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Matrix Approach to The Direct Computation Method for The Solution of Fredholm Integro-Differential Equations of The Second Kind With Degenerate Kernels

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ABSTRACT

In this paper, a matrix approach to the direct computation method for solving Fredholm Integro-Differential Equations (FIDEs) of the second kind with degenerate kernels is presented. Our approach consists of reducing the problem to a set of linear algebraic equations by approximating the kernel with a finite sum of products and determining the unknown constants by the matrix approach. The proposed method is simple, efficient and accurate; it approximates the solutions exactly with the closed form solutions. The result of this research is the solution of the second type Fredholm integro-differential equation (FIDE) with a numerically accurate kernel degenerate. Some problems are considered using maple programme to illustrate the simplicity, efficiency and accuracy of the proposed method.

Keywords: Fredholm; Matrix; Direct Solution; Integro-Differential Equation; Integral

INTRODUCTION

The subject Integro-Differential Equations (IDEs) is one of the most important mathematical tools in both pure and applied mathematics. In recent year's mathematical modeling of real-life usually results in functional equations such as differential equations, integral and Integro-Differential Equations (IDEs) these equations play very important role in modern science and technology application such as the theory of signal processing, neutral networks, heat transfer, diffusion process, neutron diffusion and biological species. These equations can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of the Volterra type is a variable, while it is a fixed number for that of Fredholm type. More details and sources where these equations can be found are in the areas of Physics, biology, engineering and social sciences and have been extensively studied both at theoretical and practical level. Most integro-differential equations are usually very difficult to solve analytically and so accurate, acceptable and efficient numerical method is required to approximate the solution (see [1], [4], [5], [6], [13], [15] and [16]).

In recent times, extensive efforts have been devoted to the numerical methods of solution for Fredholm integro-differential equations by many researchers. [2] considered non-standard finite difference method for the numerical solution of linear Fredholm integro-differential equations. The method and the repeated composite

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trapezoidal quadrature method were used to transform the Fredholm integro-differential equation into a system of non-linear algebraic equations and experiments on some linear model problems showed the simplicity and efficiency of the proposed method. The Wavelet method for the numerical solution of Fredholm integro-differential equation was used in [14]. [3] developed a finite difference hybrid method by a combination of power series and the shifted Legendre polynomial to solve Fredholm integro-differential equation. A new and efficient approach for the numerical solution of Fredholm integro-differential equations (FIDEs) of the second kind with an unbounded domain with degenerate kernel based on operational matrices with respect to generalized Laguerre polynomials (GLPs) was introduced in [8].The Adomian's decomposition method which is a well-known method for solving functional equations in recent times was used to solve linear Fredholm integro-differential equations by [9]. The result obtained gives more accurate approximation as compared to two other methods. [10], applied the Legendre polynomials for the solution of the linear Fredholm integro-differential-difference equation of high order and the results obtained by the developed technique were more accurate than the results reported for the Taylor and the wavelet Galerkin methods

The purpose of this paper is to solve Fredholm Integro-differential equation of the second kind with separable kernels by the approach of matrix method for the direct computation method, this approach is considered simple, accurate and easy to implement.

METHODS

Consider the standard form of the Fredholm integro-differential equation of the second kind given by

$$\psi^{(n)}(\theta) = \omega(\theta) + \lambda \int_a^b k(\theta, \xi)\psi(\xi)d\xi \qquad (1)$$

$$\psi^{(k)}(0) = q_k, \ \ 0 \leq k \leq n-1 \qquad (2)$$

where $\psi^{(n)}(\theta)$indicates the n^{th} derivative of $\psi(\theta)$ with respect to θ. Because (1)combines differential operator and the integral operator, then it is necessary to define initial conditions given in (2) for the determination of the particular solution $\psi(\theta)$ of (1). Suppose that we wish to determine the approximate solution of the theoretical solution $\psi(\theta)$ of problem (1) at the domain $[a, b]$. Let the separable or degenerate kernel$K(\theta, \xi)$of (1) be approximated by$\sum_{k=1}^{n} \tau_k(\theta)\phi_k(\xi)$. Thus the integro-differential equation (1) may be written as

$$\psi^{(n)}(\theta) = \omega(\theta) + \lambda \int_a^b \sum_{k=1}^{n} \tau_k(\theta)\phi_k(\xi)\,\psi(\xi)d\xi, \qquad (3)$$

where $k(\theta, \xi) = \sum_{k=1}^{n} \tau_k(\theta)\phi_k(\xi)$ is a finite sum of products$\tau_k(\theta)$ and $\phi_k(\xi)$, $\tau_k(\theta)$ is a function of θ only and $\phi_k(\xi)$ is a function of ξ only.

Equation (3) can be rewritten as

$$\psi^{(n)}(\theta) = \omega(\theta) + \lambda \sum_{k=1}^{n} \tau_k(\theta) \int_a^b \phi_k(\xi)\psi(\xi)d\xi \qquad (4)$$

Discretizing the integral part of (4) by letting $\mu_k = \int_a^b \phi_k(\xi)\psi(\xi)d\xi$, (4) becomes

$$\psi^{(n)}(\theta) = \omega(\theta) + \lambda \sum_{k=1}^{n} \tau_k(\theta)\mu_k \tag{5}$$

Now, multiplying both sides of (5) by the integral operator χ^{-n} define by ($\chi^{-n}(\star) = \int(\star)d\theta$) with the application of the initial conditions (2), it follows that (5) can be written as

$$\psi(\theta) - \sum_{i=1}^{n} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \chi^{-n}\left[\omega(\theta) + \lambda \sum_{k=1}^{n} \tau_k(\theta)\mu_k\right] \tag{6}$$

Simplifying (6), we obtain

$$\psi(\theta) - \sum_{i=1}^{n} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \omega^*(\theta) + \lambda \sum_{k=1}^{n} \tau_k^*(\theta)\mu_k \tag{7}$$

Multiplying both sides of (7) by $\phi_m(\theta)$, $m = 1,2,\ldots,n$ and integrating from a to b over θ leads to a matrix equation that facilitates the determination of the μ_k's in (5). Thus equation (7) becomes

$$\int_a^b \psi(\theta)\phi_m(\theta)d\theta - \int_a^b \sum_{i=1}^{n} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0)\phi_m(\theta)d\theta$$

$$= \int_a^b \left(\omega^*(\theta) + \lambda \sum_{k=1}^{n} \tau_k^*(\theta)\mu_k\right)\phi_m(x)d\theta \tag{8}$$

If we define μ_m, ω_m, a_{mk} as $\int_a^b \psi(\theta)\phi_m(\theta)d\theta$, $\int_a^b \omega^\star(\theta)\phi_m(\theta)d\theta$ and $\int_a^b \tau^*{}_k(\theta)\phi_m(\theta)d\theta$ respectively, then (8) can be written as

$$\mu_m - \lambda \sum_{k=1}^{n} \mu_k a_{mk} = \omega_m, \quad m = 1,2,\ldots,n \tag{9}$$

Equation (9) gives a nonhomogeneous system of n linear equations in $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ unknown. ω_m and τ_{mk} are known since $\phi_m(\theta), \omega^\star(\theta)$ and $\tau^*{}_k(\theta)$ are all given.
Putting (9) in matrix equation form, it follows that

$$\left(I - \lambda \sum_{k=1}^{n} a_{mk}\right)\mu_k = \omega_m \tag{10}$$

where

$$\mu_k = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}, \qquad \omega_m = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}, \qquad a_{mk} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Equation (10) can be used to determine the unknown $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ which are then substituted into $\psi(\theta) = \omega^*(\theta) + \sum_{k=1}^{n} \tau_k^*(\theta)\mu_k$ for the particular solution of (1)

RESULTS AND DISCUSSION

We now apply the method presented in this paper to solved four model problems to illustrate the above mentioned approach and demonstrate its computational accuracy. This method differs from the direct computation method since the unknown constants are determined at once by the introduced matrix equation

Problem1

Consider the linear Fredholm integro-differential equation given in [1]

$$y'''(x) = 5ln2 - 3 - x + 4coshx - \int_0^{ln2} (x-t)y(t)dt, \quad y(0) = y''(0) = 0\,, y'(0) = 4,$$
$$0 \le \theta \le \pi$$

The analytical solution to the problem is $y(x) = 4sinh(x)$

Let

$$k(\theta,\xi) = \sum_{k=1}^{3} \tau_k(\theta)\phi_k(\xi) = (\theta - \xi)$$

$$\tau_1(\theta) = \theta, \tau_2(\theta) = -1, \phi_1(\xi) = 1, \phi_2(\xi) = \xi$$

Let the required solution be given as

$$\psi(\theta) = \omega^*(\theta) + \sum_{k=1}^{3} \tau_k^*(\theta)\mu_k$$

Multiplying the given FIDE through with the integral operatorχ^{-3} to obtain

$$\psi(\theta) - \sum_{i=1}^{3} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \omega^*(\theta) + \sum_{k=1}^{3} \tau_k^*(\theta)\mu_k$$

or

$$\psi(\theta) - \sum_{i=1}^{3} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = 2e^{\theta} + \frac{5}{6}\theta^3 ln2 - \frac{1}{2}\theta^3 - \frac{1}{24}\theta^4 - \frac{2}{e^{\theta}} - \left(\frac{1}{24}\theta^4\mu_1 - \frac{1}{6}\theta^3\mu_2\right)$$

where

$$\omega_1 = \tfrac{1}{5}ln2^5 - \tfrac{1}{8}ln2^4 + 1, \quad \omega_2 = 5ln2 - \tfrac{1}{10}ln2^5 + \tfrac{23}{144}ln2^6 - 3$$

$$a_{11} = \tfrac{1}{120}ln2^5, a_{12} = -\tfrac{1}{24}ln2^4, a_{21} = \tfrac{1}{120}ln2^6, a_{22} = -\tfrac{1}{30}ln2^5$$

Using (10) to solve for$\mu_{k,}$ $k = 1,2$, we have $\mu_1 = 1$, $\mu_2 = 5ln2 - 3$ substituting in

$$\psi(\theta) = 2e^{\theta} + \frac{5}{6}\theta^3 ln2 - \frac{1}{2}\theta^3 - \frac{1}{24}\theta^4 - \frac{2}{e^{\theta}} - \left(\frac{1}{24}\theta^4\mu_1 - \frac{1}{6}\theta^3\mu_2\right)$$

we have $\psi(\theta) = 4\sinh(\theta)$as the exact solution. Table 1 below reveals the performance of the proposed method for problems 1

Table 1: The performance results of the proposed method for problem 1

values $(x = \theta)$	Exact solution	Proposed Method	Absolute error
0.00	0.000000000	0.000000000	0.00E+00
0.10	0.400667000	0.400667000	0.00E+00
0.20	0.805344010	0.805344010	0.00E+00
0.30	1.218081174	1.218081174	0.00E+00
0.40	1.643009303	1.643009303	0.00E+00
0.50	2.084381222	2.084381222	0.00E+00
0.60	2.546614328	2.546614328	0.00E+00
0.70	3.034334807	3.034334807	0.00E+00
0.80	3.552423929	3.552423929	0.00E+00
0.90	4.106066904	4.106066904	0.00E+00
1.00	4.700804776	4.700804776	0.00E+00

Problem2

Consider the linear Fredholm integro-differential equation in [1] given by

$$y'''(x) = e - 2 - x + e^x(3 + x) + \int_0^1 (x - t)y(t)dt, \quad y(0) = 0, y'(0) = 1, y''(0) = 2,$$

$$0 \le \theta \le \pi$$

The analytical solution to the problem is $y(x) = xe^x$

Let

$$k(\theta, \xi) = \sum_{k=1}^{3} \tau_k(\theta)\phi_k(\xi) = (\theta - \xi)$$

$$\tau_1(\theta) = \theta, \tau_2(\theta) = -1, \phi_1(\xi) = 1, \phi_2(\xi) = \xi$$

The required solution is given as

$$\psi(\theta) = \omega^*(\theta) + \sum_{k=1}^{3} \tau_k^*(\theta)\mu_k$$

Multiplying the given FIDE with the integral operatorχ^{-3}, we have

$$\psi(\theta) - \sum_{i=1}^{3} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \omega^*(\theta) + \sum_{k=1}^{3} \tau_k^*(\theta)\mu_k$$

or

$$\psi(\theta) - \sum_{i=1}^{3} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \frac{1}{6}e\theta^3 - \frac{1}{3}\theta^3 - \frac{1}{24}\theta^4 + \theta e^\theta + \left(\frac{1}{24}\theta^4\mu_1 - \frac{1}{6}\theta^3\mu_2\right)$$

where

$$\omega_1 = \frac{109}{120} + \frac{1}{24}e, \quad \omega_2 = -\frac{1493}{720} + \frac{31}{30}e$$

$$a_{11} = \frac{1}{120}, \qquad a_{12} = -\frac{1}{24}, \qquad a_{21} = \frac{1}{144}, \qquad a_{22} = -\frac{1}{30}$$

Using (10) to solve for $\mu_{k,}$ $k = 1, 2$ and substituting into

$$\psi(\theta) = \frac{1}{6}e\theta^3 - \frac{1}{3}\theta^3 - \frac{1}{24}\theta^4 + \theta e^\theta + \left(\frac{1}{24}\theta^4\mu_1 - \frac{1}{6}\theta^3\mu_2\right)$$

we obtain$\psi(\theta) = \theta e^\theta$giving the same result as the exact solution.

Problem 3

Consider the linear Fredholm integro-differential equation in [1] given as

$$y''(x) = 4x - \sin(x) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x - t)^2 y(t)dt, \quad y(0) = 0, y'(0) = 1, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

The analytical solution to the problem is $y(x) = sin(x)$

Let

$$k(\theta, \xi) = \sum_{k=1}^{2} \tau_k(\theta)\phi_k(\xi) = (\theta - \xi)^2$$

$$\tau_1(\theta) = \theta^2, \tau_2(\theta) = -2\theta, \tau_3(\theta) = 1 \; \phi_1(\xi) = 1, \phi_2(\xi) = \xi, \phi_3(\xi) = \xi^2$$

The required solution is given as

$$\psi(\theta) = \omega^*(\theta) + \sum_{k=1}^{2} \tau_k^*(\theta)\mu_k$$

Multiplying the given FIDE with the integral operatorχ^{-2}, we have

$$\psi(\theta) - \sum_{i=1}^{2} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \omega^*(\theta) + \sum_{k=1}^{3} \tau_k^*(\theta)\mu_k$$

or

$$\psi(\theta) - \sum_{i=1}^{2} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = sin\theta + \frac{2}{3}\theta^3 + \left(\frac{1}{2}\theta^2\mu_3 - \frac{1}{3}\theta^3\mu_2 + \frac{1}{12}\theta^4\mu_1\right)$$

where

$$\omega_1 = 0, \qquad \omega_2 = \tfrac{1}{12}\pi^5 + 2, \qquad \omega_2 = 0$$

$$a_{11} = \tfrac{1}{960}\pi^5, \qquad a_{12} = 0, \qquad a_{13} = \tfrac{1}{24}\pi^3$$

$$a_{21} = 0, \qquad a_{22} = -\tfrac{1}{240}\pi^5, \qquad a_{23} = 0$$

$$a_{31} = \tfrac{1}{5376}\pi^7, a_{32} = 0, \qquad a_{33} = \tfrac{1}{160}\pi^5$$

Using (10) to solve for $\mu_{k,}$ $k = 1, 2, 3$ and substituting in

$$\psi(\theta) = sin\theta + \frac{2}{3}\theta^3 + \left(\frac{1}{2}\theta^2\mu_3 - \frac{1}{3}\theta^3\mu_2 + \frac{1}{12}\theta^4\mu_1\right)$$

gives $\psi(\theta) = sin(\theta)$ having the same result as the exact solution.

Problem 4

Consider the linear Fredholm integro-differential equation given by

$$y^{(iv)}(x) = 2x - \pi + \sin(x) + cos(x) - \int_0^{\frac{\pi}{2}} (x - 2t)y(t)dt, \quad y(0) = y'(0) = 1,$$

$$y''(0) = y'''(0) = -1, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

The analytical solution to the problem is $y(x) = sin(x) + cos(x)$

Let

$$k(\theta, \xi) = \sum_{k=1}^{4} \tau_k(\theta)\phi_k(\xi) = (\theta - 2\xi)$$

$\tau_1(\theta) = \theta, \tau_2(\theta) = -2, \tau_3(\theta) = 0, \tau_4(\theta) = 0, \phi_1(\xi) = 1, \phi_2(\xi) = \xi, \phi_3(\xi) = 0, \phi_4(\xi) = 0$

The required solution is given as

$$\psi(\theta) = \omega^*(\theta) + \gamma \sum_{k=1}^{4} \tau_k^*(\theta)\mu_k$$

Multiplying the given FIDE with the integral operatorχ^{-4}, we have

$$\psi(\theta) - \sum_{i=1}^{4} \frac{\theta^{(i-1)}}{(i-1)!}\psi^{(i-1)}(0) = \omega^*(\theta) + \sum_{k=1}^{4} \tau_k^*(\theta)\mu_k$$

or

$$\psi(\theta) - \sum_{i=1}^{4} \frac{\theta^{(i-1)}}{(i-1)!} \psi^{(i-1)}(0) = cos\theta + sin\theta - \frac{1}{24}\pi\theta^4 + \frac{1}{60}\theta^5 + \left(\frac{1}{12}\theta^4\mu_2 - \frac{1}{120}\theta^5\mu_1\right)$$

where

$$\omega_1 = 2 - \frac{1}{4608}\pi^6, \quad \omega_2 = \frac{1}{2}\pi - \frac{29}{322560}\pi^7$$

$$a_{11} = -\frac{1}{46080}\pi^6, \quad a_{12} = \frac{1}{1920}\pi^5$$

$$a_{21} = -\frac{1}{107520}\pi^7, \quad a_{22} = \frac{1}{4608}\pi^6$$

Using (10) to solve for $\mu_{k,}$ $k = 1, 2$ and substituting in

$$\psi(x) = cos\theta + sin\theta - \frac{1}{24}\pi\theta^4 + \frac{1}{60}\theta^5 + \left(\frac{1}{12}\theta^4\mu_2 - \frac{1}{120}\theta^5\mu_1\right)$$

we obtain $\psi(\theta) = cos(\theta) + sin(\theta)$ giving the same result with the exact solution.

CONCLUSIONS

This paper deals with the solution of linear Fredholm integro-differential equations of the second kind with separable kernels. Our approach was based on the matrix approach which reduces the Fredholm integro- differential equation into a set of linear algebraic equations for the determination of the unknown constants. The advantage of this method over the direct computation method is that the constants in this method are obtained at once instead of the successive substitution approach inherent in the direct computation method. The method was tested on some model problems from the literature; the results obtained by the technique developed were the same with the exact solution revealing the effectiveness of the proposed method.

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The algorithm for the implementation of the given problems using the Maple programme is outline below*:*

$with(Student[Calculus1])$:

$n := ?$:

$\lambda := ?$

$f := f(x)$

$g := (\lambda) sum(\tau[\mathrm{i}] \cdot \mathrm{mu}[\mathrm{i}], i = 1 \,..n)$

$\tau[i] := \tau[i](x), i = 1 \,..n$

$\phi[i] := \phi[i](t), i = 1 \,..n$

$f^{\circledast} := int(int(f, x), x)$:

$g^{\circledast} := int(int(\mathrm{g}, x), x)$:

$f[j] := int(\phi[j] \cdot f^{\circledast}, x = a \,..b), j = 1 \,..n$:

$$A := Matrix([[seq(int(\phi[1] \cdot int(int(int(int(\tau[\mathrm{j}], x), x), x), x), x = \mathrm{a} \,..\mathrm{b}), j = 1 \,..n)], [seq(int(\phi[2] \cdot int(int(int(int(\tau[\mathrm{j}], x), x), x), x), x = \mathrm{a} \,..\mathrm{b}), j = 1 \,..n)], \ldots, [seq(int(\phi[n] \cdot int(int(int(int(\tau[\mathrm{j}], x), x), x), x), x = \mathrm{a} \,..\mathrm{b}), j = 1 \,..n)]])$$

$with(LinearAlgebra)$:

$B := Matrix(n, n, shape = identity) - (\lambda) A$:

$C := simplify(MatrixInverse(B))$

$F := Matrix(n, 1, [f[j], j = 1 \,..n])$

$\% := MatrixVectorMultiply(\mathrm{C}, F)$:

$v := simplify(\%)$:

$Vector(n, 1, [\mathrm{mu}[j], \mathrm{j} = 1 \,..\mathrm{n}]) = Matrix(n, 1, [v])$

$y := eval(f^{\circledast} + g^{\circledast}, \{mu[j] = ?, j=1..n\})$