 \\ m+1, & \text{for } m \ge 3 \end{cases}$$

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