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# Basis Existence of Internal n-Graph Space

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## ABSTRACT

Graph of real valued continuous function with special addition and multiplication has already proven that is isomorphic to real number system. Furthermore, the graph of continous real valued function forms a field. The aim of this research was to generalize such concept to its n-tuple Cartesian Product and to prove that interchange of basis still able to be executed. The result of this research is n-tuple Cartesian Product of graph function forms a vector space over $\mathbb{R}$ and interchange of basis still able to be executed.

**Keywords**: n-Graph Space; General Vector Space; Internal n-Graph Space

## INTRODUCTION

By associating the special operation to a graph of continuous function, such graph can be claimed as a vector space. This follows from the fact that every graph of continuous real valued function has a bijection to its domain i.e. the real number system. Furthermore, they are homeomorphic.

Graph of real valued continuous function has a unique characteristic. It has continuous shape of curve along real line $\mathbb{R}$. More detail will be described as follows [1]

**Definition 1.** Let real valued function $f: \mathbb{R} \to \mathbb{R}$. The graph of $f$ is defined as

$$\mathbb{R}_f = \{(x, f(x)): x \in \mathbb{R}\} \tag{1}$$

Continuity of $f$ indicates that $\mathbb{R}_f$ topologically equivalent to $\mathbb{R}$. Its described briefly as follows;

**Definition 2** [2]**.** For each $U \subset \mathbb{R}$. Define the image of $U$ over $\mathbb{R}_f$ as

$$U \times f(U) = \{(x, f(x)): x \in U\} \tag{2}$$

define the associated topology for $\mathbb{R}_f$ as

$$\tau_f = \{U \times f(U): U\ open\} \tag{3}$$

It can be shown trivially that such topology implies $\mathbb{R}_f$ and $\mathbb{R}$ are homeomorphics . The fact that $\mathbb{R}_f$ and $\mathbb{R}$ are homeomorphics describes that even though their graph geometrically has different shapes, but they still have similarity in views of topology[3]. It motivates us to explore more special properties of $\mathbb{R}_f$.

Now let see it further.

**Definition 3.** Let real valued function $f: \mathbb{R} \to \mathbb{R}$. Define the addition for $\mathbb{R}_f$ as follow

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$$x_f \oplus y_f = \big(x+y, f(x+y)\big)$$

for any $x_f, y_f \in \mathbb{R}_f$.
Define the scalar multiplication as follow

$$\begin{aligned}(\alpha)x_f &= T_f\left(\alpha T^{-1}{}_f\big(x, f(x)\big)\right)\\ &= \big(\alpha x, f(\alpha x)\big)\end{aligned}$$

For any $x_f \in \mathbb{R}_f$ and scalar $\alpha$ , and $T_f$ is natural bijection generated by real valued function $f$ such as $T_f(x) = \big(x, f(x)\big)$.

By associating $\mathbb{R}_f$ with those operations, $\mathbb{R}_f$ become real vector space. Moreover $\mathbb{R}_f$ and $\mathbb{R}$ are isomorphics [4].

## METHODS

The method of this research is done by following method: first was to prove that $\mathbb{R}_f$ has dimension 1 and isomorphics to $\mathbb{R}$. Next step was to analyize more general space i.e. n tuple Cartesian Product of $\mathbb{R}_f$ denoted by $\mathbb{R}_f^n$. In generalization of $\mathbb{R}_f^n$ was to prove that such space is a vector space over $\mathbb{R}$ and was able to change of bases. Finally, was to find out the briefly method to change of bases as well as in real vector spaces.

## RESULTS AND DISCUSSION

It was proven previously that for each continuous function $f: \mathbb{R} \to \mathbb{R}$, the space $\mathbb{R}_f$ generated by $f$ forms a field. This fact becomes the basic to generalize the idea by constructing new n-tuple Cartesian Product of $\mathbb{R}_f$ which preserves vector space properties [5]. Before any further discussion, first define some necessary terms in order to help in generalization. We mean linear combination is $z_f = (\propto)x_f + (\beta)y_f$ for each $z_f \in \mathbb{R}_f$ and some scalars $\alpha, \beta \in \mathbb{R}$ [6]. The set of linear combinations of $x_f, y_f$ is named as Span $\{x_f, y_f\}$ [7]. The set $U \subset \mathbb{R}_f$ is said to be linearly independent if none of its members is able to expressed as linear combination of other members. Here is definition of basis:

**Definition 4.** The set $\{x_f^1, x_f^2, \ldots, x_f^n\}$ is called basis of subspace $U \subset \mathbb{R}_f$ if $\{x_f^1, x_f^2, \ldots, x_f^n\}$ are linearly independent and Span$\{x_f^1, x_f^2, \ldots, x_f^n\} = U$.

Recall that, dimension of $U$ is defined as base cardinality of $U$.
Next theorem is an important result.

**Theorem 1.** $\mathbb{R}_f$ has dimension 1.

Proof:
Chose $1_f \in \mathbb{R}_f$. For each $x_f \in \mathbb{R}_f$, it's obvious that

$$\begin{aligned}\mathrm{x}_f &= \big(x, f(x)\big)\\ &= T_f(x)\\ &= T_f(x.1)\\ &= \big(x.1, f(x.1)\big).\\ &= (x) \times 1_f\end{aligned}$$

By last equation, we conclude that $Span\{1_f\} = \mathbb{R}_f$.
Furthermore, based on the above results, vector space theory of $\mathbb{R}_f$ can be developed:

**Definition 5.** Let $S = \{f_1, f_2, \dots, f_n\}$ be a finite collection of real valued continuous functions, define **n-graph space** as

$$\mathbb{R}^n_{f_p} = \prod_{k=1}^{n} \mathbb{R}_{f_k} = \left\{\vec{v}: \{1,2,\dots,n\} \to \bigcup_{k=1}^{n} \mathbb{R}_{f_k}: \vec{v}(i) \in \mathbb{R}_{f_i}\right\}. \quad (4)$$

**Definition 6.** Let $S = \{f_1, f_2, \dots, f_n\}$ be a finite collection of real valued continuous functions. Define addition on $\mathbb{R}^n_{f_p}$ as

.
$$\vec{v} \oplus \vec{w} = (\vec{v}(1) \oplus \vec{w}(1), \vec{v}(2) \oplus \vec{w}(2), \dots, \vec{v}(n) \oplus \vec{w}(n)) \quad (5)$$

and scalar multiplication as

.
$$(\propto)\vec{v} = \left((\propto)\vec{v}(1), (\propto)\vec{v}(2), \dots, (\propto)\vec{v}(n)\right) \quad (6)$$

for each $\alpha \in \mathbb{R}$.

**Theorem 2.** Let $S = \{f_1, f_2, \dots, f_n\}$ be a finite collection of real valued continuous function. n-graph space $\mathbb{R}^n_{f_p}$ is a vector space over $\mathbb{R}$ under operation $\oplus$ and scalar multiplication $(\propto)\vec{v}$.

In special case which $f_1 = f_2 = f_3 = \cdots = f_n$, graph space $\mathbb{R}^n_{f_p}$ is called internal n-graph space i.e. Cartesian Product n-tuple of $\mathbb{R}_f$ itself. One can write

$$\mathbb{R}^n_{f_p} = \mathbb{R}^n_f.$$

Euclidean Space $\mathbb{R}^n$ is one of finest example of graph space which *f* is defined as identity mapping.

One of the most important tools to analyze relation between two internal n-graph space is linear transformation[8], here we still able to define linear transformation as well as done on commonly vector spaces.

**Definition 7.** Let two graph spaces $\mathbb{R}^m_f, \mathbb{R}^n_f$. Mapping $L: \mathbb{R}^m_f \to \mathbb{R}^n_f$ is linear transformation if the following properties hold

$$L(\vec{v} \oplus \vec{w}) = L(\vec{v}) \oplus L(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m_f$$

$$L\left((\propto)\vec{v}\right) = (\propto)L(\vec{v}), \quad \forall \propto \in \mathbb{R}$$

The set of all linear transformation from $\mathbb{R}^m_f$ to $\mathbb{R}^n_f$ is denoted as $Lin(\mathbb{R}^m_f, \mathbb{R}^n_f)$ [7]. One of necessary example of linear transformation is linear mapping from $\mathbb{R}^m_f$ to its coordinate i.e.

$$\overrightarrow{v_i}: \mathbb{R}^m_f \to \mathbb{R}$$

by specific formula

$\overrightarrow{v_i}(\vec{v}) = \vec{v}(i), \quad i = 1,2,3,\dots,m.$

The term of linear transformation is very useful in constructing theory change bases in graph space[9]. On internal graph space $\mathbb{R}^m_f$ change of bases still able to be constructed.

**Definition 8.** Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{R}^m_f$ are said to be linearly independent if for each $\vec{v}_i$ can't be expressed as linear combination of some others.

The term of linear combination refers to equation

$$\vec{v} = (\propto_1)\overrightarrow{v_1} \oplus \dots \oplus (\propto_k)\overrightarrow{v_k} \quad (7)$$

Another way to express coordinate transformation is by defining linear transformation which maps a vector to its scalars corresponding to the basis used to.

**Definition 9**. Let $V$ be a nontrivial vector space over field $F$. Let $S = \{\overrightarrow{s_1}, \overrightarrow{s_2}, \overrightarrow{s_3}, \dots, \overrightarrow{s_m}\}$ be a basis of $V$. Define coordinate transformation as $\emptyset_S : V \to F^m$ such that for each $\vec{v} = v_1\overrightarrow{s_1} + v_2\overrightarrow{s_2} + \cdots + v_m\overrightarrow{s_m}$, the map

$$\emptyset_S(\vec{v}) = [\vec{v}]_S = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

It's easy to check that coordinate transformation is an isomorphism [10]. The concept of coordinate mapping above is used to construct the concept of change of basis. First, we have to decide standard basis which lies on $\mathbb{R}_f^m$.

Standard basis of $\mathbb{R}_f^m$ is

$$\overrightarrow{v_i} = \left(1_f, 0_f, 0_f, 0_f, \dots, 0_f\right)$$
$$\overrightarrow{v_2} = \left(0_f, 1_f, 0_f, 0_f, \dots, 0_f\right)$$
$$\overrightarrow{v_m} = \left(0_f, 0_f, 0_f, 0_f, \dots, 1_f\right)$$

Now let's see how the change of basis works:

For each $\vec{v} \in \mathbb{R}_f^m$. By previous definition, one can write

$$\vec{v} = \left(\vec{v}(1), \vec{v}(2), \dots, \vec{v}(m)\right) \tag{8}$$

which $\vec{v}(i) \in \mathbb{R}_f$ for each $i$ = 1,2, . . ., $m$. Therefore

$$\vec{v} = \vec{v}(1)\vec{v}_1 \oplus \vec{v}(2)\vec{v}_2 \oplus \dots \oplus \vec{v}(m)\vec{v}_m$$

for corresponding coordinate $\vec{v}(i) \in \mathbb{R}_f$.

This is how the above coordinate and corresponding scalars related, let's see it as follows:

$$\begin{aligned} \vec{v}(1).1_f = \vec{v}(1) &= T_f\left(T_f^{-1}(\vec{v}(1))\right) \\ &= T_f\left(T_f^{-1}(\vec{v}(1).1\right) \\ &= \left(T_f^{-1}(\vec{v}(1).1), f(\vec{v}(1).1)\right) \\ &= \left(T_f^{-1}(\vec{v}(1).1\right).1_f \end{aligned}$$

Hence

$$\vec{v} = \left(T_f^{-1}(\vec{v}(1)\right).\overrightarrow{v_1} \oplus \left(T_f^{-1}(\vec{v}(2)\right).\overrightarrow{v_2} \oplus \dots \oplus \left(T_f^{-1}(\vec{v}(m)\right).\overrightarrow{v_m} \tag{9}$$

for some $\left(T_{f_k}^{-1}(\vec{v}(1)\right) \in \mathbb{R}$.

The last equation can be expressed in term of more visual vector addition as follows:

$$\begin{aligned} \vec{v} &= \left(T_f^{-1}(\vec{v}(1)\right).\overrightarrow{v_1} \oplus \dots \oplus \left(T_f^{-1}(\vec{v}(m)\right).\overrightarrow{v_m} \\ &= \left(T_f^{-1}(\vec{v}(1)\right).\begin{bmatrix} 1_f \\ 0_f \\ \vdots \\ 0_f \end{bmatrix} \oplus \dots \oplus \left(T_f^{-1}(\vec{v}(m)\right)\begin{bmatrix} 0_f \\ 0_f \\ \vdots \\ 1_f \end{bmatrix} \\ &= \begin{bmatrix} 1_f & 0_f & \dots & 0_f \\ 0_f & \dots & \dots & 0_f \\ \vdots & \dots & \dots & \vdots \\ 0_f & 0_f & \dots & 1_f \end{bmatrix} \begin{bmatrix} T_f^{-1}(\vec{v}(1)) \\ \dots \\ \dots \\ T_f^{-1}(\vec{v}(m)) \end{bmatrix} = I_f T_f^{-1}(\vec{v}) \end{aligned}$$

The above equation gives a consequence that $\mathbb{R}_f^m$ is isomorphic to $\mathbb{R}^m$.

The next theorem will ensure that $\mathbb{R}_f^m$ has more than just standard basis.

**Teorema 3.** If $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_m\}$ is basis for $\mathbb{R}^m$ then $T_f(S) = \left\{T_f(\boldsymbol{s_1}), T_f(\boldsymbol{s}_2), \dots, T_f(\boldsymbol{s}_m)\right\}$ forms a basis for $\mathbb{R}_f^m$.

Bukti:

Suppose that $\vec{v} \in \mathbb{R}_f^m$. Let's see the following fact, another fine expression for vector in $\mathbb{R}_f^m$ is $\vec{v} = \begin{bmatrix} \vec{v}(1) \\ \vec{v}(2) \\ \vdots \\ \vec{v}(m) \end{bmatrix}$. Thus we have $T_f^{-1}(\vec{v}) = \begin{bmatrix} T_f^{-1}(\vec{v}(1)) \\ T_f^{-1}(\vec{v}(2)) \\ \vdots \\ T_f^{-1}(\vec{v}(m)) \end{bmatrix}$ is actually lies in $\mathbb{R}^m$. Therefore

$T_f^{-1}(\vec{v}) = \begin{bmatrix} T_f^{-1}(\vec{v}(1)) \\ T_f^{-1}(\vec{v}(2)) \\ \vdots \\ T_f^{-1}(\vec{v}(m)) \end{bmatrix} = (\propto_1) \begin{bmatrix} s_{11} \\ s_{12} \\ \vdots \\ s_{1m} \end{bmatrix} + \ldots + (\propto_m) \begin{bmatrix} s_{m1} \\ s_{m2} \\ \vdots \\ s_{mm} \end{bmatrix}$, by $\boldsymbol{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{im} \end{bmatrix}$ for each $i$.

One can have $\vec{v} = T_f\left(T_f^{-1}(\vec{v})\right) = T_f\big((\alpha_1)\boldsymbol{s}_1 + (\alpha_2)\boldsymbol{s}_2 + \ldots + (\alpha_m)\boldsymbol{s}_m\big)$

$$= \Big((\alpha_1)\boldsymbol{s}_1 + (\alpha_2)\boldsymbol{s}_2 + \ldots + (\alpha_m)\boldsymbol{s}_m, f\big((\alpha_1)\boldsymbol{s}_1 + (\alpha_2)\boldsymbol{s}_2 + \ldots + (\alpha_m)\boldsymbol{s}_m\big)\Big)$$

$$= (\alpha_1)T_f(\boldsymbol{s}_1) \oplus (\alpha_2)T_f(\boldsymbol{s}_2) \oplus \ldots \oplus (\alpha_m)T_f(\boldsymbol{s}_m).$$

In other words, $\{T_f(\boldsymbol{s}_1), T_f(\boldsymbol{s}_2), \ldots, T_f(\boldsymbol{s}_m)\}$ spans $\mathbb{R}_f^m$. The rest is to prove that $T_f(\boldsymbol{s})$ is linearly independent.

Suppose the statement is not true, then there exist $T_f(\boldsymbol{s})$ which become linear combination of other members. Let's assume that is $T_f(\boldsymbol{s_i})$ then we have

$$T^{-1}\left(T_f(\boldsymbol{s}_i)\right) = \boldsymbol{s}_i = (\alpha_1)\boldsymbol{s}_1 + \ldots + (\alpha_{i-1})\boldsymbol{s}_{i-1} + (\alpha_{i+1})\boldsymbol{s}_{i+1} + \ldots + (\alpha_m)\boldsymbol{s}_m$$

But since $\{\boldsymbol{s_1}, \boldsymbol{s_2}, \ldots, \boldsymbol{s_m}\}$ is linearly independent, then it should be a contradiction. ∎

The above theorem indirectly explains that the internal graph space $\mathbb{R}_f^m$ has infinitely many vectors that can form a basis for $\mathbb{R}_f^m$. The next theorems will be discussing about how the basis related each other.

Let $S = \{\overrightarrow{s_1}, \overrightarrow{s_2}, \ldots, \overrightarrow{s_m}\}$ be a basis for $\mathbb{R}_f^m$. Then for each $\vec{v} \in \mathbb{R}_f^m$, one can write

$$\vec{v} = (\alpha_1^s)\overrightarrow{s_1} + (\alpha_2^s)\overrightarrow{s_2} + \ldots + (\alpha_m^s)\overrightarrow{s_m} \quad \text{dengan } (\alpha_i^s) \in \mathbb{R}. \tag{10}$$

$$= (\alpha_1^s) \begin{bmatrix} \overrightarrow{s_1(1)} \\ \overrightarrow{s_1(2)} \\ \vdots \\ \overrightarrow{s_1(m)} \end{bmatrix} + (\alpha_2^s) \begin{bmatrix} \overrightarrow{s_2(1)} \\ \overrightarrow{s_2(2)} \\ \vdots \\ \overrightarrow{s_2(m)} \end{bmatrix} + \ldots + (\alpha_m^s) \begin{bmatrix} \overrightarrow{s_m(1)} \\ \overrightarrow{s_m(2)} \\ \vdots \\ \overrightarrow{s_m(m)} \end{bmatrix}$$

$$= \begin{bmatrix} \overrightarrow{s_1(1)} & \overrightarrow{s_2(1)} & & \overrightarrow{s_m(1)} \\ \overrightarrow{s_1(2)} & \overrightarrow{s_2(2)} & \ldots & \overrightarrow{s_m(2)} \\ \vdots & \vdots & & \vdots \\ \overrightarrow{s_1(m)} & \overrightarrow{s_2(m)} & & \overrightarrow{s_m(m)} \end{bmatrix} \begin{bmatrix} (\alpha_1^s) \\ (\alpha_2^s) \\ \vdots \\ (\alpha_m^s) \end{bmatrix}$$

$$= [M]_{S,f}[\alpha]_S$$

by $[M]_{S,f}$ is matrics which for each entry lies in $\mathbb{R}_f^m$ and $[\alpha]_S$ is real scalar vector.

Then we have

$$T_f^{-1}(\vec{v}) = \begin{bmatrix} T_f^{-1}(\overrightarrow{s_1(1)}) & T_f^{-1}(\overrightarrow{s_2(1)}) & & T_f^{-1}(\overrightarrow{s_m(1)}) \\ T_f^{-1}(\overrightarrow{s_1(2)}) & T_f^{-1}(\overrightarrow{s_2(2)}) & \ldots & T_f^{-1}(\overrightarrow{s_m(1)}) \\ \vdots & \vdots & & \vdots \\ T_f^{-1}(\overrightarrow{s_1(m)}) & T_f^{-1}(\overrightarrow{s_2(m)}) & & T_f^{-1}(\overrightarrow{s_m(m)}) \end{bmatrix} \begin{bmatrix} (\alpha_1^s) \\ (\alpha_2^s) \\ \vdots \\ (\alpha_m^s) \end{bmatrix}$$

Therefore,

$$T_f^{-1}(\vec{v})=\left[T_f^{-1}(S)\right][\alpha]_S \tag{11}$$

The above equation explains how the basis $S$ in internal graph space $\mathbb{R}_f^m$ mapped to $\mathbb{R}^m$. It will make us easier to change basis from the old to the new one. Let's pay attention to the following discussion;
Let $S = \{\overrightarrow{s_1}, \overrightarrow{s_2}, \ldots, \overrightarrow{s_m}\}$ be a basis of $\mathbb{R}_f^m$ and $W = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_m}\}$ another basis. We want to change the old basis $S$ into the new one $W$. We have

$$\vec{v} = [M]_{S,f}[\alpha]_S$$

On the other hand

$$\vec{v} = [M]_{W,f}[\alpha]_W$$

Therefore

$$\begin{aligned} T_f^{-1}(\vec{v}) &= \left[T_f^{-1}(S)\right][\alpha]_S \\ \phi_{T_f^{-1}(S)} \circ T_f^{-1}(\vec{v}) &= [\alpha]_S \end{aligned}$$

There exists transition metrics $[M]_{T_f^{-1}(S),T_f^{-1}(W)}$ such that the following works

$$\begin{aligned} \phi_{T_f^{-1}(W)} \circ T_f^{-1}(\vec{v}) &= [M]_{T_f^{-1}(S),T_f^{-1}(W)}[\alpha]_S \\ &= [\alpha]_W. \end{aligned}$$

Now we send it back through the map

$$\phi_{T_f^{-1}(W)}^{-1} \circ \phi_{T_f^{-1}(W)} \circ T_f^{-1}(\vec{v}) = \left[T_f^{-1}(W)\right][\alpha]_W.$$

Hence

$$\vec{v} = [M]_{S,f}[\alpha]_S = [M]_{W,f}[\alpha]_W.$$

That's how the basis change. For more understanding, let's see the example

Let $f(x) = e^x$. The corresponding isomorphism is $T_f(x) = (x, e^x)$ for each $x \in \mathbb{R}$.
The graph is $\mathbb{R}_f = \{a_f = (a, e^a) : a \in \mathbb{R}\}$. By applying the method to find the corresponding addition, we have

$$a_f \oplus b_f = (a + b, e^{a+b})\ , \quad a, b \in \mathbb{R} \tag{12}$$

And the scalar multiplication

$$(\alpha)(x_f) = (\alpha x, e^{\alpha x}),\ x \in \mathbb{R} \tag{13}$$

Now the internal graph space dimension 2 has the form

$$\mathbb{R}_f^2 = \left\{\begin{pmatrix} x_f \\ y_f \end{pmatrix} : x_f, y_f \in \mathbb{R}_f\right\} \tag{14}$$

Now let see how the basis change $\mathbb{R}_f^2$. Choose a basis

$$S = \left\{\begin{pmatrix} (2, e^2) \\ (0,1) \end{pmatrix}, \begin{pmatrix} (0,1) \\ (2, e^2) \end{pmatrix}\right\}$$

and

$$W = \left\{\begin{pmatrix} (1, e) \\ (0,1) \end{pmatrix}, \begin{pmatrix} (1, e) \\ (-1, e^{-1}) \end{pmatrix}\right\}.$$

Suppose $\vec{v} = \begin{pmatrix} (5, e^5) \\ (2, e^2) \end{pmatrix}$. It will be shown that $\vec{v}$ is linear combination of $S$.
Let's pay attention to this

$$\vec{v} = (\alpha)\begin{pmatrix} (2, e^2) \\ (0,1) \end{pmatrix} \oplus (\beta)\begin{pmatrix} (0,1) \\ (2, e^2) \end{pmatrix}$$

$$=\begin{pmatrix}(2\alpha, e^{2\alpha})\\(0,1)\end{pmatrix} \oplus \begin{pmatrix}(0,1)\\(2\beta, e^{2\beta})\end{pmatrix}$$
$$= \begin{pmatrix}(2\alpha, e^{2\alpha})\\(2\beta, e^{2\beta})\end{pmatrix}$$
$$=\begin{pmatrix}(5, e^{5})\\(2, e^{2})\end{pmatrix}$$

hence $2\alpha = 5, 2\beta = 2, e^{2\alpha} = e^5, e^{2\beta}=e^2$. Those imply $\alpha = \frac{5}{2}$ and $\beta = 1$.
Therefore

$$\vec{v} = \begin{bmatrix}2_f & 0_f\\0_f & 2_f\end{bmatrix}\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}$$
$$=[M]_{S,f}[\alpha]_S$$

On the other hand,

$$\vec{v} = (\alpha)\begin{pmatrix}(1, e)\\(0,1)\end{pmatrix} \oplus (\beta)\begin{pmatrix}(1, e)\\(-1, e^{-1})\end{pmatrix}$$
$$=\begin{pmatrix}(\alpha, e^{\alpha})\\(0,1)\end{pmatrix} \oplus \begin{pmatrix}(\beta, e^{\beta})\\(-\beta, e^{-\beta})\end{pmatrix}$$
$$= \begin{pmatrix}(\alpha+\beta, e^{\alpha+\beta})\\(-\beta, e^{-\beta})\end{pmatrix}$$
$$=\begin{pmatrix}(5, e^{5})\\(2, e^{2})\end{pmatrix}$$

We have $\alpha + \beta = 5, -\beta = 2, e^{\alpha+\beta} = e^5, e^{-\beta}=e^2$. Those imply $\alpha = 7$ and $\beta = -2$.
Therefore

$$\vec{v} = \begin{bmatrix}1_f & 1_f\\0_f & -1_f\end{bmatrix}\begin{bmatrix}7\\-2\end{bmatrix}$$
$$=[M]_{W,f}[\alpha]_W.$$

Its already shown how the vector $\vec{v}$ to be expressed as linear combination of $S$ and $W$. To change basis from $S$ to $W$, we must transfer all members of $S$ to the $\mathbb{R}^2$ i.e. $T_f^{-1}(S) = \left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\}$ and $T_f^{-1}(W) = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}$. By elementary calculation we have

$$\begin{pmatrix}2\\0\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}0\\-1\end{pmatrix}$$
$$\begin{pmatrix}0\\2\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} - 2\begin{pmatrix}0\\-1\end{pmatrix}$$

thus, the transition metrics is

$$\phi_{T_f^{-1}(W)} \circ T_f^{-1}(\vec{v}) = \begin{bmatrix}2 & 0\\2 & -2\end{bmatrix}\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}$$
$$=\begin{bmatrix}7\\-2\end{bmatrix}.$$ That's the new coordinate.

## CONCLUSIONS

For each real valued function *f*, the corresponding internal graph space forms a vector space over $\mathbb{R}$. The concept of linear mapping and linear combinations still can be adapted from the graph space and still well defined.

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