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The First Zagreb Index, The Wiener Index, and The Gutman Index of The Power of Dihedral Group

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ABSTRACT

Research on graphs combined with groups is an interesting topic in the field of combinatoric algebra where graphs are used to represent a group. One type of graph representation of a group is a power graph. A power graph of the group G is defined as a graph whose vertex set is all elements of G and two distinct vertices a and b are adjacent if and only one vertice is the power of other vertice. In addition to mathematics, graph theory can be applied to various fields of science, one of which is chemistry, which is related to topological indices. In this study, the topological indexes will be discussed, namely the Zagreb index, the Wiener index, and the Gutman index of the power graph of the dihedral group where n is a prime power. The method used in this research is a literature review. For the main result, we gives the first Zagreb index, Wiener index, and Gutman index of the power graph of the dihedral group.

Keywords: first Zagreb index; Wiener index; Gutman index; power graph; dihedral group

INTRODUCTION

In mathematics, graph theory has many uses, especially in algebraic structures where graphs are used to represent a group. Many types of graphs are developed from a group, one of which is a power graph. The first power graph introduced by Kalarev in 2013 [1] is to define a directed power graph of a semigroup. And motivated by this, Askin and Buyukkose discuss the undirected power graph of semigroups and groups [2]. Recently, there have been many studies discussing the power graph of a group, one of which is the study by Asmarani et al. which deals with the power graph of a dihedral group when $n = p^m$ where p is a prime number and an m is a natural number [3]–[5].

Besides mathematics, graph theory has benefits in other fields, one of which is chemistry, which is related to topological indices. Topological indices represent chemical structures and are useful for predicting the chemical and physical properties of molecular structures. Not only researching graphs related to chemical structures but over time research on topological indices has developed to examine graphs in general. Several types

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of topological indexes are interesting to discuss. Some of them are the first Zagreb index, the Wiener index, and the Gutman index [6].

Research on the topological index of a graph, especially graphs related to groups is interesting to do. Several previous research results discuss the topological index of graphs related to groups, namely the topological index of the non-commuting graph of a dihedral group, the Szeged and Wiener indices for the coprime graph of the dihedral group [7], and connectivity indices of the coprime graph of generalized quaternion group [8]. For other graph representations of a group, see [8]–[14]. Recently, not many studies have investigated the topological indices of graphs associated with groups, especially the power graphs of dihedral groups. Therefore, in this study, topological indices will be discussed, namely the first Zagreb index, Wiener index, and the Gutman index of the power graph of a dihedral group when $n = p^m$ where p is a prime number and an m is a natural number.

METHODS

This study uses a deductive proof method to find new knowledge from an algebraic structure from a previous study. We start by studying the algebraic structure for several cases looking for a pattern. And with the foundation of the pattern, we stated the conjecture for a general case, if the conjecture is proven by deductive proof, the conjecture is stated as a theorem.

RESULTS AND DISCUSSION

Preliminaries

In this section, we present some definitions and theorems that are needed in this research.

Definition 1 [15] Group G is said to be a dihedral group of order $2n$, $n \geq 3$, and $n \in \mathbb{N}$, is a group composed of two elements $a, b \in G$ with the property

$$G = \langle a, b | a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle$$

The dihedral group of order 2n is denoted by D_{2n}.

Definition 2 [5] Power graph of group G denoted by $\mathcal{G}(G)$ is an undirected graph whose vertex set is G and two vertices $a, b \in G$ are adjacent if and only if $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m.

We will give some topological indices of the power graph such as the Zagreb index, Wiener index, and Gutman index. The definitions are as follows

Definition 3 [16] Let $\mathcal{G}$ be a simple connected graph. The first Zagreb index of $\mathcal{G}$, denoted by $M_1(\mathcal{G})$, is defined as

$$M_1(\mathcal{G}) = \sum_{v \in V(\mathcal{G})} (\deg(v))^2$$

where $deg(v)$ is the number of edges that incident to v.

Definition 4 [2] Let $\mathcal{G}$ be a simple connected graph. The Wiener index of $\mathcal{G}$, denoted by $W(\mathcal{G})$, is defined as

$$W(\mathcal{G}) = \sum_{u,v \in V(\mathcal{G})} d(u,v)$$

where $d(u,v)$ is the shortest distance between vertex u and v.

Definition 5 [17] Let $\mathcal{G}$ be a simple connected graph. The Gutman index of $\mathcal{G}$, denoted by $Gut(\mathcal{G})$, is defined as

$$Gut(\mathcal{G}) = \sum_{u,v \in V(\mathcal{G})} \deg(u)\deg(v)\, d(u,v)$$

where $deg(u)$ and $\deg(v)$ are the number of edges that incident to u and v and $d(u,v)$ are the shortest distance between vertex u and v.

Theorem 1 [3] If $n = p^m$ with p prime numbers and an m natural numbers, then the power graph of a dihedral group is a graph that has two non-disjoint subgraphs, namely a complete subgraph and a star subgraph.

Example 1 Power graph of the dihedral group $D_{2.3}$ as shown in the following figure

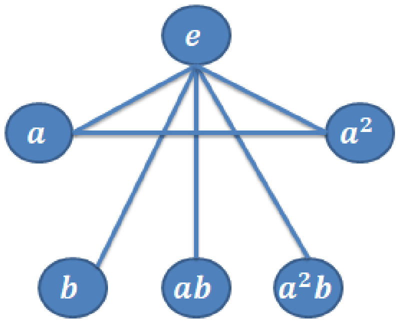

FIGURE 1. Power graph of the dihedral group $D_{2.3}$

Theorem 2 [3] The vertex degree of the power graph of a dihedral group when $n = p^m$ where p prime numbers and an m natural numbers are

a. $\deg(e) = 2n - 1$
b. $\deg(a^i) = n - 1$ for every $i \in \mathbb{Z}, 1 \leq i \leq n - 1$
c. $\deg(a^j b) = 1$ for every $j \in \mathbb{Z},\ 0 \leq j \leq n - 1$

Main Result

If $n = p^m$ where p is prime and an m natural number then the first Zagreb index, Wiener index, and Gutman index of the power graph of a dihedral group, respectively is $n^2(n+1), \frac{7n^2}{2} - \frac{5n}{2}, \frac{1}{2}(n^4 + n) + \frac{3}{2}(n^3 - n^2)$ as shown in the following theorem.

Theorem 3 If $n = p^m$ with p prime numbers and an m natural numbers, then the Zagreb index of the power graph of the dihedral group D_{2n} is $n^2(n-1)$.

Proof.

$$M_1(\mathcal{G}(D_{2n})) = \sum_{u \in V(\mathcal{G}(D_{2n}))} \deg(u)^2$$

$$
\begin{aligned}
&= \deg(e)^2 . 1 + \sum_{i=1}^{n-1} \deg\left(a^i\right)^2 + \sum_{i=0}^{n-1} \deg\left(a^i b\right)^2 \\
&= (2n-1)^2 . 1 + (n-1)^2 (n-1) + 1^2 . n \\
&= 4n^2 - 4n + 1 + (n-1)^3 + n \\
&= 4n^2 - 4n + 1 + n^3 - 3n^2 + 3n - 1 + n \\
&= n^3 + n^2 \\
&= n^2 (n+1)
\end{aligned}
$$

Theorem 4 If $n = p^m$ with p prime numbers and an m natural numbers then the Wiener index of the power graph of the dihedral group D_{2n} is $\frac{7n^2}{2} - \frac{5n}{2}$.

Proof. Let $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b\}$, a dihedral group with $n = p^m$ where p is a number prime and m natural number then the dihedral group can be partitioned into 3 partitions namely $V_1 = \{e\}$, $V_2 = \{a, a^2, \dots, a^{n-1}\}$ and $V_3 = \{b, ab, a^2 b, \dots, a^{n-1}b\}$. To prove the Wiener index of the power graph of a dihedral group can be divided into 4 cases.

Case 1.

For $e \in V_1$ and $x \in V(\mathcal{G}(D_{2n})$ where $e \neq v$, obtained

$$
\begin{aligned}
\sum_{x \in D_{2n}^*} d(e, x) &= (2n-1). 1 \\
&= 2n - 1
\end{aligned}
$$

Case 2

For $a^i, a^j \in V_2$ where $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, and $i \neq j$ obtained

$$
\begin{aligned}
\sum_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1 \\ i \neq j}} d\left(a^i, a^j\right) &= \binom{n-1}{2} . 1 \\
&= \frac{(n-1)!}{2!\,(n-3)!} \\
&= \frac{(n-1)(n-2)}{2} \\
&= \frac{n^2}{2} - \frac{3n}{2} + 1
\end{aligned}
$$

Case 3

For $a^c b, a^d b \in V_3$ where $0 \leq c \leq n-1$, $0 \leq d \leq n-1$, and $c \neq d$ obtained

$$
\begin{aligned}
\sum_{\substack{0 \leq c \leq n-1 \\ 0 \leq d \leq n-1 \\ c \neq d}} d(a^c b, a^d b) &= \binom{n}{2} . 2 \\
&= \frac{n!}{2!\,(n-2)!} . 2 \\
&= n(n-1) \\
&= n^2 - n
\end{aligned}
$$

Case 4

For $a^e \in V_2$ and $a^f b \in V_3$ where $1 \leq e \leq n-1$, $0 \leq f \leq n-1$ obtained

$$\sum_{\substack{1\le e\le n-1\\0\le f\le n-1}} d(a^e, a^f b) \quad = (n-1)n.2$$

$$= 2n^2 - 2n$$

Based on definition 4 and the four cases, the Wiener index of the power graph of the dihedral group D_{2n} when $n = p^m$ for a prime number p and m natural numbers is:

$$W((\mathcal{G}(D_{2n}) = \sum_{u,v\in V(\mathcal{G}(D_{2n})} d(u,v)$$

$$= \sum_{x\in D_{2n}^*} d(e,x) + \sum_{\substack{1\le i\le n-1\\1\le j\le n-1\\i\ne j}} d(a^i, a^j) + \sum_{\substack{0\le c\le n-1\\0\le d\le n-1\\c\ne d}} d(a^c b, a^d b)$$

$$+ \sum_{\substack{1\le e\le n-1\\0\le f\le n-1}} d(a^e, a^f b)$$

$$= (2n-1) + \left(\frac{n^2}{2} - \frac{3n}{2} + 1\right) + (n^2 - n) + (2n^2 - 2n)$$

$$= \frac{7n^2}{2} - \frac{5n}{2}$$

Teorema 5 If $n = p^m$ with p prime numbers and an m natural numbers then the Gutman index of the power graph of the dihedral group D_{2n} is $\frac{1}{2}(n^4 + n) + \frac{3}{2}(n^3 - n^2)$.

Proof. Let $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b\}$, a dihedral group with $n = p^m$ where p is a number prime and an m natural number then the dihedral group can be partitioned into 3 partitions namely $V_1 = \{e\}$, $V_2 = \{a, a^2, \dots, a^{n-1}\}$ and $V_3 = \{b, ab, a^2 b, \dots, a^{n-1}b\}$. To prove the Gutman index of the power graph of a dihedral group can be divided into 5 cases.

Case 1

For $e \in V_1$ and $a^i \in V_2$ where $1 \le i \le n-1$, obtained

$$\sum_{1\le i\le n-1} \left(\deg(e) . \deg(a^i)\right). d(e, a^i) = (((2n-1)(n-1))1)(n-1)$$

$$= 2n^3 - 5n^2 + 4n - 1$$

Case 2

For $e \in V_1$ and $a^j b \in V_3$ where $0 \le j \le n-1$, obtained

$$\sum_{0\le j\le n-1} \left(\deg(e) . \deg(a^j b)\right). d(e, a^j b) = \left(((2n-1)1)1\right) n = 2n^2 - n$$

Case 3

For $a^k, a^l \in V_2$ where $1 \le k \le n-1$, $1 \le l \le n-1$, and $k \ne l$, obtained

$$\sum_{\substack{1\le k\le n-1\\1\le l\le n-1\\k\ne l}} (\deg(a^k) . \deg(a^l)). d(a^k, a^l) = (((n-1)(n-1))1) \binom{n-1}{2}$$

$$= (n-1)(n-1) \frac{(n-1)!}{2!\,(n-3)!}$$

$$= \frac{(n-1)^3(n-2)}{2}$$

$$= \frac{n^4}{2} - \frac{5n^3}{2} + \frac{9n^2}{2} - \frac{7n}{2} + 1$$

Case 4

For $a^c b, a^d b \in V_3$ where $0 \le c \le n-1, 0 \le d \le n-1$, and $c \ne d$, obtained

$$\sum_{\substack{0\le c\le n-1 \\ 0\le d\le n-1 \\ c\ne d}} (\deg(a^c b) \,.\, \deg(a^d b)).\, d(a^c b, a^d b) = (1.1.2)\binom{n}{2}$$

$$= 2\binom{n}{2}$$

$$= 2\frac{n!}{2!\,(n-2)!}$$

$$= n(n-1)$$

$$= n^2 - n$$

Case 5

For $a^e \in V_2$ and $a^f b \in V_3$ where $1 \le e \le n-1, 0 \le f \le n-1$, obtained

$$\sum_{\substack{1\le e\le n-1 \\ 0\le f\le n-1}} (\deg(a^e) \,.\, \deg(a^f b)).\, d(a^e, a^f b) = (((n-1)1)2)\,(n-1)n$$

$$= 2n(n-1)(n-1)$$

$$= 2n(n^2 - 2n + 1)$$

$$= 2n^3 - 4n^2 + 2n$$

Based on definition 5 and the five cases, the Gutman index of the power graph of the dihedral group D_{2n} when $n = p^m$ for a prime number p and m natural numbers is:

$$Gut\big(\mathcal{G}(D_{2n})\big) = \sum_{u,v\in V(\mathcal{G}(D_{2n})} (\deg(u) \,.\, \deg(v)).\, d(u,v)$$

$$= \sum_{1\le i\le n-1} \big(\deg(e) \,.\, \deg\big(a^i\big)\big).\, d\big(e, a^i\big)$$

$$+ \sum_{0\le j\le n-1} \big(\deg(e) \,.\, \deg\big(a^j b\big)\big).\, d\big(e, a^j b\big)$$

$$+ \sum_{\substack{1\le k\le n-1 \\ 1\le l\le n-1 \\ k\ne l}} (\deg(a^k) \,.\, \deg(a^l)).\, d(a^k, a^l)$$

$$+ \sum_{\substack{0\le c\le n-1 \\ 0\le d\le n-1 \\ c\ne d}} (\deg(a^c b) \,.\, \deg(a^d b)).\, d(a^c b, a^d b)$$

$$+ \sum_{\substack{1\le e\le n-1 \\ 0\le f\le n-1}} (\deg(a^e) \,.\, \deg(a^f b)).\, d(a^e, a^f b)$$

$$= (2n^3 - 5n^2 + 4n - 1) + (2n^2 - n) + (\frac{n^4}{2} - \frac{5n^3}{2} + \frac{9n^2}{2} - \frac{7n}{2} + 1) + (n^2 - n) + (2n^3 - 4n^2 + 2n)$$

$$= \frac{n^4}{2} + \frac{3n^3}{2} - \frac{3n^2}{2} + \frac{n}{2}$$

$$= \frac{1}{2}(n^4 + n) + \frac{3}{2}(n^3 - n^2)$$

CONCLUSIONS

The results obtained from this study are the first Zagreb index, Wiener index, and Gutman index of the power graph for the dihedral group D_{2n} where $n = p^m$, p is prime and m a natural number respectively is $n^2(n+1), \frac{7n^2}{2} - \frac{5n}{2}, \ \frac{1}{2}\ (n^4 + n) + \frac{3}{2}(n^3 - n^2)$.

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