CAUCHY –Jurnal Matematika Murni dan Aplikasi
Volume 7(3) (2022), Pages 445-463
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Mathematical Model of Iteroparous and Semelparous Species Interaction

Arjun Hasibuan*, Asep Kuswandi Supriatna, and Ema Carnia

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Jl. Raya Bandung-Sumedang KM 21, Jatinangor, Sumedang, West Java Province, 45363, Indonesia

Email: arjun17001@mail.unpad.ac.id

ABSTRACT

To study the survival of a species in an ecosystem it is very important to consider the dynamics of the species. A species can be categorized based on its reproductive strategy either semelparous or iteroparous. In this paper, we examine the dynamics involving both categories of species in an ecosystem. We focus on one semelparous and one iteroparous species influenced by density-dependent and also by harvesting factors in which there are two age classes for each species. We study two different models, i.e competitive and non-competitive models. We also consider two type of competition, i.e intraspecific and interspecific competition. The approach that we use in this research is the multispecies Leslie matrix model. In addition, we use *M*-Matrix theory to obtain the locally stable asymptotically of the model. Our results show that the level of competition both intraspecific and interspecific competition affect the co-existence equilibrium point and the stability of the equilibrium point. We also present explicitly the conditions for all equilibrium points to exist and to be locally stable asymptotically. This theory can be applied to study the dynamics of natural resource models including the effects of different management to the growth of the resources, such as in fisheries.

Keywords: density-dependent; harvesting; multispecies; Leslie matrix; age-structured model

INTRODUCTION

In an ecosystem, the survival of a species is an important thing to study. Species in the same ecosystem have reciprocal relationships between one species and other. The survival of each species can be affected by density-dependent, harvesting, competition, predator-prey, and so on. Of course, the important thing to do is to ensure the survival of these species to survive. The survival of a species can be studied with a system dynamics approach. In some studies, species in an ecosystem can be categorized based on their reproductive strategy, including species with semelparous and iteroparous strategy. Research on semelparous species can be seen in [1]–[3]. Then, research on iteroparous species can be seen in [4]–[6]. Semelparous species are species that reproduce only once in their lifetime shortly before dying. Then, iteroparous species are species that reproduce more than once in the lifetime of the species. Both species allow to live together and interact in the same ecosystem. In this research, we focus on studying the growth of multispecies cases consisting of one semelparous species and one iteroparous species

Submitted: June 16, 2022 Reviewed: June 22, 2022 Accepted: June 27, 2022
DOI: http://dx.doi.org/10.18860/ca.v7i3.16447

using a dynamic system approach, especially using the Leslie matrix model. This model is a population growth model based on age class which was introduced in 1945 by Leslie in [7].

Research on studying the dynamics of population growth using the Leslie matrix model has been carried out by several researchers. These studies can be in the form of single species and multispecies cases. Several studies on single species cases include in [3], [8], [9], and many more. Then, several multispecies studies examine the effect of density-dependent on the Leslie matrix model which is one of the nonlinear models of the Leslie matrix model. In 1968, Pennycuick et al. [10] focused on simulating the case of single species and multispecies interacting in the form of competition and predator-prey using the Leslie matrix. In 1980, Travis et al. [11] reviewed two competing species and provided a case study on semelparous. In 2011, Kon [12] studied two semelparous species with one species containing two age classes while the other species amounting to one age class. In 2012, Kon [13] conducted a study on two semelparous species that have a predator-prey relationship and observed the effect of coprime traits from the number of age classes in both species. Coprime is a condition where two numbers have the greatest common factor of one, in which case the number is the number of age classes of each species. Then in 2017, Kon [1] examined the Leslie multispecies semelparous matrix model which has an arbitrary number of age classes. Then, there are also studies on multispecies but with other methods using the Rosenzweig-MacArthur model (See [14], [15]), the Leslie-Gower model (See [16], [17]), and the Lotka-Volterra model (See [18]–[20]).

Our aim in this paper is to study the growth dynamics of an ecosystem consisting of one semelparous species and an iteroparous species with two age classes in each species. In addition, we combined the density-dependent effect of the first age class for the two species. Then, we consider the effect of harvesting that occurs in the second age class in each species on the growth of each species. Next, we divide the case into two models consisting of without competition and with competition. Both models were analyzed and seen the influence of the level of intraspecific and interspecific competition on the equilibrium point and locally stable asymptotically for each equilibrium point.

METHODS

Leslie's Matrix Model with One Iteroparous Species and One Semelparous Species Without Competition

In this section, we present one of the models that we studied, namely the multispecies Leslie matrix model with the case of one iteroparous species (x) and one semelparous species (y) in this case each species has two age classes. In this first model, we assume that the growth of both species is influenced by density-dependent occurrence in the first age class and harvesting is carried out in the second age class. In this case, the density-dependent problem used in the model uses the classical Beverton-Holt function which is also used in Wikan's research [21]. This problem is presented in the following model and we refer to as Model 1:

$$
\begin{aligned}
x_1(t+1) &= \frac{f_{x1}}{1+x_1(t)+y_1(t)}x_1(t) + f_{x2}x_2(t) \\
x_2(t+1) &= \frac{s_{x1}(1-h_{x2})}{1+x_1(t)+y_1(t)}x_1(t) \qquad (1)
\end{aligned}
$$

$$y_1(t+1) = f_{y2} y_2(t)$$
$$y_2(t+1) = \frac{s_{y1}(1-h_{y2})}{1+x_1(t)+y_1(t)} y_1(t)$$

There are several parameters in the Model 1. First, $f_{x1} > 0$ and $f_{x2} > 0$ are the birth rates of the 1st and 2nd age classes of species x, respectively. Second, $f_{y2} > 0$ is the birth rate of the 2nd age classes of species y. Third, $0 < s_{x1}, s_{y1} < 1$ are the survival rates of the 1st age classes of species x and y, respectively. Fourth, $0 < h_{x2}, h_{y2} < 1$ are the harvesting rates of the 2nd age classes of species x and y, respectively. In addition, the variables $x_i(t)$, and $y_i(t)$ represent the total population of each species x and y for the age class $i = \{1,2\}$. Simply put, equation 1 in (1) means that the population of the first age class of species x at time $t+1$ is obtained by adding the number of newborn from the first age class and the second class at time t. The newborn of first age class is affected by density-dependent while the newborn of the second age class is not affected by density-dependent. Then, equation 2 in (1) means that the number of population of the second age class of species x at time $t+1$ is obtained from the number of surviving populations which is influenced by density-dependent of the first age class at time t. Equations 3 and 4 in (1) have the same meaning as equations 1 and 2 in (1) but in species y there is no birth in the first age class. Model 1 is constructed based on research conducted by Leslie [7], Travis et al. [11], and Wikan [21]. In addition, Model 1 is adjusted based on the assumptions and simplifications in this research.

Leslie Matrix Model with One Iteroparous Species and One Semelparous Species with Competition Effect

In this section, we present a model which is an extension of the previous model. The problems raised in this section involve the effect of competition between the same species, also known as intraspecific competition, and competition between different species, also known as interspecific competition. The following is an extension of the Model 1 and we refer to it as Model 2.

$$\begin{aligned}
x_1(t+1) &= \frac{f_{x1}}{1+ax_1(t)+by_1(t)} x_1(t) + f_{x2} x_2(t) \\
x_2(t+1) &= \frac{s_{x1}(1-h_{x2})}{1+ax_1(t)+by_1(t)} x_1(t) \\
y_1(t+1) &= f_{y2} y_2(t) \\
y_2(t+1) &= \frac{s_{y1}(1-h_{y2})}{1+bx_1(t)+ay_1(t)} y_1(t)
\end{aligned} \tag{2}$$

The description of the parameters and variables in the Model 2 is the same as in Model 1 where the difference is only in parameters $a > 0$ and $b > 0$. In Model 2, to simplify the problem, we assume that the level of competition between the first age class in species x and species y has the same value, namely a. Then, the level of competition between the first age class in species y against species x and vice versa has the same value, namely b. The level of competition within the same species is referred to as intraspecific competition (a) and the level of competition between different species is referred to as interspecific competition (b). Details of Model 2 are the same as Model 1 but there are differences in the first age class birth rate, first age class survival rate and second age class harvesting rate in species x which are influenced by density-dependent and competition. Then, species y is only affected by density-dependent and competition on the survival rate of the first age class and the level of harvesting of the second age class. Model 2 is constructed based on research conducted by Leslie [7], Travis et al. [11], Wikan [21], and Cushing [22]. In addition, Model 2 is adjusted based on the assumptions and simplifications in this research.

Local Stability Criteria Using M-Matrix

In this section, we present the definition of M-matrix and the theorem that ensures a matrix has absolute eigenvalues less than one to determine the locally stable asymptotically of the model.

Definition 1. (See [11] or [23]) **(M-Matrix)**

A square matrix of size n, for example, $M = (m_{ij})$ $(1 \leq i, j \leq n)$ is called an M-matrix if it is satisfied that $m_{ij} \leq 0$ $\forall i \neq j$ and if any of the following things are true:

1. All minor principals of the M matrix are positive
2. All eigenvalues of the M matrix have a positive real part
3. The matrix M is a non-singular matrix and M^{-1} is positive
4. There is a vector $v > 0$ so that it meets $Mv > 0$ or
5. There is a vector $u > 0$ s so that it satisfies $M^T u > 0$

Theorem 1. (See [11])

Suppose a matrix J has the following form

$$J = \begin{bmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{bmatrix}$$

and the matrix $G = I - SJS^{-1}$ is an M-matrix with

$$S = I \quad \text{if } B \text{ and } C \geq 0$$

or

$$S = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} \quad \text{if } B \text{ and } C \leq 0,$$

where I, I_m, and I_n are identity matrices with sizes $m+n$, m, and n, respectively, then matrix J has a spectral radius of less than one. The spectral radius is the largest modulus of all the eigenvalues.

Theorem 1 and Definition 1 are used to determine the locally stable asymptotically of Model 1 and Model 2 in the Results and Discussion section.

RESULTS AND DISCUSSION

In the previous section, we have presented Model 1, Model 2, Definition 1 and Theorem 1. In this section, the two models, definition, and theorem will then be used to analyze the equilibrium point and the locally stable asymptotically of each equilibrium point.

Equilibrium Point of Model 1

The equilibrium point of Model 1 can be obtained by expressing the variables x and y to the left of the Model 1 depending on time t. The equilibrium Model 1 is obtained as follows:

$$
\begin{aligned}
x_1(t) &= \frac{f_{x1}}{1 + x_1(t) + y_1(t)} x_1(t) + f_{x2} x_2(t) \\
x_2(t) &= \frac{s_{x1}(1 - h_{x2})}{1 + x_1(t) + y_1(t)} x_1(t) \\
y_1(t) &= f_{y2} y_2(t) \\
y_2(t) &= \frac{s_{y1}(1 - h_{y2})}{1 + x_1(t) + y_1(t)} y_1(t)
\end{aligned}
\tag{3}
$$

Then by determining the solution of equation (3), the equilibrium point of the Model 1 is obtained as follows:

1. The equilibrium point with both species going extinct is

$$
E_0 = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

2. The equilibrium point with species x exists while species y is extinct, i.e

$$
E_x = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_x - 1 \\ \dfrac{(R_x - 1) s_{x1}(1 - h_{x2})}{R_x} \\ 0 \\ 0 \end{bmatrix}.
$$

   The condition E_x exists if it is fulfilled $R_x = f_{x2} s_{x1}(1 - h_{x2}) + f_{x1} > 1$. R_x is referred to as the expected number of offspring per individual per lifetime when density-dependent effects are neglected on harvest-influenced growth of species x.

3. The equilibrium point with species y exists while species x is extinct, i.e

$$
E_y = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R_y - 1 \\ \dfrac{R_y - 1}{f_{y2}} \end{bmatrix}.
$$

   The condition E_y exists if it is fulfilled $R_y = f_{y2} s_{y1}\left(1 - h_{y2}\right) > 1$. R_y is referred to as the expected number of offspring per individual per lifetime when density-dependent effects are neglected on harvest-influenced growth of species y.

It can be seen that in the model (1) there is no equilibrium point where the two species survive or co-existence equilibrium point.

Locally Stable Asymptotically at the Equilibrium Point of Model 1

In this section, we perform a locally stable asymptotically analysis of the Model 1. The equilibrium point is said to be asymptotically stable if it is satisfied that the spectral radius of the Jacobian matrix at the equilibrium point is less than one. The locally stable asymptotically of each equilibrium point of the Model 1 is stated in the following theorem.

Theorem 2. (Locally Stable Asymptotically at the Equilibrium Point Model 1)

For the Leslie multispecies matrix model with the case of one iteroparous species and one semelparous species in which there are two classes each whose growth is influenced by density-dependent, harvesting and without the influence of competition described in the Model 1, among others:

1. The equilibrium point E_0 is locally stable asymptotically if $R_x < 1$ and $R_y < 1$.
2. The equilibrium point E_x is locally stable asymptotically if $R_x > R_y$ and $R_x > 1$.
3. The equilibrium point E_y is locally stable asymptotically if $R_y > R_x$ and $R_y > 1$.

Proof:

In determining the stability of all equilibrium points of the Model 1, it can be obtained by determining the linearization of the Model 1, namely

$$J(E) = J\left(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} \dfrac{f_{x1}(1+y_1)}{(1+x_1+y_1)^2} & f_{x2} & -\dfrac{f_{x1}x_1}{(1+x_1+y_1)^2} & 0 \\ P_x(1+y_1) & 0 & -P_x x_1 & 0 \\ 0 & 0 & 0 & f_{y2} \\ -P_y y_1 & 0 & P_y(1+x_1) & 0 \end{bmatrix} \tag{4}$$

where

$$P_x = \frac{s_{x1}(1-h_{x2})}{(1+x_1+y_1)^2} \text{ and } P_y = \frac{s_{y1}(1-h_{y2})}{(1+x_1+y_1)^2}.$$

The reason for using the M-Matrix theory in this study is due to the complexity of determining the spectral radius matrix $J(E)$ at the corresponding equilibrium point E. The use of Definition 1 and Theorem 1 guarantee that the spectral radius value of the $J(E)$ matrix at the corresponding equilibrium is less than one.

Based on the $J(E)$ matrix in (4) the elements of rows 1-2 columns 3-4 and rows 3-4 columns 1-2 are non-positive because the elements of the equilibrium point are guaranteed to be zero or positive. Theorem 1 says matrix S for matrix $J(E)$ is

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The next step is to substitute all the equilibrium points in (4) and analyze their stability one by one.

1. The Jacobian matrix for the equilibrium point E_0 is

$$J(E_0) = \begin{bmatrix} f_{x1} & f_{x2} & 0 & 0 \\ s_{x1}(1-h_{x2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{y2} \\ 0 & 0 & s_{y1}(1-h_{y2}) & 0 \end{bmatrix}.$$

Next, determine the matrix $G = I - S\big(J(E_0)\big)S^{-1}$ and examine the element g_{ij} for $i \neq j$ that is non-positive and that all minor principals of G are positive. If these conditions are met, then the G matrix is an M-Matrix so that the spectral radius $J(E_0)$ is less than one. As a result, the equilibrium point E_0 is locally stable asymptotically. Here we present the obtained matrix

$$G = \begin{bmatrix} 1-f_{x1} & -f_{x2} & 0 & 0 \\ -s_{x1}(1-h_{x2}) & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{y2} \\ 0 & 0 & -s_{y1}(1-h_{y2}) & 1 \end{bmatrix}$$

and all minor principals of G obtained are

$$PM_1 = |g_{11}| = 1 - f_{x1}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1 - R_x,$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = 1 - R_x, \text{and } PM_4 = |G| = (1 - R_x)\big(1 - R_y\big).$$

In this paper, we define PM_i $(i = 1,2,3,4)$ as the i-th minor principal of the matrix G for each equilibrium point under consideration. It is clear that the element $g_{ij} < 0$ for $i \neq j$ in the G matrix is nonpositive by recalling the previously defined parameters. Then, PM_2 and PM_4 will be positive if met $R_x < 1$. In addition, $R_x < 1$ implicitly results in $f_{x1} < 1$ so that $PM_1 > 0$. Furthermore, PM_4 is positive if $R_y < 1$ because it must be fulfilled that $R_x < 1$. Therefore, G is an M-Matrix, that is if it is filled with $R_x < 1$ and $R_y < 1$. Hence, according to Theorem 1, the equilibrium point E_0 is locally stable asymptotically if $R_x < 1$ and $R_y < 1$.

2. The Jacobian matrix for the equilibrium point E_x is

$$J(E_x) = \begin{bmatrix} \dfrac{f_{x1}}{R_x^2} & f_{x2} & \dfrac{f_1(1 - R_x)}{R_x^2} & 0 \\ \dfrac{s_{x1}(1 - h_{x2})}{R_x^2} & 0 & \dfrac{s_{x1}(1 - h_{x2})(1 - R_x)}{R_x^2} & 0 \\ 0 & 0 & 0 & f_{y2} \\ 0 & 0 & \dfrac{s_{y1}\big(1 - h_{y2}\big)}{R_x} & 0 \end{bmatrix}$$

Next, determine the matrix $G = I - S\big(J(E_x)\big)S^{-1}$ and make sure the matrix G is an M-Matrix. Here we present the obtained matrix

$$G = \begin{bmatrix} \frac{R_x^2 - f_{x1}}{R_x^2} & -f_{x2} & \frac{f_1(1-R_x)}{R_x^2} & 0 \\ \frac{-s_{x1}(1-h_{x2})}{R_x^2} & 1 & \frac{s_{x1}(1-h_{x2})(1-R_x)}{R_x^2} & 0 \\ 0 & 0 & 1 & -f_{y2} \\ 0 & 0 & -\frac{s_{y1}(1-h_{y2})}{R_x} & 1 \end{bmatrix}$$

and all minor principals of G obtained are

$$PM_1 = |g_{11}| = \frac{R_x^2 - f_{x1}}{R_x^2}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = -\frac{1 - R_x}{R_x},$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = -\frac{1 - R_x}{R_x}, \text{and } PM_4 = |G| = -\frac{(1 - R_x)(R_x - R_y)}{R_x}.$$

Based on the previously defined parameters, the element g_{ij} for $i \neq j$ will be non-positive if $R_x > 1$ is satisfied. Because of $R_x > 1$, consequently PM_2 and PM_3 are positive. Besides that, PM_4 is also positive but with the additional condition that is $R_x > R_y$. Then, it is clear that $f_{x1} < R_x^2 = (f_{x2}s_{x1}(1 - h_{x2}) + f_{x1})^2$ so that $PM_1 > 0$. Therefore, G is an M-Matrix, if it is fulfilled $R_x > R_y$ and $R_x > 1$. Hence, according to Theorem 1, the equilibrium point E_x is locally stable asymptotically if $R_x > R_y$ and $R_x > 1$.

3. The Jacobian matrix for the equilibrium point E_y is

$$J(E_y) = \begin{bmatrix} \dfrac{f_{x1}}{R_y} & f_{x2} & 0 & 0 \\ \dfrac{s_{x1}(1-h_{x2})}{R_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{y2} \\ -\dfrac{s_{y1}(1-h_{y2})(R_y-1)}{R_y^2} & 0 & \dfrac{1}{f_4 R_y} & 0 \end{bmatrix}$$

Next, determine the matrix $G = I - S\left(J(E_y)\right)S^{-1}$ and make sure the matrix G is an M-Matrix. Here we present the obtained matrix

$$G = \begin{bmatrix} \dfrac{R_y - f_{x1}}{R_y} & -f_{x2} & 0 & 0 \\ \dfrac{-s_{x1}(1-h_{x2})}{R_y} & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{y2} \\ \dfrac{s_{y1}(1-h_{y2})(1-R_y)}{R_y^2} & 0 & -\dfrac{1}{f_4 R_y} & 1 \end{bmatrix}$$

and all minor principals of G obtained are

$$PM_1 = |g_{11}| = \frac{R_y - f_{x1}}{R_y}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \frac{R_y - R_x}{R_y},$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \frac{R_y - R_x}{R_y}, \text{and } PM_4 = |G| = \frac{(1-R_y)(R_x - R_y)}{R_y}.$$

Note that all elements g_{ij} for $i \neq j$ are nonpositive except for g_{41}. Then, g_{41} will be negative if $R_y > 1$. Next, focus on the minor principal terms of the G matrix. PM_2 and PM_3 are positive if $R_y > R_x$. Because of $R_y > 1$ and $R_y > R_x$, consequently PM_4 are positive. Then, since $R_y > R_x$ where $R_y = f_{y2}s_{y1}\left(1 - h_{y2}\right)$ and $R_x = f_{x2}s_{x1}(1 - h_{x2}) + f_{x1}$ it follows that PM_1 is positive because $R_y > f_{x1}$. Hence, G is an M-Matrix and the equilibrium point E_y is locally stable asymptotically if $R_y > R_x$ and $R_y > 1$.

■

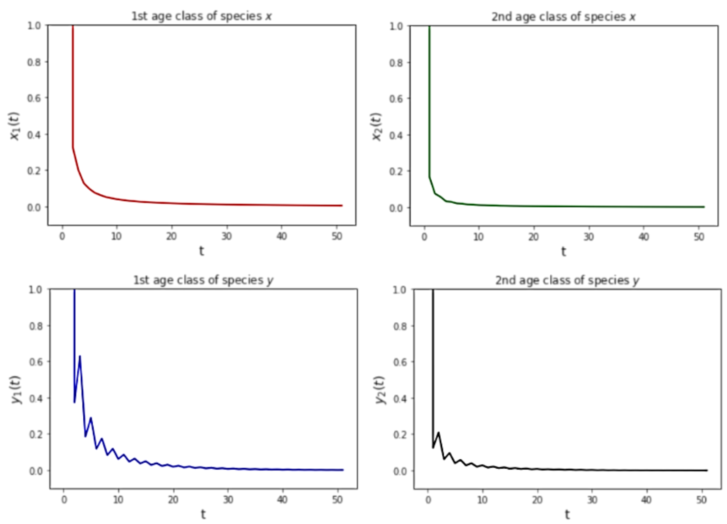

Figure 1. Population growth graph for each age class of each species x and y in case i Model 1

Equilibrium Point of Model 2

With the same treatment as determining the equilibrium point in the Model 1, the equilibrium Model 2 is obtained as follows:

$$\begin{aligned} x_1(t) &= \frac{f_{x1}}{1 + ax_1(t) + by_1(t)} x_1(t) + f_{x2}x_2(t) \\ x_2(t) &= \frac{s_{x1}(1 - h_{x2})}{1 + ax_1(t) + by_1(t)} x_1(t) \\ y_1(t) &= f_{y2}y_2(t) \\ y_2(t) &= \frac{s_{y1}(1 - h_{y2})}{1 + bx_1(t) + ay_1(t)} y_1(t) \end{aligned} \tag{5}$$

Then, there are four equilibrium points from the Model 2, namely

1. The equilibrium point with both species going extinct is
$$E_0 = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
2. The equilibrium point with species x exists while species y is extinct, i.e
$$E_x = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{R_x - 1}{a} \\ \dfrac{(R_x - 1)s_{x1}(1 - h_{x2})}{aR_x} \\ 0 \\ 0 \end{bmatrix}.$$
The condition E_x exists if it is fulfilled $R_x = f_{x2}s_{x1}(1 - h_{x2}) + f_{x1} > 1$.
3. The equilibrium point with species y exists while species x is extinct, i.e
$$E_y = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{R_y - 1}{a} \\ \dfrac{R_y - 1}{af_{y2}} \end{bmatrix}.$$
The condition E_y exists if it is fulfilled $R_y = f_{y2}s_{y1}\left(1 - h_{y2}\right) > 1$.
4. The equilibrium point with species x and y exists if one of them is satisfied, namely $a^2 > b^2$ or $a > b$, $a\left(R_y - 1\right) > b(R_x - 1)$, and $a(R_x - 1) > b\left(R_y - 1\right)$, or $a^2 < b^2$ or $a < b$, $a\left(R_y - 1\right) < b(R_x - 1)$, and $a(R_x - 1) < b\left(R_y - 1\right)$ with

$$E_{xy} = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{a(R_x - 1) - b(R_y - 1)}{(a^2 - b^2)} \\ \dfrac{s_{x1}(1 - h_{x2})\left(a(R_x - 1) - b(R_y - 1)\right)}{(a^2 - b^2)R_x} \\ \dfrac{a(R_y - 1) - b(R_x - 1)}{(a^2 - b^2)} \\ \dfrac{a(R_y - 1) - b(R_x - 1)}{(a^2 - b^2)f_{y2}} \end{bmatrix}.$$

In this second model, we obtain four equilibrium points where an equilibrium point appears with all species existing or a co-existence equilibrium point. The level of competition in both species affects the existence of a co-existence equilibrium point.

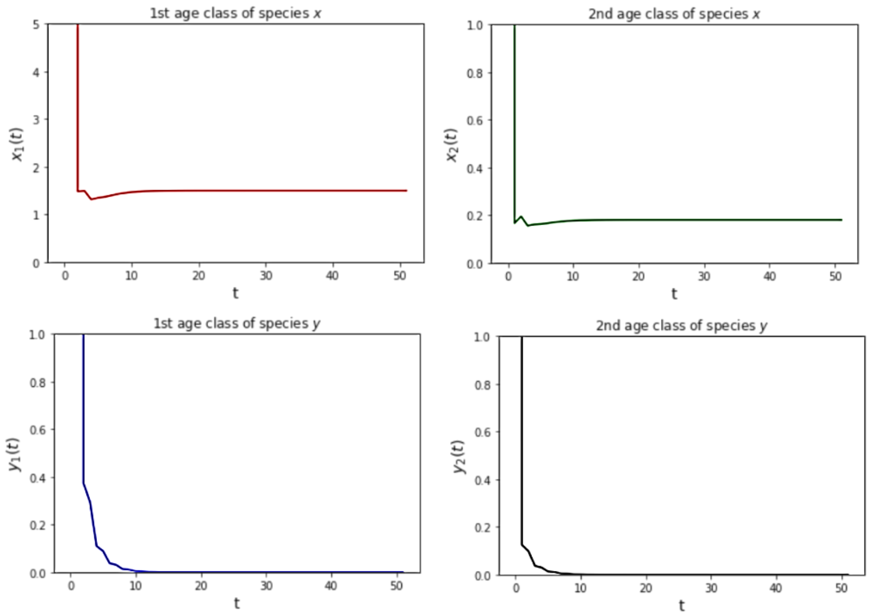

Figure 2. Population growth graph for each age class of each species x and y in case ii Model 1

Locally Stable Asymptotically at the Equilibrium Point of Model 2

This section discusses the locally stable asymptotically of Model 2 which is presented in Theorem 3 below.

Theorem 3. (Locally Stable Asymptotically at the Equilibrium Point of Model 2)

For the system in the case of one iteroparous species and one semelparous species, each of which consists of two classes whose growth is affected by density-dependent, harvesting and competition which is specifically described in the Model 2, among others:

1. The equilibrium point E_0 is locally stable asymptotically if $R_x < 1$, and $R_y < 1$.
2. The equilibrium point E_x is locally stable asymptotically if $R_x > 1$ and $a(R_y - 1) < b(R_x - 1)$.
3. The equilibrium point E_y is locally stable asymptotically if $R_y > 1$ and $a(R_x - 1) < b(R_y - 1)$.
4. The equilibrium point E_{xy} is locally stable asymptotically if $a > b$, $a(R_y - 1) >$

$b(R_x - 1), a(R_x - 1) > b\big(R_y - 1\big)$, and $f_{x1}\big(a(a-b) - b(bR_x - aR_y)\big) < (a^2 - b^2)R_x^2$.

Proof:

The steps to determine the stability of all equilibrium points of the Model 2 can be carried out as in Model 1. Linearization of the Model 2, namely

$$J(E) = J\left(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} \dfrac{f_{x1}(1+by_1)}{(1+ax_1+by_1)^2} & f_{x2} & -\dfrac{f_{x1}x_1 b}{(1+ax_1+by_1)^2} & 0 \\ P_x(1+by_1) & 0 & -P_x b x_1 & 0 \\ 0 & 0 & 0 & f_{y2} \\ -P_y b y_1 & 0 & P_y(1+bx_1) & 0 \end{bmatrix} \quad (6)$$

with

$$P_x = \frac{s_{x1}(1-h_{x2})}{(1+ax_1+by_1)^2} \text{ and } P_y = \frac{s_{y1}(1-h_{y2})}{(1+bx_1+ay_1)^2}.$$

The elements of rows 1-2 columns 3-4 and rows 3-4 columns 1-2 in (6) are non-positive because the elements of the equilibrium point are guaranteed to be zero or positive, so Theorem 1 says matrix S for matrix $J(E)$ is

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The next step is to substitute all the equilibrium points in (6) and analyze its stability one by one.

1. The Jacobian matrix for the equilibrium point E_0 is

$$J(E_0) = \begin{bmatrix} f_{x1} & f_{x2} & 0 & 0 \\ s_{x1}(1-h_{x2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{y2} \\ 0 & 0 & s_{y1}(1-h_{y2}) & 0 \end{bmatrix}.$$

   Next, determine the matrix $G = I - S\big(J(E_0)\big)S^{-1}$ and make sure the matrix G is an M-Matrix. Here we present the obtained matrix

$$G = \begin{bmatrix} 1-f_{x1} & -f_{x2} & 0 & 0 \\ -s_{x1}(1-h_{x2}) & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{y2} \\ 0 & 0 & -s_{y1}(1-h_{y2}) & 1 \end{bmatrix}$$

   and all minor principals of G obtained are

$$PM_1 = |g_{11}| = 1 - f_{x1}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1 - R_x,$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = 1 - R_x, \text{and } PM_4 = |G| = (1-R_x)\big(1-R_y\big).$$

   By considering the matrix G, it is clear that the values of all g_{ij} for $i \neq j$ are nonpositive. Next is focus on determining the conditions for $PM_i > 0$ $(i = 1,2,3,4)$. PM_2 and PM_3 are positive if $R_x < 1$ is satisfied. Because of $R_x < 1$ so that PM_1 is positive and an additional condition for PM_4 to be positive is $R_y < 1$. Therefore, G is an M-matrix, and the equilibrium point E_0 is locally stable asymptotically if $R_x < 1$ and $R_y < 1$.

2. The Jacobian matrix for the equilibrium point E_x is

$$J(E_x) = \begin{bmatrix} \frac{f_{x1}}{R_x^2} & f_{x2} & \frac{bf_1(1-R_x)}{aR_x^2} & 0 \\ \frac{s_{x1}(1-h_{x2})}{R_x^2} & 0 & \frac{bs_{x1}(1-h_{x2})(1-R_x)}{aR_x^2} & 0 \\ 0 & 0 & 0 & f_{y2} \\ 0 & 0 & \frac{as_{y1}(1-h_{y2})}{a+b(R_x-1)} & 0 \end{bmatrix}.$$

Next, determine the matrix $G = I - S\big(J(E_x)\big)S^{-1}$ and make sure the matrix G is an M-Matrix. Here we present the obtained matrix

$$G = \begin{bmatrix} \frac{R_x^2 - f_{x1}}{R_x^2} & -f_{x2} & \frac{bf_1(1-R_x)}{aR_x^2} & 0 \\ \frac{-s_{x1}(1-h_{x2})}{R_x^2} & 1 & \frac{bs_{x1}(1-h_{x2})(1-R_x)}{aR_x^2} & 0 \\ 0 & 0 & 1 & -f_{y2} \\ 0 & 0 & -\frac{as_{y1}(1-h_{y2})}{a+b(R_x-1)} & 1 \end{bmatrix}$$

and all minor principals of G obtained are

$$PM_1 = |g_{11}| = \frac{R_x^2 - f_{x1}}{R_x^2}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = -\frac{1-R_x}{R_x},$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = -\frac{1-R_x}{R_x}, \text{and } PM_4 = |G| = \frac{(1-R_x)\Big(a\big(R_y-1\big) - b(R_x-1)\Big)}{a+b(R_x-1)}.$$

In the matrix G, it can be seen that all g_{ij} for $i \neq j$ are non-positive because $R_x > 1$ which is a condition for E_x to exist. Therefore, the next step is to focus on determining the positive terms of the minor principal of the G matrix. It is clear that PM_2 and PM_3 are positive because $R_x > 1$. Then, it is clear that PM_1 is positive because in fact $f_{x1} < R_x^2 = (f_{x2}s_{x1}(1-h_{x2}) + f_{x1})^2$. Since $R_x > 1$, PM_4 is positive if $a\big(R_y-1\big) - b(R_x-1) < 0$ or $a\big(R_y-1\big) < b(R_x-1)$. Therefore, G is an M-matrix, and the equilibrium point E_x is locally stable asymptotically if $R_x > 1$ and $a\big(R_y-1\big) < b(R_x-1)$.

3. The Jacobian matrix for the equilibrium point E_y is

$$J\big(E_y\big) = \begin{bmatrix} \frac{af_{x1}}{a+b\big(R_y-1\big)} & f_{x2} & 0 & 0 \\ \frac{as_{x1}(1-h_{x2})}{a+b\big(R_y-1\big)} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{y2} \\ \frac{bs_{y1}\big(1-h_{y2}\big)\big(1-R_y\big)}{aR_y^2} & 0 & \frac{1}{f_4 R_y} & 0 \end{bmatrix}.$$

Next, determine the matrix $G = I - S\Big(J\big(E_y\big)\Big)S^{-1}$ and make sure the matrix G is an M-Matrix. Here we present the obtained matrix

$$G=\begin{bmatrix} -\dfrac{a(f_{x1}-1)-b(R_y-1)}{a+b(R_y-1)} & -f_{x2} & 0 & 0 \\ \dfrac{-as_{x1}(1-h_{x2})}{a+b(R_y-1)} & 1 & 0 & 0 \\ 0 & 0 & 1 & -f_{y2} \\ \dfrac{bs_{y1}(1-h_{y2})(1-R_y)}{aR_y^2} & 0 & -\dfrac{1}{f_4R_y} & 1 \end{bmatrix}.$$

and all minor principals of G obtained are

$$PM_1=|g_{11}|=-\frac{a(f_{x1}-1)-b\big(R_y-1\big)}{a+b\big(R_y-1\big)},$$

$$PM_2=\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}=-\frac{a(R_x-1)-b\big(R_y-1\big)}{a+b\big(R_y-1\big)},$$

$$PM_3=\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}=-\frac{a(R_x-1)-b\big(R_y-1\big)}{a+b\big(R_y-1\big)},$$

and

$$PM_4=|G|=\frac{\Big(a(R_x-1)-b\big(R_y-1\big)\Big)(1-R_y)}{a+b(R_y-1)}.$$

The equilibrium point of E_y is exist if $R_y>1$ consequently all elements of g_{ij} for $i\neq j$ are non-positive. Next is the focus on determining the conditions so that all the principal minor matrices G are positive. Since $R_y>1$, so we have PM_2, PM_3, and PM_4 are positive if $a(R_x-1)-b\big(R_y-1\big)<0$ or $a(R_x-1)<b\big(R_y-1\big)$. In addition, $a(R_x-1)<b\big(R_y-1\big)$ the result is satisfied $a(f_{x1}-1)-b\big(R_y-1\big)<0$ or $a(f_{x1}-1)<b\big(R_y-1\big)$. Then, because of $R_y>1$ and $a(f_{x1}-1)<b\big(R_y-1\big)$ so that PM_1 is fulfilled with a positive value. Therefore, G is an M-matrix and the equilibrium point E_y is locally stable asymptotically if $R_y>1$ and $a(R_x-1)<b\big(R_y-1\big)$.

4. The Jacobian matrix for the equilibrium point E_{xy} is

$$J\big(E_{xy}\big)=\begin{bmatrix} \dfrac{A_xf_{x1}}{CR_x^2} & f_{x2} & -\dfrac{B_xbf_{x1}}{CR_x^2} & 0 \\ \dfrac{s_{x1}(1-h_{x2})A_x}{CR_x^2} & 0 & -\dfrac{bs_{x1}(1-h_{x2})B_x}{CR_x^2} & 0 \\ 0 & 0 & 0 & f_{y2} \\ -\dfrac{bB_y}{CR_yf_{y2}} & 0 & \dfrac{A_y}{Cf_{y2}R_y} & 0 \end{bmatrix}.$$

Next is the G matrix for the equilibrium point E_{xy} is

$$G=\begin{bmatrix} 1-\dfrac{A_xf_{x1}}{CR_x^2} & -f_{x2} & -\dfrac{B_xbf_{x1}}{CR_x^2} & 0 \\ \dfrac{A_xs_{x1}(h_{x2}-1)}{CR_x^2} & 1 & -\dfrac{bs_{x1}(1-h_{x2})B_x}{CR_x^2} & 0 \\ 0 & 0 & 1 & -f_{y2} \\ -\dfrac{bB_y}{CR_yf_{y2}} & 0 & -\dfrac{A_y}{CR_yf_{y2}} & 1 \end{bmatrix}$$

and the principal minor of the matrix G are

$$PM_1 = |g_{11}| = 1 - \frac{A_x f_{x1}}{CR_x^2}, \qquad PM_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \frac{aB_x}{CR_x}$$

$$PM_3 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \frac{aB_x}{CR_x}, \qquad \text{and} \qquad PM_4 = |G| = \frac{B_x B_y}{CR_x R_y}$$

where

$$A_x = \left(a(a-b) - b(bR_x - aR_y)\right), \qquad A_y = \left(a(a-b) - b\left(bR_y - aR_x\right)\right)$$

$$B_x = \left(a(R_x - 1) - b\left(R_y - 1\right)\right), B_y = \left(a\left(R_y - 1\right) - b(R_x - 1)\right), \text{and } C = \left(a^2 - b^2\right).$$

In this case, the conditions that meet the requirements will be determined so that G is called the M-Matrix. First focus on PM_4 is positive. Because of E_{xy} exists if $B_x, B_y, C > 0$ or $B_x, B_y, C < 0$. However, PM_4 is positive if it is fulfilled $B_x, B_y, C > 0$. Because $B_x, B_y, C > 0$ consequently fulfilled $g_{13}, g_{23}, g_{31} < 0$, and $PM_2, PM_3 > 0$. Then, PM_1 is positive if $f_{x1}A_x < CR_x^2$. Finally, all the conditions for G to be called an M-matrix have been fulfilled. Therefore, G is an M-matrix, and the equilibrium point E_{xy} is locally stable asymptotically if $f_{x1}\left(a(a-b) - b\left(bR_y - aR_x\right)\right) < (a^2 - b^2)R_x^2$, $a\left(R_y - 1\right) > b(R_x - 1)$, $a(R_x - 1) > b\left(R_y - 1\right)$, and $a^2 > b^2$ or $a > b$.

■

Numerical Simulations of Model 1 and Model 2

In the previous subsection, an analysis of the existing condition and local stability asymptotically from each equilibrium point has been carried out on Model 1 and Model 2. In this section, we perform a numerical simulation of the results from the analysis of Model 1 and Model 2. In this case, we divide the two models into two cases and each case is divided into as many subcases as the asymptotically local stability conditions of Theorem 2 and Theorem 3.

In the simulation of Model 1, we assume for all subcases of the Model 1 case, including:

1. $s_{x1} = 0.6$ and $s_{y1} = 0.4$ respectively that if there are 10 individuals in the first age class of species x and y then only 6 individuals and 4 individuals are able to survive from species x and y.
2. $h_{x2} = 0.5$ and $h_{y2} = 0.3$ respectively that if there are 10 individuals in the first age class of species x and y then there are only 5 individuals and 3 individuals harvested from species x and y.

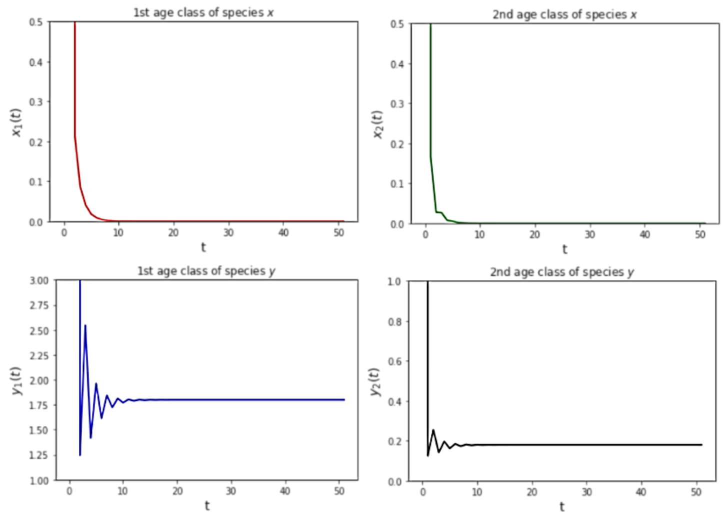

Figure 3. Population growth graph for each age class of each species x and y in case iii Model 1

Then, the birth rate for the simulation in the Model 1 case is divided into 3 subcases based on Theorem 1, including:

i. $f_{x1} = 0.7$, $f_{x2} = 0.9$, and $f_{y2} = 3$ consequently $R_x = 0.97 < 1$ and $R_y = 0.84 < 1$.
ii. $f_{x1} = 1$, $f_{x2} = 5$, and $f_{y2} = 3$ consequently $R_x = 2.5 > 1$ and $R_y = 0.84 < R_x$.
iii. $f_{x1} = 0.7$, $f_{x2} = 0.9$, and $f_{y2} = 10$ consequently $R_x = 0.97 < 1$ and $R_y = 2.8 > R_x$.

The results of the Model 1 simulation for each subcase i-iii are presented in Figure 1-3. Figures 1-3 respectively for the parameters given in each subcase i-iii of the Model 1 simulation show that the locally stable asymptotically towards the equilibrium point $E_0 = [0,0,0,0]^T$, $E_x = [1.5,0.18,0,0]^T$, and $E_y = [0,0,1.8,0.18]^T$.

In the simulation of Model 2, we assume for all subcases of Model 2 for survival and harvesting rates are equal to Model 1. Then, the levels of intraspecific and interspecific competition are $a = 0.2$ and $b = 0.1$, respectively. Then, the birth rate for the simulation in the Model 1 case is divided into 4 subcases based on Theorem 2, including:

i. $f_{x1} = 0.5$, $f_{x2} = 1$, and $f_{y2} = 3$ consequently $R_x = 0.8$ and $R_y = 0.84$.
ii. $f_{x1} = 30$, $f_{x2} = 20$, and $f_{y2} = 60$ consequently $R_x = 36$ and $R_y = 16.8$.
iii. $f_{x1} = 5$, $f_{x2} = 20$, and $f_{y2} = 100$ consequently $R_x = 11$ and $R_y = 28$.
iv. $f_{x1} = 20$, $f_{x2} = 20$, and $f_{y2} = 60$ consequently $R_x = 26$ and $R_y = 16.8$.

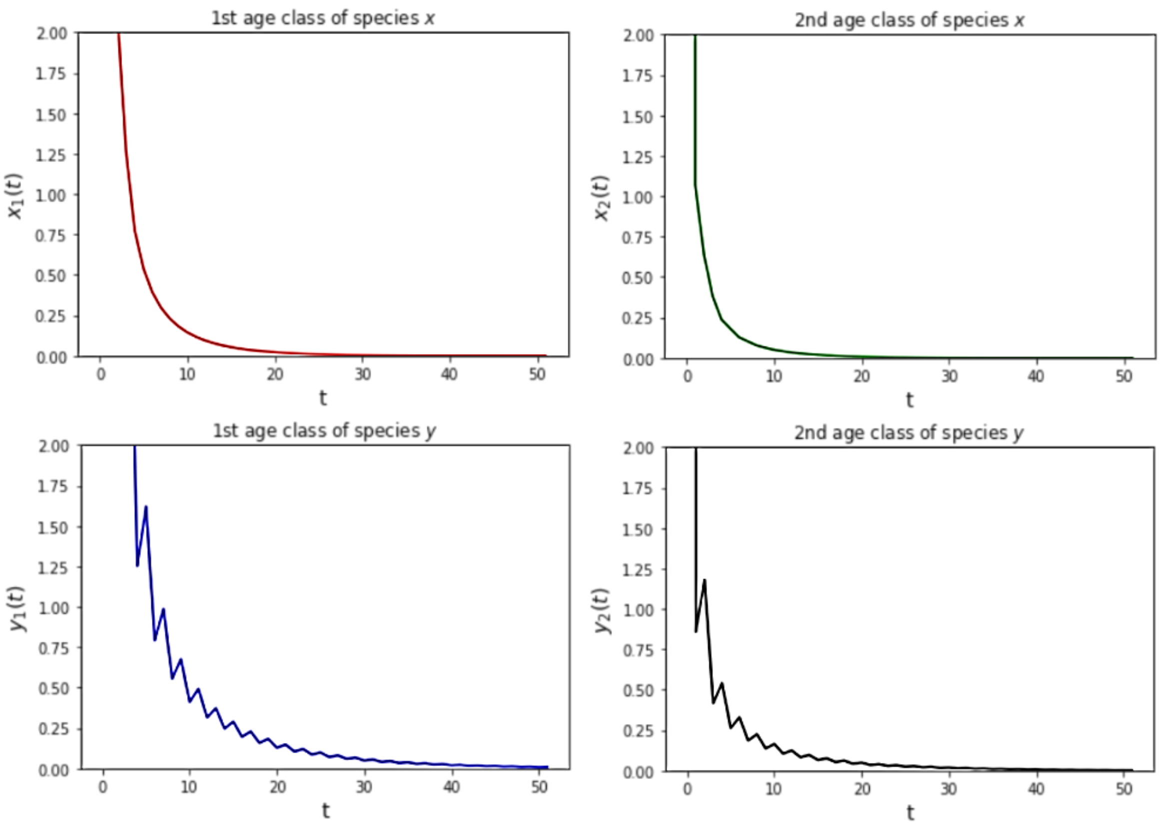

Figure 4. Population growth graph for each age class of each species x and y in case i Model 2

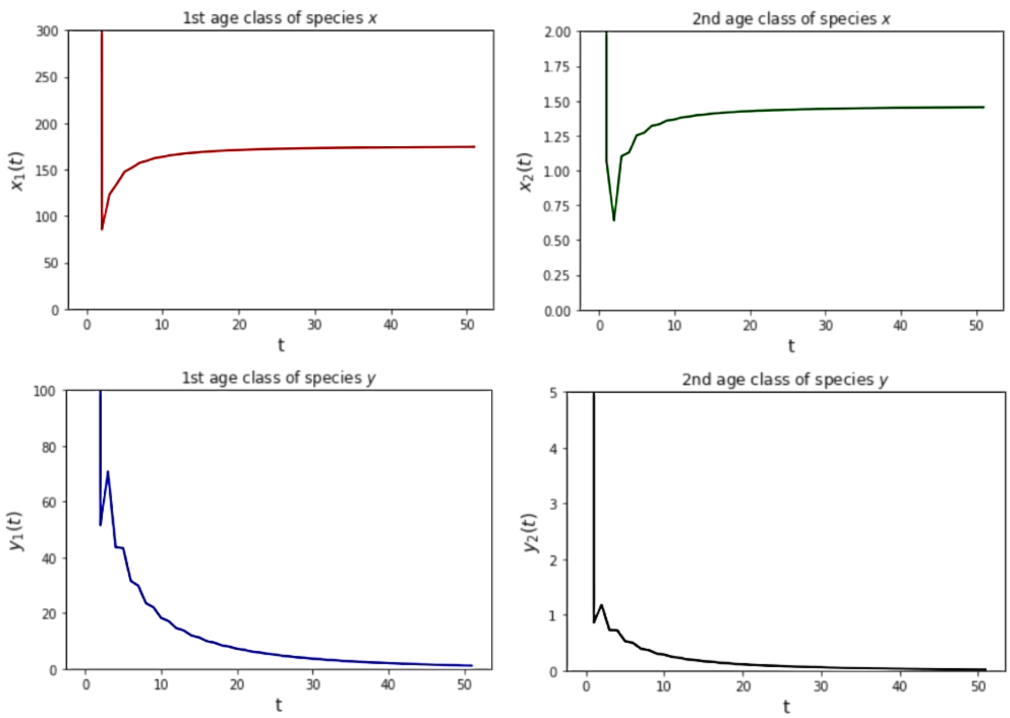

Figure 5. Population growth graph for each age class of each species x and y in case ii Model 2

Figure 4-7 is the simulation result of Model 2 for each subcase i-iv. Figure 4-7 for each parameter that satisfies Theorem 2 conditions in subcases i-iv of the Model 2 simulation that sequentially locally stable asymptotically towards the equilibrium point $E_0 = [0,0,0,0]^T$, $E_x = [175,1.46,0,0]^T$, $E_y = [0,0,135,1.35]^T$ dan $E_{xy} = [114,1.32,22,0.36]^T$.

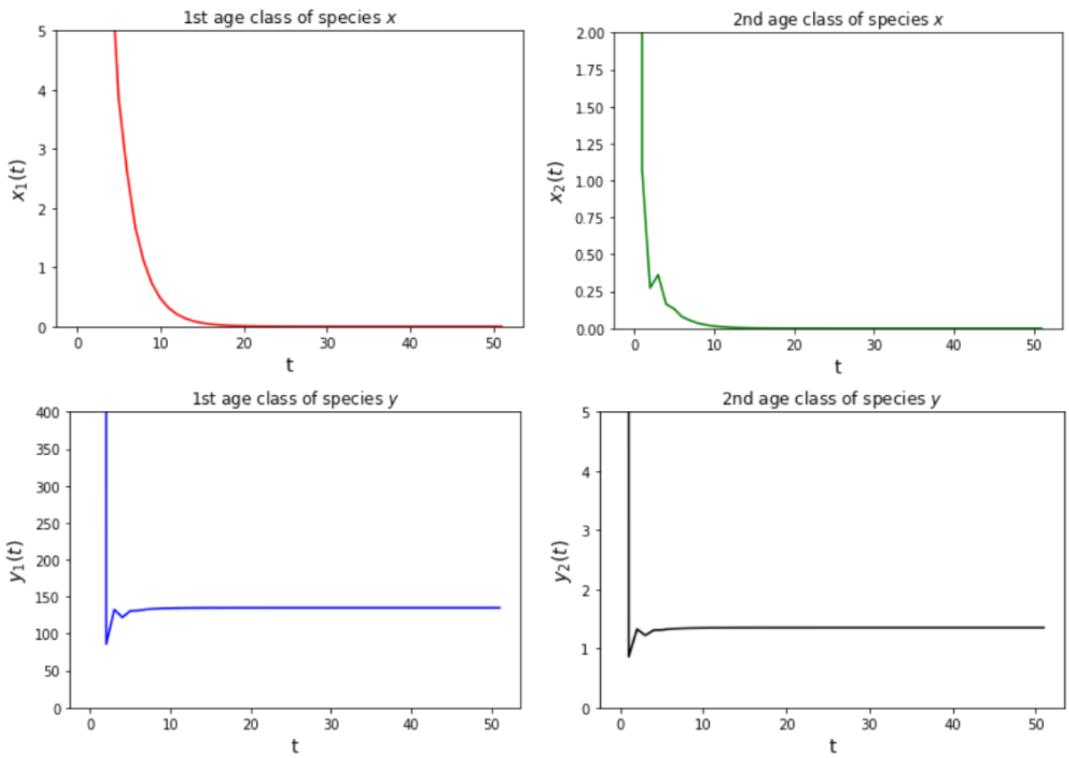

Figure 6. Population growth graph for each age class of each species x and y in case iii Model 2

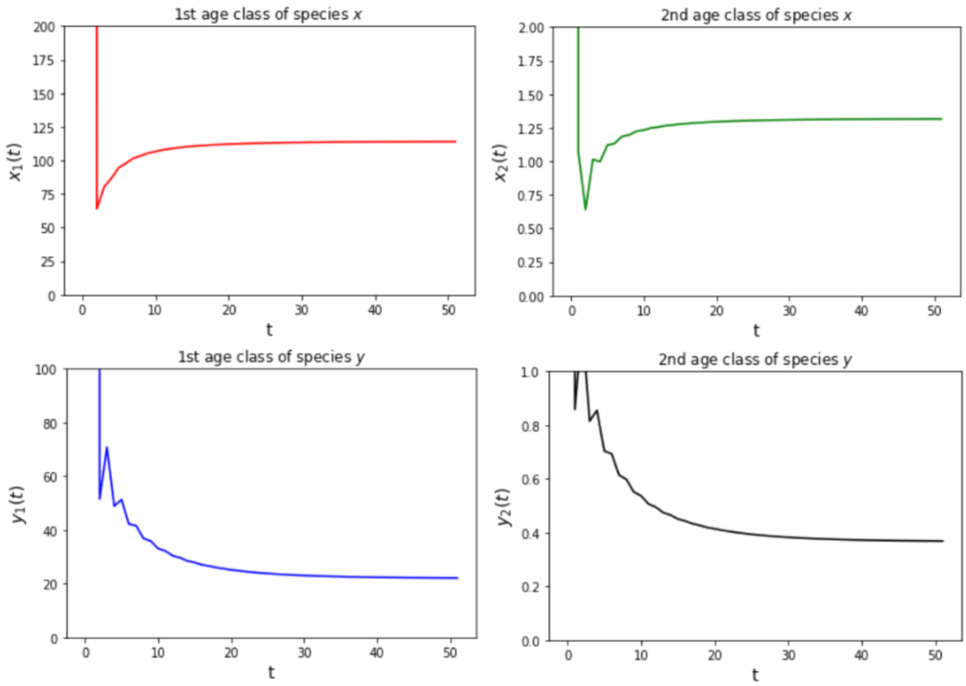

Figure 7. Population growth graph for each age class of each species x and y in case iv Model 2

CONCLUSIONS

In this paper, we compare two different models: Model 1 and Model 2. Our focus is to compare the presence and absence of the influence of intraspesific and interspecific competition in the equilibrium point and its local stability of Model 1 and Model 2. Mathematically the conditions under which the positive/non-trivial equilibrium point exists and the local stability of this equilibrium of the model is easy to interpret. However, biologically only some conditions can be interpreted because of the complexity of conditions. Simply put, the results of our study show that the level of competition has a role in the equilibrium point and its local stability of the Model 1 and Model 2. Model 1 shows that there is no coexistence equilibrium point so model 1 is never locally stable at the point where both species exist. In Model 2, one of the conditions that is easily interpreted is that the coexistence equilibrium point occurs when $a > b$ which means the intensity of the intraspecific competition level is greater than the intensity of the interspecific level competition. The inequality of $a > b$ is one of the locally stable asymptotically conditions of the co-existence equilibrium point in Model 2. The results of this study can be applied to problems that have similarities mathematical structure to this case. There still some limitation in this model to fit in a realistic real case, and hence we think that this research should be further developed, for example by increasing. the number of species, the number of classes, and so on, which is mathematically interesting and realistically important. This theory can be applied to study the dynamics of natural resource models including the effects of different management to the growth of the resources, such as in fisheries.

ACKNOWLEDGMENTS

The work is part of the research done by first author in the Master of Mathematics Study Program at Padjadjaran University. The first author would like to thank for the tuition fee during the candidature covered by ALG-Unpad Research Grant with contract number 1959/UN6.3.1/PT.00/2021. The preparation and the APC (Article Processing Charge) for manuscript publication is funded by "Penelitian Tesis Magister (PTM)" from the Indonesian Ministry of Education, Culture, Research and Technology in 2022 with contract number 1318/UN6.3.1/PT.00/2022.

REFERENCES

[1] R. Kon, "Stable bifurcations in multi-species semelparous population models," 2017, doi: 10.1007/978-981-10-6409-8_1.

[2] R. Kon, "Bifurcations of cycles in nonlinear semelparous Leslie matrix models," *J. Math. Biol.*, vol. 80, no. 4, 2020.

[3] Y. Chow and R. Kon, "Global dynamics of a special class of nonlinear semelparous Leslie matrix models," *J. Differ. Equations Appl.*, vol. 26, no. 5, 2020, doi: 10.1080/10236198.2020.1777288.

[4] A. Wikan and Ø. Kristensen, "Nonstationary and Chaotic Dynamics in Age-Structured Population Models," *Discret. Dyn. Nat. Soc.*, vol. 2017, 2017.

[5] B. J. Hanley, E. M. Bunting, and K. L. Schuler, "How can we augment the few that remain? Using stable population dynamics to aid reintroduction planning of an iteroparous species," *PeerJ*, vol. 2019, no. 5, 2019.

[6] M. D. Hatch, F. Abadi, W. J. Boeing, S. Lois, M. D. Porter, and D. E. Cowley, "Sustainability management of short-lived freshwater fish in human-altered ecosystems should focus on adult survival," *PLoS One*, vol. 15, no. 5, 2020.

[7] P. H. Leslie, "On the use of matrices in certain population mathematics.," *Biometrika*, vol. 33, 1945, doi: 10.1093/biomet/33.3.183.

[8] J. M. Cushing and S. M. Henson, "Stable bifurcations in semelparous Leslie models," *J. Biol. Dyn.*, 2012, [Online]. Available: https://www.tandfonline.com/doi/abs/10.1080/17513758.2012.716085.

[9] O. Diekmann and R. Planqué, "The winner takes it all: how semelparous insects can become periodical," *J. Math. Biol.*, vol. 80, no. 1–2, 2020, doi: 10.1007/s00285-019-01362-3.

[10] C. J. Pennycuick, R. M. Compton, and L. Beckingham, "A computer model for simulating the growth of a population, or of two interacting populations," *J. Theor. Biol.*, vol. 18, no. 3, 1968, doi: 10.1016/0022-5193(68)90081-7.

[11] C. C. Travis, W. M. Post, D. L. DeAngelis, and J. Perkowski, "Analysis of compensatory Leslie matrix models for competing species," *Theor. Popul. Biol.*, vol. 18, no. 1, pp. 16–30, 1980, doi: 10.1016/0040-5809(80)90037-4.

[12] R. Kon, "Age-structured Lotka-Volterra equations for multiple semelparous populations," *SIAM J. Appl. Math.*, vol. 71, no. 3, pp. 694–713, 2011, doi: 10.1137/100794262.

[13] R. Kon, "Permanence induced by life-cycle resonances: The periodical cicada problem," *J. Biol. Dyn.*, vol. 6, no. 2, 2012, doi: 10.1080/17513758.2011.594098.

[14] L. K. Beay, A. Suryanto, I. Darti, and Trisilowati, "Stability of a stage-structure Rosenzweig-MacArthur model incorporating Holling type-II functional response," 2019, doi: 10.1088/1757-899X/546/5/052017.

[15] J. Li, X. Liu, and C. Wei, "The impact of role reversal on the dynamics of predator-

prey model with stage structure," *Appl. Math. Model.*, vol. 104, 2022.
[16] J. Wang, Y. Cai, S. Fu, and W. Wang, "The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge," *Chaos*, vol. 29, no. 8, 2019.
[17] E. Rahmi, I. Darti, A. Suryanto, and Trisilowati, "A modified leslie–gower model incorporating beddington–deangelis functional response, double allee effect and memory effect," *Fractal Fract.*, vol. 5, no. 3, 2021.
[18] T. Hidayati and W. Kurniawan, "Stability Analysis of Lotka-Volterra Model in The Case of Interaction of Local Religion and Official Religion," *Int. J. Educ. Res. Soc. Sci.*, vol. 2, no. 3, 2021.
[19] W. W. Mohammed, E. S. Aly, A. E. Matouk, S. Albosaily, and E. M. Elabbasy, "An analytical study of the dynamic behavior of Lotka-Volterra based models of COVID-19," *Results Phys.*, vol. 26, 2021.
[20] Y. Lin, Q. Din, M. Rafaqat, A. A. Elsadany, and Y. Zeng, "Dynamics and Chaos Control for a Discrete-Time Lotka-Volterra Model," *IEEE Access*, vol. 8, 2020.
[21] A. Wikan, "Dynamical consequences of harvest in discrete age-structured population models," *J. Math. Biol.*, vol. 49, no. 1, 2004, doi: 10.1007/s00285-003-0251-5.
[22] J. M. Cushing, "Three stage semelparous Leslie models," *J. Math. Biol.*, vol. 59, no. 1, 2009, doi: 10.1007/s00285-008-0208-9.
[23] C. C. Travis and W. M. Post, "Dynamics and comparative statics of mutualistic communities," *J. Theor. Biol.*, vol. 78, no. 4, 1979, doi: 10.1016/0022-5193(79)90190-5.