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On Transitive Closures of Two-dimensional Strongly Positive Arithmetical Sets[1]

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Abstract

The notions of positive and strongly positive arithmetical set are considered in ([1]-[3]). It is noted in [3] that the transitive closure of any 2-dimensional strongly positive set is primitive recursive. In this article a more strong statement is proved: the transitive closure of any 2-dimensional strongly positive set is defined by an arithmetical formula in the signature $(0, =, <, S)$, where $S(x) = x + 1$. Besides, it is proved that the class of two-dimensional strongly positive sets and the class of transitive closures of such sets do not coincide with the class of two-dimensional arithmetical sets expressible by the formulas in the signature $(0, =, <, S)$.

Keywords: Positive, Strongly positive, Arithmetical set, Dimension, Signature.

1. Introduction

The notion of strongly positive arithmetical set is defined and investigated in [3]. It is proved in [3] that for any $n \geq 3$ there exists a $2n$-dimensional strongly positive set such that its transitive closure is not recursive. It is noted in [3] (without a proof) that the transitive closure of any 2-dimensional strongly positive set is primitive recursive. Below a stronger statement is proved: the transitive closure of any 2-dimensional strongly positive set can be defined by an arithmetical formula in the signature $(0, =, <, S)$, where $S(x) = x + 1$(see below, Theorem 1). It is proved also that the class of the mentioned transitive closures does not coincide with the class of sets expressible by arithmetical formulas in the signature $(0, =, <, S)$. For example, it is proved that

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the set $\{(x, y)/y = x + 2\}$ is not strongly positive and cannot be represented as the transitive closure of some strongly positive set (see below, Theorem 2).

2. Main Definitions and Results

By N we denote the set of all non-negative integers, $N = \{0,1,2, \dots\}$. By N^n, where $n \geq 1$, we denote the set of n-tuples $(x_1, x_2, \dots x_n)$, where $x_i \in N$ for $1 \leq i \leq n$.
An n-dimensional arithmetical set, where $n \geq 1$, is defined as any subset of N^n. An n-dimensional arithmetical predicate is defined as a predicate P, which is true on some set $A \subseteq N^n$ and false on the set $N^n \backslash A$. If the mentioned relation between A and P takes place, then we say that P is the representing predicate for A, and A is the set of truth for P.

The notions of primitive recursive set and recursive set are defined in a usual way (see [4]-[6]).

The notion of arithmetical formula on a given signature (on the base of logical operations $\&, \vee, \supset, \neg, \forall, \exists$) is defined in a usual way, ([2]-[6]). We will consider arithmetical formulas in the signatures $(0, =, S)$ and $(0, =, <, S)$, where $S(x) = x + 1$ for $x \in N$.

The deductive systems in the signatures $(0, =, S)$ and $(0, =, <, S)$ are defined as in [6]; we will denote these deductive systems correspondingly by $\mathrm{Ded_S}$ and $\mathrm{Ded_L}$. As it is proved in [6], these deductive systems are complete. We say that the formulas F and G in the corresponding signatures are equivalent if the formula $(F \supset G)\&(G \supset F)$ is deducible in the corresponding deductive system. We will consider the formulas in the mentioned signatures up to their equivalence.

The relation "n-dimesional arithmetical set A is defined by an arithmetical formula F" is given in a usual way (see [2]-[6]) (in [2] this relation is called as follows: "k-dimensional arithmetical set A is represented (or representable) by a formula F").

The notion of transitive closure A^* for an arithmetical set A having an even dimension $2k$ (where $k \geq 1$) is defined in a usual way (see, for example, [3], [8]). Let us recall that the following statement holds (see [3], lemma 3.4, and [8], p.72): if A is a $2k$-dimensional set, $A \subseteq N^{2k}$, where $k \geq 1$, then $(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k) \in A^*$ if and only if there exists a sequence $(Q_1, Q_2, \dots, Q_m)$ of k-tuples such that $m \geq 2$, $Q_1 = (x_1, x_2, \dots, x_k)$, $Q_m = (y_1, y_2, \dots, y_k)$, $(Q_i, Q_{i+1}) \in A$ for $1 \leq i \leq m - 1$. Below we will say that the sequence $(Q_1, Q_2, \dots, Q_m)$, having the mentioned properties is a sequence establishing the value $(Q_1, Q_m) \in A^*$ of the transitive closure A^* (or, shortly, ETC-sequence). In what follows we will consider ETC-secuences only for the case $k = 1$.

The notion of strongly positive arithmetical set is defined as in [3]. Let us recall that an m-dimensional arithmetical set, where $m \geq 1$, is said to be strongly positive if it is defined by an arithmetical formula F which is constructed by logical operators & and $\vee$ from subformulas having the forms $(x = a)$ (where a is a constant, $a \in N$), $x = y$, $y = S(x)$, $\neg(x = 0)$, where x and y are variables.

Theorem 1: *The transitive closure of any 2-dimensional strongly positive set can be defined by an arithmetical formula in the signature* $(0,=,<,S)$.

Theorem 2: *The set* $\{(x,y)/\ y=S(S(x))\}$ *is not strongly positive and cannot be represented as a transitive closure of some strongly positive set.*

3. Proofs of Theorems

We consider the properties of 2-dimensional strongly positive sets. Let π be any set of such kind. By η we denote the representing predicate for π. Using the definition of strongly positive set we conclude that the predicate η can be expressed by an arithmetical formula F having the form $F_1 \vee F_2 \vee \ldots \vee F_m$, where any F_i is the conjuction of subformulas having the following forms: $x=a$, $y=b$, (where a and b are constants, $a \in N$, $b \in N$), $x=y$, $y=S(x)$, $x=S(y)$, $\neg(x=0)$, $\neg(y=0)$. The predicate expressed by the formula F_i we denote by η_i; the set of truth for η_i we denote by π_i. The following equalities hold:

$$\eta(x,y) \equiv \eta_1(x,y) \vee \eta_2(x,y) \vee \ldots \vee \eta_m(x,y);$$

$$\pi = \pi_1 \cup \pi_2 \cup \ldots \cup \pi_m.$$

Clearly, all the predicates $\eta, \eta_1, \eta_2, \ldots, \eta_m$, and all the sets $\pi, \pi_1, \pi_2, \ldots, \pi_m$ are expressible by arithmetical formulas in the signature $(0,=,S)$. Let us note that if some F_i includes simultaneously some two subformulas of the forms $x=y$, $y=S(x)$, $x=S(y)$, then the corresponding predicate η_i is identically false, hence, F_i can be deleted from the structure of F, similarly, F_i can be deleted from the structure of F if it includes subformulas of the forms $x=a_1$ and $x=a_2$ where $a_1 \neq a_2$ or subformulas of the forms $y=b_1$ and $y=b_2$ where $b_1 \neq b_2$.

Let us consider possible forms of the formula F_i. We do not consider the cases mentioned above when F_i can be deleted from the structure of F.

(Case 1). F_i contains the subformulas $x=a$, $y=S(x)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair $(a,a+1)$ (for example, π_i is empty if F_i has the form $(x=0\ \&\ y=S(x) \& \neg(x=0))$).

(Case 2). F_i contains the subformulas $y=b$, $y=S(x)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty or contains the single pair $(b-1,b)$, where $b>0$.

(Case 3). F_i contains the subformulas $y=S(x)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$ (we suppose that F_i contains no subformula having one of the forms $x=a$, $y=b$, $x=y$, $x=S(y)$). In this case all pairs of numbers having the form $(x,x+1)$, where $x>0$, belong to π_i. The statement $(0,1) \in \pi_i$ is true if and only if the subformula $\neg(x=0)$ is not included in F_i.

(Case 4). F_i contains the subformulas $y=b$, $x=S(y)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair $(b+1,b)$.

(Case 5). F_i contains the subformulas $x=a$, $x=S(y)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair $(a,a-1)$, where $a>0$.

(Case 6). F_i contains the subformulas $x=S(y)$, and, possibly, $\neg(x=0)$, $\neg(y=0)$ (we suppose that F_i contains no subformulas having one of the forms $x=a$, $y=b$, $x=y$, $y=S(x)$).

In this case all pairs of numbers having the form $(x+1,x)$, where $x>0$, belong to π_i. The statement $(1,0) \in \pi_i$ is true if and only if the subformula $\neg(y=0)$ is not included in F_i.

(Case 7). F_i contains the subformulas $x=a$, $y=b$, and, possibly, $x=y$ $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair (a,b).

(Case 8). F_i contains the subformulas $x=a$, $x=y$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair (a,a).

(Case 9). F_i contains the subformulas $y=b$, $x=y$, and, possibly, $\neg(x=0)$, $\neg(y=0)$. In this case π_i is either empty, or contains the single pair (b,b).

(Case 10). F_i contains the subformulas $x=a$, and, possibly, $\neg(x=0)$, $\neg(y=0)$ (we suppose that F_i contains no subformulas having one of the forms, $y=b$, $x=y$, $y=S(x)$, $x=S(y)$). In this case π_i is empty when $a=0$ and the subformula $\neg(x=0)$ is included in F_i. In the opposite case π_i contains all pairs (a,y), where y>0. The statement $(a,0) \in \pi_i$ is true (for $a>0$) if and only if the subformula $\neg(y=0)$ is not included in F_i.

(Case 11). F_i contains the subformulas $y=b$, and, possibly, $\neg(x=0)$, $\neg(y=0)$ (we suppose that F_i contains no subformulas having one of the forms, x$=a$, $x=y$, $y=S(x)$, $x=S(y)$). In this case π_i is empty when $b=0$, and the subformula $\neg(y=0)$ is included in F_i. In the opposite case π_i contains all pairs (x,b), where x>0. The statement $(0,b) \in \pi_i$ is true (for $b \neq 0$) if and only if the subformula $\neg(x=0)$ is not included in F_i.

(Case 12). F_i contains the subformulas $x=y$, and, possibly, $\neg(x=0)$, $\neg(y=0)$ (we suppose that F_i contains no subformulas having one of the forms, x= a, $y=b$, $y=S(x)$, $x=S(y)$). In this case π_i contains all the pairs having the form (x,x), where $x>0$. The statement $(0,0) \in \pi_i$ is true if and only if the subformulas $\neg(x=0)$ and $\neg(y=0)$ are not included in F_i.

It is easily seen that all the variants of the structure of π_i are exhausted in the cases 1-12.

Now we will consider the variants of the structure of π^*. As it is proved in [3] (see [3], Lemma 3.4) the statement $(x,y) \in \pi^*$is true if and only if there exists an ETC-sequence $(q_1,q_2,\dots,q_r)$ such that $q_1=x$, $q_r=y$, $(q_i,q_{i+1}) \in \pi$ for $1 \leq i < r$. Without loss of generality we may suppose that any considered ETC-sequence $(q_1,q_2,\dots,q_r)$ where $r \geq 3$, satisfies the condition $q_i \neq q_j$ when $i \neq j$ (otherwise the given ETC-sequence may be replaced by a shorter sequence having the same properties).

Let us consider the number d such that $d=d_1+1$, where d_1 is the maximum of the numbers a and b in the formulas $x=a$ and $y=b$ included in F. If no formula of such forms is included in F, then we admit $d=3$.

We will use below some classification of pairs (x,y) such that $x \in N$, $y \in N$. We say that (x,y) belongs to the subset $S1$ if $x \leq d$, $y \leq d$. In a similar way we define the subsets $S2$, $S3$, $S4$ as sets of pairs (x,y) such that $(x,y) \in S2$ if $x>d$, $y \leq d$; $(x,y) \in S3$ if $x \leq d$, $y>d$; $(x,y) \in S4$ if $x>d$, $y>d$.

The sets $S1^*$, $S2^*$, $S3^*$, $S4^*$ are defined coorrespondingly as $S1 \cap \pi^*$, $S2 \cap \pi^*$, $S3 \cap \pi^*$, $S4 \cap \pi^*$.

A pair (x,y) is said to be increasing if $x<y$ and decreasing if $x>y$.

Lemma 3.1: *If the number h satisfies the condition $h \geq d$, and the pair $(x,y) \in \pi^*$ satisfies the conditions $x \leq h$, $y \leq h$, then there exists an ETC-sequence $(q_1,q_2,\dots,q_r)$ for the value $(x,y) \in \pi^*$ such that $q_1=x$, $q_r=y$, $(q_i,q_{i+1}) \in \pi$ for $1 \leq i < r$, $q_i \leq h$ for $1 \leq i \leq r$.*

Proof: As it follows from the condition $(x,y) \in \pi^*$, there exists an ETC- sequence $(q_1, q_2, \dots, q_r)$ such that $q_1 = x, q_r = y, (q_i, q_{i+1}) \in \pi$ for $1 \le i < r$. If $q_i \le h$ for $1 \le i \le r$, then the statement of Lemma is satisfied. In the opposite case let k be the minimal index in the sequence $(q_1, q_2, \dots, q_r)$ such that $k > 1$, $q_k > h$. Let l be the minimal index such that $l \ge k$, $q_{l+1} \le h$. Clearly, any number q_j, where $k \le j \le l$ satisfies the condition $q_j > h$(note that the case $k = l$ is not excluded). The following statements hold: $q_{k-1} \le h$, $q_{l+1} \le h$, the pair (q_{k-1}, q_k) is increasing, the pair (q_l, q_{l+1}) is decreasing. But any increasing pair $(x,y) \in \pi$ such that $x \ge h$, $y \ge h$, $x < y$ should satisfy the conditions of (Case 3) or (Case 10) mentioned above. Therefore, either $q_{k-1} = h$, $q_k = h + 1$ (Case 3) or $q_{k-1} = a$, where a is the number contained in a formula $x = a$ included in F. Similarly, any decreasing pair $(x,y) \in \pi$ such that $x \ge h$, $y \ge h$, $x > y$, satisfies the conditions of (Case 6) or (Case 11) mentioned above. Therefore either $q_l = h + 1$, $q_{l+1} = h$ (Case 6) or $q_{l+1} = b$, where b is the number contained in a formula $y = b$ included in F (Case 11). Now if $q_{k-1} = q_{l+1} = h$, $q_k = q_l = h + 1$, (Case 3, Case 6) then the segment $(q_{k-1}, q_k, q_{k+1}, \dots, q_l, q_{l+1})$ of the ETC-sequence $(q_1, q_2, \dots, q_r)$ can be replaced by the single number $q_{k-1} = q_{l+1} = h$. If $q_{k-1} = a$, then the mentioned segment can be replaced by the segment (a, q_{l+1}) (Case 10). If $q_{l+1} = b$, then the mentioned segment can be replaced by the segment (q_{k-1}, b) (Case 11). Clearly the sequence obtained by these replacements is an ETC-sequence for the value $(x,y) \in \pi^*$.

Transforming in this way any segment of the sequence $(q_1, q_2, \dots, q_r)$ containing members greater than h, we obtain the ETC-sequence satisfying the conditions of Lemma. This completes the proof.

Corollary 1: *There is only finite number of ETC-sequences obtained by the transformations described in Lemma 3.1. Indeed, the length of such sequence (without repetitions of members) is* $\le h + 1$, *and any member of such sequence is* $\le h$.

Corollary 2: *The set* $S1^*$ *can be defined by arithmetical formula in the signature* $(0, =, <, S)$ *(even in* $(0, =, S)$*). Indeed, applying Corollary 1 to the case when* $h = d$, *we conclude that the set of pairs* (x,y) *such that* $x \le d$, $y \le d$, $(x,y) \in \pi^*$ *is finite, hence, it can be defined by a formula having the form*

$$\big((x = x_1)\&(y = y_1)\big) \vee \big((x = x_2)\&(y = y_2)\big) \vee \dots \vee \big((x = x_m)\&(y = y_m)\big),$$

where all x_i *and* y_i *are constants. If this set is empty, then it can be defined by the formula* $(x = y)\&(x = S(y))$.

Note. If $x \le d$, $y \le d$, then the statement $(x,y) \in \pi^*$ may be tested constructively. The method of testing is actually given in Corollary 1.

Lemma 3.2: *If some pair* (x_0, y_0), *where* $x_0 > d$, $y_0 \le d$ *belongs to* π^*, *then any pair* (x, y_0), *where* $x > d$, *belongs to* π^*.

Proof: If some pair (x_0, y_0) satisfies the conditions of Lemma, then there exists an ETC-sequence $(q_1, q_2, \dots, q_r)$ such that $q_1 = x_0$, $q_r = y_0$, $(q_i, q_{i+1}) \in \pi$ for $1 \le i < r$. Let k be the minimal index in the sequence $(q_1, q_2, \dots, q_r)$ such that $k < r$, $q_{k+1} \le d$ (the case $k = 1$ is not excluded). Without loss of generality we may suprose that $q_i \le d$ for $k + 1 \le i \le r$ (indeed, in the opposite case the segment $(q_{k+1}, q_{k+2}, \dots, q_r)$ can be transformed by the method described in the proof of Lemma 3.1).

The pair (q_k, q_{k+1}) is decreasing, hence, either $q_{k+1} = b$, where b is the number in a formula $y = b$ included in F, (Case 11 considered above), or $q_{k+1} = q_k - 1$ (Case 6 considered above). Now the ETC-sequence for the pair (x, y_0) where $x > d$ is obtained from the sequence $(q_1, q_2, \dots, q_r)$ as follows: if $q_{k+1} = b$, then the segment $(q_1, q_2, \dots, q_{k+1})$ is replaced by the segment (x, q_{k+1}) (see Case 11); if $q_k = q_{k+1} + 1$, then any pair $(x + 1, x)$, where $x > 0$, belongs to π (see Case 6) hence, we can obtain the required ETC-sequence replacing in the sequence $(q_1, q_2, \dots, q_r)$ the segment $(q_1, q_2, \dots, q_{k+1})$ by the segment $(x, x - 1, \dots, q_{k+1} + 1, q_{k+1})$. It is easily seen that the sequence obtained by the mentioned replacements is an ETC-sequence for the value $(x, y_0) \in \pi^*$. This complets the proof.

Corollary: The set $S2^*$ can be defined by an arithmetical formula in the signature $(0, =, <, S)$. Indeed, applying Lemma 3.1 and its corollaries to the case when $h = d + 1$, we obtain the complete list of pairs $(x, y) \in \pi^*$ such that $x \le d + 1$, $y \le d + 1$. In particular we obtain the complete list of pairs having the property $(d + 1, y_0) \in S2^*$, where $y_0 \le d$. Using Lemma 3.2 we conclude that any pair $(d + 1, y_0) \in S2^*$ where $y_0 \le d$ generates the set $\{(x, y)/(x > d)\&(y = y_0)\}$, contained in $S2^*$, so the set $S2^*$ is the union of sets having this form for all $y_0 \le d$ such that $(d + 1, y_0) \in S2^*$. But any set $\{(x, y)/(x > d)\&(y = y_0)\}$ is defined by the formula $(x > d)\&(y = y_0)$, so the set $S2^*$ is defined by the disjunction of these formulas. This completes the proof.

Lemma 3.3: *If some pair (x_0, y_0), where $x_0 \le d$, $y_0 > d$, belongs to π^*, then any pair (x_0, y), where $y > d$ belongs to π^*.*

The proof is similar to that of Lemma 3.2. We use the ETC-sequence $(q_1, q_2, \dots, q_r)$ such that $q_1 = x_0$, $q_r = y_0$, $(q_i, q_{i+1}) \in \pi$ for $1 \le i < r$. Let l be the maximal index in the sequence $(q_1, q_2, \dots, q_r)$ such that $q_l \le d$. Without loss of generality we may suppose that $q_i \le d$ when $1 \le i \le l$ (otherwise the segment $(q_1, q_2, \dots, q_l)$ may be transformed by the method used in the proof of Lemma 3.1).

The pair (q_l, q_{l+1}) is increasing, therefore either $q_l = a$, where a is the number in a formula $x = a$ included in F (Case 10) or $q_{l+1} = q_l + 1$ (Case 3). The ETC-sequence for establishing the statement $(x_0, y) \in \pi^*$ (where $y > d$) is obtained from the sequence $(q_1, q_2, \dots, q_r)$ by the following replasements: either the segment $(q_l, q_{l+1}, \dots, q_r)$ is replaced by the segment (a, y) (Case 10), or this segment is replased by the segment $(q_l, q_l + 1, \dots, y - 1, y)$ (Case 3). This completes the proof.

Corollary: *The set $S3^*$ can be defined by an arithmetical formula in the signature* $(0, =, <, S)$.

The proof is similar to the proof of corollary of Lemma 3.2.

In what follows we will say that a formula having the form $y = S(x)$, $x = S(y)$, or $x = y$ is **contained in a special way** in some F_i if this formula is contained in F_i and the conditions described correspondingly in (Case 3), (Case 6) or (Case 12) mentioned above are satisfied.

Lemma 3.4: *If the formula $y = S(x)$ is contained in a special way in some F_i then any pair (x, y) such that $x < y$, $x > 0$, belongs to π^*.*

Indeed, for establishing this statement it is sufficient to consider the ETC-sequence $(x, x + 1, \dots, y - 1, y)$.

Lemma 3.5: *If the formula $x = S(y)$ is contained in a special way in some F_i, then any pair (x, y) such that $x > y$, $y > 0$ belongs to π^*.*

For establishing this statement it is sufficient to consider the ETC-sequence $(x, x-1, \dots, y-1, y)$.

Lemma 3.6: *If the formula $y = S(x)$ is contained in a special way in some F_i, and the formula $x = S(y)$ is contained in a special way in some F_j, where $i \neq j$, then any pair (x, y), where $x > 0, y > 0$ belongs to π^*.*

For establishing this statement it is sufficient to consider the ETC-sequence $(x, x+1, \dots, z, \dots, y-1, y)$, where $z = \max(x, y) + 1$.

Lemma 3.7: *The set $S4^*$ can be defined by an arithmetical formula in the signature* $(0, =, <, S)$.

Proof: If the set $S4^*$is empty, then it is defined, for example, by the formula $(x = y)\&(y = S(x))$. Otherwise, there exists a pair $(x_0, y_0) \in S4^*$, that is $x_0 > d$, $y_0 > d$, $(x_0, y_0) \in \pi^*$. Hence, there exists an ETC-sequence $(q_1, q_2, \dots, q_r)$ such that $q_1 = x_0$, $q_r = y_0$, $(q_i, q_{i+1}) \in \pi$ for $1 \le i < r$. We will distinguish two cases:

(α) There exists such i that $1 \le i \le r$, $q_i \le d$.

(β) $q_i > d$ for any i, where $1 \le i \le r$.

Let us consider the case (α).We denote the number q_i such that $q_i \le d$ by z. The pair (x_0, z), where $x_0 > d$ belongs to π^*, therefore, using Lemma 3.2 we conclude that any pair (x, z), where $x > d$, belongs to π^*. The pair (z, y_0), where $y_0 > d$, belongs to π^*, therefore, using Lemma 3.3 we conclude that any pair (z, y), where $y > d$ belongs to π^*. Hence, any pair (x, y), where $x > d$, $y > d$, belongs to π^*.

So in the case (α) the set $S4^*$ is defined by the formula $(x > d)\&(y > d)$.

Let us note that similar conclusion concerning the set $S4^*$ can be made if there exists any ETC-sequence $(q_1, q_2, \dots, q_r)$ such that $q_1 > d$, $q_r > d$ and $q_i \le d$ for some i, $1 < i < r$.

Now let us consider the case (β). We will investigate the properties of all ETC-sequences $(q_1, q_2, \dots, q_r)$ such that $q_i > d$ for $1 \le i \le r$, and $(q_i, q_{i+1}) \in \pi$ for $1 \le i < r$. We distingmish the following subcases: (β_1), (β_2), (β_3), (β_4).

(β_1) In some ETC-sequence $(q_1, q_2, \dots, q_r)$ of the mentioned kind there exists an index i such that $1 \le i < r$, $q_{i+1} = q_i + 1$, but there is no ETC-sequence of the mentioned kind containing an index j such that $q_{j+1} = q_j - 1$.

(β_2) In some ETC-sequence $(q_1, q_2, \dots, q_r)$ of the mentioned kind there exists an index i such that $1 \le i < r$, $q_{i+1} = q_i - 1$, but there is no ETC-sequence of the mentioned kind containing an index j such that $q_{j+1} = q_j + 1$.

(β_3) In some ETC-sequence $(q_1, q_2, \dots, q_r)$ of the mentioned kind there exists an index i such that $1 \le i < r$, $q_{i+1} = q_i + 1$; besides, in some ETC-sequence $(q_1, q_2, \dots, q_t)$ of the mentioned kind there exists an index j such that $1 \le j < t$, $q_{j+1} = q_j - 1$.

(β_4) There is no ETC-sequence of the mentioned kind satisfying the conditions described in the subcases (β_1)-(β_3).

Clearly, the subcase (β_1) takes place if some F_i in the structure of F has the form $(y = S(x))$, but there is no F_j having the form $(x = S(y))$. Similarly, the subcase (β_2) takes place if some F_i in the structure of F has the form $(x = S(y))$, but there is no F_j having the form $(y = S(x))$. The subcase (β_3) takes place if some F_i and F_j in the structure of F have the forms, correspondingly

$(y = S(x))$ and $(x = S(y))$. The subcase (β_4) takes place if the formula F contains no F_i having one of the mentioned forms.

It is easily seen that in the subcase (β_1) the set $S4^*$ is defined by the formula $(x > d)\&(y > d)\&(x < y)$ or by the formula $(x > d)\&(y > d)\&(x \leq y)$ (see Lemma 3.4). In the subcase (β_2) the set $S4^*$ is defined by the formula $(x > d)\&(y > d)\&(x > y)$ or by the formula $(x > d)\&(y > d)\&(x \geq y)$ (see Lemma 3.5). Let us note that the inequalities $x \leq y$ and $x \geq y$ are obtained in the subcases (β_1) and (β_2) if some F_i in the structure of F has the form $x = y$ (see Case 12 mentioned above). In the subcase (β_3) the set $S4^*$ is defined by the formula $(x > d)\&(y > d)$ (see Lemma 3.6). In the subcase (β_4) the set $S4^*$ is either empty or is defined by the formula $(x > d)\&(y > d)\&(x = y)$. This completes the proof.

Proof of Theorem 1.

As it is established in Lemmas 3.1-3.7, the sets $S1^*$, $S2^*$, $S3^*$, $S4^*$, are defined by formulas in the signature $\{0, =, <, S\}$. Hence, the set $\pi^* = S1^* \cup S2^* \cup S3^* \cup S4^*$ is defined by the disjunction of the mentioned formulas. This completes the proof.

Proof of Theorem 2.

Let A be the set $\{(x, y)/y = S(S(x))\}$, let B be any 2-dimensional strongly positive set, let B^* be the transitive closure of B. We define the number d for the set B by the method given above. By D we denote the set $\{(x, y)/(x > d)\&(y > d)\}$. Using Lemma 3.7 we conclude that the set $B^* \cap D$ either is empty or is defined by one of the following formulas: $(x > d)\&(y > d)\&(x > y)$, $(x > d)\&(y > d)\&(x < y)$, $(x > d)\&(y > d)\&(x = y)$, or by the disjunction of some of these formulas. Similarly, using Lemmas 3.4-3.6 we conclude that the set $B \cap D$ either is empty or is defined by one of the following formulas: $(x > d)\&(y > d)\&(y = S(x))$, $(x > d)\&(y > d)\&(x = S(y))$, $(x > d)\&(y > d)\&(x = y)$, or by the disjuction of some of these formulas. Therefore, in all the cases the set $A \cap D$ is different form $B \cap D$ and $B^* \cap D$. Hence, $A \neq B$ and $A \neq B^*$. This completes the proof.

Note 1. The statement of Theorem 2 is true also for any set defined by the formula $y = S(S \dots S(x) \dots)$, where the symbol S is repeated $n \geq 2$ times. The proof is similar to that of Theorem 2.

Note 2. Obviously, any set defined by a formula in the signature $(0, =, <, S)$ is primitive recursive, however, the reverse is not true (for example, the set of even numbers is primitive recurcive, but it cannot be defined by arithmetical formula in the signature $(0, =, <, S)$ (see [6])). So the statement of Theorem 1 is stronger than the statement of Theorem 2 in [3].

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Խիստ պոզիտիվ երկչափ թվաբանական բազմությունների տրանզիտիվ փակումների մասին

Ս. Մանուկյան

Ամփոփում

Պոզիտիվ և խիստ պոզիտիվ թվաբանական բազմությունների գաղափարները սահմանված են [1]-[3] հոդվածներում: [3] հոդվածում նշված է, որ ցանկացած երկչափ խիստ պոզիտիվ բազմության տրանզիտիվ փակումը պարզագույն անդրադարձ է: Այս հոդվածում ապացուցվում է ավելի ուժեղ պնդում, այսինքն՝ ցանկացած երկչափ խիստ պոզիտիվ բազմության տրանզիտիվ փակումը նկարագրվում է թվաբանական բանաձևի միջոցով $(0,=,<,S)$ սիգնատուրայում (որտեղ $S(x) = x + 1$): Բացի դրանից ապացուցվում է, որ երկչափ խիստ պոզիտիվ բազմությունների դասը և այդ բազմությունների տրանզիտիվ փակումների դասը չեն համընկնում $(0,=,<,S)$ սիգնատուրայում արտահայտվող թվաբանական բազմությունների դասի հետ:

О транзитивных замыканиях строго позитивных арифметических множеств размерности 2

С. Манукян

Аннотация

Понятия позитивного и строго позитивного множества рассматриваются в [1]-[3]. В [3] указано, что транзитивное замыкание всякого строго позитивного множества размерности 2 примитивно рекурсивно. В этой статье доказывается более сильное утверждение: транзитивное замыкание всякого строго позитивного множества размерности 2 задается арифметической формулой в сигнатуре $(0, =, <, S)$, где $S(x) = x + 1$. Доказывается также, что класс строго позитивных множеств размерности 2 и класс транзитивных замыканий таких множеств не совпадают с классом арифметических множеств размерности 2, задаваемых посредством арифметических формул в сигнатуре $(0, =, <, S)$.