



The Rule of Hessenberg Matrix for Computing the Determinant of Centrosymmetric Matrices

Nur Khasanah¹, Agustin Absari Wahyu Kuntarini²

^{1,2}Department of Mathematics, Faculty of Science and Technology UIN Walisongo
Semarang

Email: khasanah.nur@walisongo.ac.id

ABSTRACT

The application of centrosymmetric matrix on engineering takes their part, particularly about determinant rule. This basic rule needs a computational process for determining the appropriate algorithm. Therefore, the algorithm of the determinant kind of Hessenberg matrix is used for computing the determinant of the centrosymmetric matrix more efficiently. This paper shows the algorithm of lower Hessenberg and sparse Hessenberg matrix to construct the efficient algorithm of the determinant of a centrosymmetric matrix. Using the special structure of a centrosymmetric matrix, the algorithm of this determinant is useful for their characteristics.

Key Words : Hessenberg; Determinant; Centrosymmetric

INTRODUCTION

One of the widely used studies in the use of centrosymmetric matrices is how to get determinant from centrosymmetric matrices. Besides this special matrix has some applications [1], it also has some properties used for determinant purpose [2]. Special characteristic centrosymmetric at this entry is evaluated at [3] resulting in the algorithm of centrosymmetric matrix at determinant. Due to sparse structure of this entry, the evaluation of the determinant matrix has simpler operations than full matrix entries.

One special sparse matrix having rules on numerical analysis and arise at centrosymmetric the determinant matrix is the Hessenberg matrix. The role of Hessenberg matrix decomposition is the important role of computing the eigenvalue matrix. In the discussion, the recursive algorithm is explained to compute the n-per-n determinant of the Hessenberg matrix [4]. This study is the evaluation of some researches before about the determinant Hessenberg matrix [5,6].

The rule of the Hessenberg matrix for computing the determinant of general centrosymmetric matrix based on block matrix has been done by [7]. Moreover, this study continued by [8] on evaluating the previous work and shows only lower Hessenberg matrix as block centrosymmetric matrix can use this algorithm. Then necessary and sufficient condition for this algorithm is constructed also that can be evaluated for the determinant process [9]. A Numerical example also shows for more understanding at a special centrosymmetric matrix on its block.

Furthermore, the determinant of sparse Hessenberg matrix is also constructed for resulting in the new algorithm for a pentadiagonal matrix[10]. Some studies are also

focussing on the determinant algorithm for computational process [11-17]. Based on the previous study, we propose a new form of pentadiagonal matrix with construction entries based on a centrosymmetric matrix called pentadiagonal centrosymmetric matrix. We then continue applying the previous algorithm to compute our new form of pentadiagonal centrosymmetric matrix for determinant purpose. Finally, this paper shows two algorithms of centrosymmetric matrix determinant by using the algorithm of the determinant of the Hessenberg matrix.

PRELIMINARIES

Before we discuss the main result, let introduce some definitions and properties used to explain the determinant of centrosymmetric matrix using Hessenberg’s determinant.

Definition 1 [3] The form of n -by- n lower Hessenberg matrix is written as the matrix with the entries as follow

$$\mathbf{H} = \begin{pmatrix}
 h_{11} & h_{1,2} & 0 & 0 & 0 \\
 h_{21} & h_{2,2} & h_{2,3} & 0 & 0 \\
 \vdots & \vdots & \ddots & \ddots & 0 \\
 h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} \\
 h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n}
 \end{pmatrix}. \tag{1}$$

Definition 2 [10] The sparse Hessenberg matrix with order n is the matrix with construction as

$$\mathbf{S} = \begin{pmatrix}
 h_{11} & h_{1,2} & h_{1,3} & & & & & & & & \\
 h_{21} & h_{2,2} & h_{2,3} & h_{2,4} & & & & & & & \\
 & h_{3,2} & h_{3,3} & h_{3,4} & h_{3,5} & & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & & h_{n-2,n} \\
 & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & h_{n-1,n-2} & h_{n-1,n-1} & h_{n-1,n} & \\
 & & & & & & & h_{n,n-1} & h_{n,n} & & \\
 & & & & & & & & & &
 \end{pmatrix}. \tag{2}$$

Definition 3 [2] Let $\mathbf{A} = (a_{ij})_{n \times n} \in R^{n \times n}$ be a centrosymmetric matrix, if the entries of the matrix satisfy $a_{ij} = a_{n-i+1,n-j+1}$ for $1 \leq i \leq n, 1 \leq j \leq n$ or

$$\mathbf{A} = \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1,n} \\
 a_{21} & a_{22} & \cdots & a_{2,n} \\
 \vdots & \vdots & \vdots & \vdots \\
 a_{2,n} & \cdots & a_{22} & a_{21} \\
 a_{1,n} & \cdots & a_{12} & a_{11}
 \end{pmatrix}. \tag{3}$$

Definition 4 [7] The centrosymmetric matrix has a special structure that can be constructed as $\mathbf{J}_n \mathbf{A} \mathbf{J}_n = \mathbf{A}$, where $\mathbf{J}_n = (\mathbf{e}_n, \mathbf{e}_{n-1}, \dots, \mathbf{e}_1)$ and \mathbf{e}_i is the unit vector with the i -th element 1 and others 0.

Definition 5 [8] The centrosymmetric matrix with n order where $n = 2m$ is even number order can be partitioned with the form

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{B} & \mathbf{J}_m \mathbf{C} \mathbf{J}_m \\ \hline \mathbf{C} & \mathbf{J}_m \mathbf{B} \mathbf{J}_m \end{array} \right). \quad (4)$$

RESULTS AND DISCUSSION

This part shows the rule of the determinant of lower Hessenberg and sparse Hessenberg matrix for determinant concept of the centrosymmetric matrix. Further discussion is also given for explaining the result and deeper understanding.

Determinant Lower Hessenberg Matrix for Centrosymmetric Matrix's Determinant

The algorithm of the determinant of lower Hessenberg is presented based on the definition and properties of this matrix. The following is the Lemma of the determinant of lower Hessenberg, which is constructed as a block matrix.

Lemma 6 [7] Let $\tilde{\mathbf{H}} = \left(\begin{array}{c|c} \mathbf{e}_1^T & 0 \\ \hline \mathbf{H} & \mathbf{e}_n \end{array} \right) = \left(\begin{array}{cccc|c} 1 & 0 & \dots & \dots & 0 \\ h_{11} & h_{1,2} & 0 & 0 & 0 \\ h_{21} & h_{2,2} & h_{2,3} & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h_{n-1,1} & h_{n-1,2} & \dots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & \dots & h_{n,n-1} & h_{n,n} \end{array} \right)$ is the partition

matrix and $\tilde{\mathbf{H}} = \left(\begin{array}{c|c} \mathbf{e}_1^T & 0 \\ \hline \mathbf{H} & \mathbf{e}_n \end{array} \right) = \left(\begin{array}{cccc|c} 1 & 0 & \dots & \dots & 0 \\ h_{11} & h_{1,2} & 0 & 0 & 0 \\ h_{21} & h_{2,2} & h_{2,3} & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h_{n-1,1} & h_{n-1,2} & \dots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & \dots & h_{n,n-1} & h_{n,n} \end{array} \right)$ as the inverse matrix of

its matrix, then the determinant of this matrix is written as $\det(\mathbf{H}) = (-1)^n h \cdot \prod_{i=1}^{n-1} h_{i,i+1}$.

This algorithm is used for evaluating the algorithm of the determinant of centrosymmetric matrix with lower Hessenberg as a block matrix. The algorithm of the determinant of the centrosymmetric matrix with lower Hessenberg as block matrix can be proposed as follow.

Input : Matrix $\mathbf{A} = \left(\begin{array}{c|c} \mathbf{B} & \mathbf{J}_m \mathbf{C} \mathbf{J}_m \\ \hline \mathbf{C} & \mathbf{J}_m \mathbf{B} \mathbf{J}_m \end{array} \right)$

Output : $\det(\mathbf{A})$

1) Operate $\mathbf{P}^T \mathbf{A} \mathbf{P} = \left(\begin{array}{cc} \mathbf{B} - \mathbf{J}_m \mathbf{C} & 0 \\ 0 & \mathbf{B} + \mathbf{J}_m \mathbf{C} \end{array} \right) = \left(\begin{array}{cc} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{array} \right)$

2) Construct \mathbf{M}, \mathbf{N} are Hessenberg matrices become $\tilde{\mathbf{M}} = \begin{pmatrix} \mathbf{e}_1^T & 0 \\ \mathbf{M} & \mathbf{e}_m \end{pmatrix}$,

$$\tilde{\mathbf{N}} = \begin{pmatrix} \mathbf{e}_1^T & 0 \\ \mathbf{N} & \mathbf{e}_m \end{pmatrix}$$

3) Let inverse matrices $\tilde{\mathbf{M}}^{-1} = \begin{pmatrix} \boldsymbol{\alpha}_M & \mathbf{L}_M \\ h_M & \boldsymbol{\beta}_M^T \end{pmatrix}$, $\tilde{\mathbf{N}}^{-1} = \begin{pmatrix} \boldsymbol{\alpha}_N & \mathbf{L}_N \\ h_N & \boldsymbol{\beta}_N^T \end{pmatrix}$

4) Compute $\det(\mathbf{H}) = \det(\mathbf{P}) \cdot \det \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix} \cdot \det(\mathbf{P}^T) = \det(\mathbf{M}) \cdot \det(\mathbf{N})$

5) $\det(\mathbf{H}) = (-1)^m h_M \cdot \prod_{i=1}^{m-1} g_{i,i+1} \cdot (-1)^m h_N \cdot \prod_{i=1}^{m-1} q_{i,i+1}$.

Or we can write this algorithm by the following theorem for computing the determinant of centrosymmetric matrix with applying the algorithm of the determinant of lower Hessenberg matrix. This algorithm is only applied on a specific centrosymmetric matrix based on the special block matrix.

Theorem 7 [9] Let centrosymmetric matrix with its block matrix, lower Hessenberg matrix, and its notation, such $\mathbf{H}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \tilde{\mathbf{M}}^{-1}, \tilde{\mathbf{N}}^{-1}$, then the determinant of the centrosymmetric matrix is written as

$$\det(\mathbf{H}) = h_N \cdot h_M \cdot \prod_{i=1}^{m-1} (g_{i,i+1} \cdot q_{i,i+1}). \quad (5)$$

Proof. From the definition of the centrosymmetric matrix, this matrix can be formed as a block matrix $\mathbf{A} = \left(\begin{array}{c|c} \mathbf{B} & \mathbf{J}_m \mathbf{C} \mathbf{J}_m \\ \hline \mathbf{C} & \mathbf{J}_m \mathbf{B} \mathbf{J}_m \end{array} \right)$. By the orthogonal matrix $\mathbf{P} = \frac{\sqrt{2}}{2} \begin{pmatrix} \mathbf{I}_m & \mathbf{I}_m \\ -\mathbf{J}_m & \mathbf{J}_m \end{pmatrix}$, this block matrix can be constructed as $\mathbf{P}^T \mathbf{H} \mathbf{P} = \begin{pmatrix} \mathbf{B} - \mathbf{J}_m \mathbf{C} & 0 \\ 0 & \mathbf{B} + \mathbf{J}_m \mathbf{C} \end{pmatrix} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix}$. Based on this matrix, the determinant of centrosymmetric matrix only calculate on block matrices the determinant. These block matrices \mathbf{M}, \mathbf{N} are Hessenberg form, therefore the algorithm of determinant lower Hessenberg can be applied for this determinant. This step becomes the main necessary for the next algorithm.

By the algorithm of lower Hessenberg matrix, let's form $\tilde{\mathbf{M}} = \begin{pmatrix} \mathbf{e}_1^T & 0 \\ \mathbf{M} & \mathbf{e}_m \end{pmatrix}$, $\tilde{\mathbf{N}} = \begin{pmatrix} \mathbf{e}_1^T & 0 \\ \mathbf{N} & \mathbf{e}_m \end{pmatrix}$, and assume $\tilde{\mathbf{M}}^{-1} = \begin{pmatrix} \boldsymbol{\alpha}_M & \mathbf{L}_M \\ h_M & \boldsymbol{\beta}_M^T \end{pmatrix}$, $\tilde{\mathbf{N}}^{-1} = \begin{pmatrix} \boldsymbol{\alpha}_N & \mathbf{L}_N \\ h_N & \boldsymbol{\beta}_N^T \end{pmatrix}$, where $\boldsymbol{\alpha}_{M,N}$, $\mathbf{L}_{M,N}$, $\boldsymbol{\beta}_{M,N}$ are matrices with the size of $n \times 1$, $n \times n$, $n \times 1$ respectively and $h_{M,N}$ is scalar. By analytical process at lower Hessenberg algorithm, then we find $h_M \neq 0$ and $h_N \neq 0$. Finally, the determinant of the centrosymmetric matrix based on lower Hessenberg form is

$$\det(\mathbf{H}) = \det(\mathbf{P}) \cdot \det \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix} \cdot \det(\mathbf{P}^T) = \det(\mathbf{M}) \cdot \det(\mathbf{N}). \blacksquare \quad (6)$$

$$\mathbf{H} = \begin{pmatrix} d_{11} & d_{12} & d_{13} & & & & & & & & \\ d_{21} & d_{22} & d_{23} & d_{24} & & & & & & & \\ & d_{32} & d_{33} & d_{34} & d_{35} & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & \ddots & \ddots & \ddots & & & & & \\ & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & d_{n-1,n-2} & d_{n-1,n-1} & d_{n-1,n} & & & \\ & & & & & & d_{n,n-1} & d_{n,n} & & & \end{pmatrix}. \quad (8)$$

The form of (8) matrix is resulted by choosing $\lambda_{i+1,i}$ from the matrix of

$$\lambda_{i+1,i} = \begin{pmatrix} I_{i-1} & & & & \\ & 1 & & & \\ & \lambda_{i+1,i} & 1 & & \\ & & & \ddots & \\ & & & & I_{n-i-1} \end{pmatrix}. \quad (9)$$

For instance, let pentadiagonal centrosymmetric matrix with the size 8×8 and choose

$\lambda_{3,2} = -\frac{d_{3,1}}{d_{2,1}}$ for transforming matrix as follows.

$$\lambda_3 \mathbf{D} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & -\frac{d_{3,1}}{d_{2,1}} & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} & d_{13} & & & & & \\ d_{21} & d_{22} & d_{23} & d_{24} & & & & \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & & & \\ & d_{42} & d_{43} & d_{44} & d_{45} & d_{56} & & \\ & & d_{53} & d_{54} & d_{55} & d_{56} & d_{57} & \\ & & & d_{64} & d_{65} & d_{66} & d_{67} & d_{68} \\ & & & & d_{75} & d_{76} & d_{77} & d_{78} \\ & & & & & d_{86} & d_{87} & d_{88} \end{pmatrix}$$

$$= \begin{pmatrix} d_{11} & d_{12} & d_{13} & & & & & \\ d_{21} & d_{22} & d_{23} & d_{24} & & & & \\ & \tilde{d}_{32} & d_{33} & d_{34} & d_{35} & & & \\ & & d_{42} & d_{43} & d_{44} & d_{45} & d_{56} & \\ & & & d_{53} & d_{54} & d_{55} & d_{56} & d_{57} \\ & & & & d_{64} & d_{65} & d_{66} & d_{67} & d_{68} \\ & & & & & d_{75} & d_{76} & d_{77} & d_{78} \\ & & & & & & d_{86} & d_{87} & d_{88} \end{pmatrix}.$$

By the same way, taking $\lambda_{4,3} = -\frac{d_{42}}{d_{32}}, \dots, \lambda_{87} = -\frac{d_{86}}{d_{76}}$ and the process will have the following Hessenberg matrix as follow.

$\alpha_n = \mathbf{d}_{n,n} - \mathbf{d}_{n,n-1} \left(\mathbf{d}_{n-2,n} \mathbf{e}^T_{n-1} \mathbf{H}^{-1}_{n-1} \mathbf{e}_{n-2} + \mathbf{d}_{n-1,n} \mathbf{e}^T_{n-1} \mathbf{H}^{-1}_{n-1} \mathbf{e}_{n-1} \right)$, where \mathbf{e} is a vector unit.

By substituting the equations $\mathbf{e}^T_{n-1} \mathbf{H}^{-1}_{n-1} \mathbf{e}_{n-2} = -\frac{\mathbf{f}_{n-3}}{\mathbf{f}_{n-1}} \mathbf{d}_{n-1,n-2}$ and

$\mathbf{e}^T_{n-1} \mathbf{H}^{-1}_{n-1} \mathbf{e}_{n-1} = -\frac{\mathbf{f}_{n-2}}{\mathbf{f}_{n-1}}$ then become

$\alpha_n = \mathbf{d}_{n,n} - \mathbf{d}_{n,n-1} \left(\frac{\mathbf{f}_{n-3}}{\mathbf{f}_{n-1}} \mathbf{d}_{n-1,n-2} \mathbf{d}_{n-2,n} - \frac{\mathbf{f}_{n-2}}{\mathbf{f}_{n-1}} \mathbf{d}_{n-1,n} \right)$. Then, by multiply the equation with

\mathbf{f}_{n-1} we have

$$\mathbf{f}_n = \alpha_n \mathbf{f}_{n-1} = \mathbf{f}_{n-1} \mathbf{d}_{n,n} + \mathbf{d}_{n,n-1} (\mathbf{f}_{n-3} \mathbf{d}_{n-1,n-2} \mathbf{d}_{n-2,n} - \mathbf{f}_{n-2} \mathbf{d}_{n-1,n}).$$

4. Construct the Algorithm of Determinant Pentadiagonal Centrosymmetric Matrix

$$\mathbf{f}_1 = \mathbf{d}_{11}$$

$$\mathbf{f}_2 = \mathbf{f}_1 \mathbf{d}_{22} - \mathbf{d}_{21} \mathbf{d}_{12}$$

$$\mathbf{f}_3 = \mathbf{f}_2 \mathbf{d}_{33} + \mathbf{d}_{32} (\mathbf{d}_{21} \mathbf{d}_{13} - \mathbf{f}_1 \mathbf{d}_{23})$$

for $i = 4, 5, \dots, n$

$$\mathbf{f}_i = \mathbf{f}_{i-1} \mathbf{d}_{ii} + \mathbf{d}_{i,i-1} (\mathbf{f}_{i-3} \mathbf{d}_{i-1,i-2} \mathbf{d}_{i-2,i} - \mathbf{f}_{i-2} \mathbf{d}_{i-1,i})$$

end

$$\det(\mathbf{D}) = \mathbf{f}_n.$$

This algorithm shows the rule of the determinant of Hessenberg matrix can be constructed for the general determinant of the pentadiagonal centrosymmetric matrix.

Based on two different algorithms above, the rule of determinant Hesseberg matrix can evaluate the determinant of the centrosymmetric matrix. For efficient algorithm, the first algorithm evaluates the determinant of centrosymmetric matrix applying the algorithm of the determinant of the lower Hessenberg matrix. It happens caused by the lower Hessenberg matrix appear as its block matrix, then the algorithm of centrosymmetric matrix only a half working. On the next algorithm, the algorithm of the sparse Hessenberg matrix is applied to construct the algorithm of determinant general pentadiagonal centrosymmetric matrix. This algorithm is used caused by the same structure of the main matrix is a sparse Hessenberg matrix, therefore is applicable.

CONCLUSIONS

The different cases of the application of determinant Hesseberg matrix are applied to a different form of a centrosymmetric matrix. It is necessary to identify the form of the main general centrosymmetric from the beginning. This part determines the best algorithm for the computational process more efficient. This paper shows the different Hessenberg appearance on a centrosymmetric matrix results different algorithms of the computing of the determinant.

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