Mathematical Problems of Computer Science 40, 34–38, 2013.

On Neyman-Pearson Principle in Multiple Hypotheses Testing

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Abstract

The aim of this paper is to newly generalize the classical Neyman-Pearson Lemma to the case of more than two simple hypotheses.

Keywords: Multiple hypotheses, Optimal statistical test, Error probability, Neyman-Pearson Lemma.

1. Introduction

The Neyman-Pearson lemma is the foundation of the mathematical theory of statistical hypothesis testing.

We call the statistical hypothesis each supposition statement which must be verified concerning the probability distribution of an observable random object. The task of statistician is to construct an algorithm (test) for effective detection of the hypothesis which is realized. The decision must be made on the base of vector of results of N independent identically distributed experiments, called a sample and denoted by $\mathbf{x} \triangleq (x_1, ..., x_n, ..., x_N)$, the elements of $\mathcal{X}^N$, where $\mathcal{X}$ is the space of possible results of each experiment.

The principle of Neyman-Pearson plays a central role in both the theory and practice of statistics.

There exists a vast literature where the theory of the hypothesis testing and the Neyman-Pearson lemma are expounded in detail [1]–[10]. The paradigm of Neyman-Pearson is frequently used in different applications [11]–[13]. But the most part of these texts is dedicated to the case of two hypotheses.

The possibility of extension of Fundamental Lemma to the case of multiple hypotheses is mentioned in [3]. Since the testing of hypothesis is actual in applications we present our version of the Lemma for the case of three, or more hypotheses.

The idea of this study was formulated in [4].

2. Problem Statement and Result Formulation

Let $\mathcal{P}(\mathcal{X})$ be the space of all probability distributions (PDs) on $\mathcal{X}$. Let X be RV taking values in $\mathcal{X}$ with one of M continuous PDs $G_m \in \mathcal{P}(\mathcal{X})$, $m = \overline{1, M}$. Let the sample $\mathbf{x} =$

$(x_1, ..., x_n, ..., x_N)$, $x_n \in \mathcal{X}$, $n = \overline{1, N}$, be a vector of results of N independent observations of X.

Based on data sample a statistician makes a decision which of the proposed hypotheses $H_m : G = G_m$, $m = \overline{1, M}$, is correct.

The procedure of decision making is a non-randomized test $\varphi_N(\mathbf{x})$, it can be defined by division of the sample space $\mathcal{X}^N$ on M disjoint subsets $\mathcal{A}_m$, $m = \overline{1, M}$. The set $\mathcal{A}_m$, $m = \overline{1, M}$, consists of vectors $\mathbf{x}$ for which the hypothesis H_m is adopted.

We study the probabilities of the erroneous acceptance of hypothesis H_l provided that H_m is true

$$\alpha_{l|m}(\varphi_N) \triangleq G_m^N(\mathcal{A}_l) = \sum_{\mathbf{x}:\ \mathbf{x} \in \mathcal{A}_l} G_m^N \mathbf{x}), \quad m, l = \overline{1, M}, \ \ m \neq l.$$

If the hypothesis H_m is true, but it is not accepted then the probability of error is the following:

$$\alpha_{m|m}(\varphi_N) \triangleq \sum_{l: l \neq m} \alpha_{l|m}(\varphi_N) = 1 - G_m^N(\mathcal{A}_m), \quad m = \overline{1, M}.$$

For the given preassigned values $0 < \alpha^*_{1|1}, \alpha^*_{2|2}, ..., \alpha^*_{M-1|M-1} < 1$ we choose numbers T_1, T_2, ..., T_{M-1} and sets $\mathcal{A}_m$, $m = \overline{1, M}$, such that

$$\mathcal{A}_1^* = \left\{ \mathbf{x} : \min \left(\frac{G_1(\mathbf{x})}{G_2(\mathbf{x})}, ..., \frac{G_1(\mathbf{x})}{G_M(\mathbf{x})} \right) > T_1 \right\}, \ \ 1 - G_1^N(\mathcal{A}_1^*) = \alpha^*_{1|1},$$

$$\mathcal{A}_2^* = \overline{\mathcal{A}_1^*} \cap \left\{ \mathbf{x} : \min \left(\frac{G_2(\mathbf{x})}{G_3(\mathbf{x})}, ..., \frac{G_2(\mathbf{x})}{G_M(\mathbf{x})} \right) > T_2 \right\}, \ \ 1 - G_2^N(\mathcal{A}_2^*) = \alpha^*_{2|2},$$

....

$$\mathcal{A}_{M-1}^* = \overline{\mathcal{A}_1^*} \cap \overline{\mathcal{A}_2^*} \cap ... \cap \overline{\mathcal{A}_{M-2}^*} \cap \left\{ \mathbf{x} : \frac{G_{M-1}(\mathbf{x})}{G_M(\mathbf{x})} > T_{M-1} \right\}, \ \ 1 - G_{M-1}^N(\mathcal{A}_{M-1}^*) = \alpha^*_{M-1|M-1},$$

and

$$\mathcal{A}_M^* = \mathcal{X}^N - (\mathcal{A}_1^* \cup \mathcal{A}_2^* \cup ... \cup \mathcal{A}_{M-1}^*) = \overline{\mathcal{A}_1^*} \cap \overline{\mathcal{A}_2^*} \cap ... \cap \overline{\mathcal{A}_{M-1}^*}.$$

The corresponding error probabilities are denoted by

$$\alpha^*_{l|m}(\varphi_N), \ \ m, l = \overline{1, M-1}.$$

Theorem: *The test determined by the sets* $\mathcal{A}_1^*$, $\mathcal{A}_2^*$, *.....,* $\mathcal{A}_M^*$ *is optimal in the sense that, for each other test defined by the set* $\mathcal{B}_1$, $\mathcal{B}_2$, *....,* $\mathcal{B}_M$ *with the corresponding error probabilities* $\beta_{l|m}$, $m, l = \overline{1, M}$,

$$\textit{if} \ \ \beta_{1|1} \leq \alpha^*_{1|1}, \quad \textit{then} \ \ \max(\beta_{1|2}, \beta_{1|3}, ..., \beta_{1|M}) \geq \max(\alpha^*_{1|2}, \alpha^*_{1|3}, ..., \alpha^*_{1|M}),$$

$$\textit{if} \ \ \beta_{2|2} \leq \alpha^*_{2|2}, \quad \textit{then} \ \ \max(\beta_{2|3}, \beta_{2|4}, ..., \beta_{2|M}) \geq \max(\alpha^*_{2|3}, \alpha^*_{2|4}, ..., \alpha^*_{2|M}),$$

....

$$\textit{and if} \ \ \beta_{M-1|M-1} \leq \alpha^*_{M-1|M-1}, \quad \textit{then} \ \ \beta_{M-1|M} \geq \alpha^*_{M-1|M}.$$

For simplicity of formulations we present the proof of the Theorem for $M = 3$.

In that case for the given values $0 < \alpha^*_{1|1}, \alpha^*_{2|2} < 1$ and chosen numbers T_1 and T_2 sets $\mathcal{A}^*_m$, $m = \overline{1,3}$, are the following:

$$\mathcal{A}^*_1 = \left\{\mathbf{x} : \min\left(\frac{G_1(\mathbf{x})}{G_2(\mathbf{x})}, \ \frac{G_1(\mathbf{x})}{G_3(\mathbf{x})}\right) > T_1\right\}, \ 1 - G_1^N(\mathcal{A}^*_1) = \alpha^*_{1|1},$$

$$\mathcal{A}^*_2 = \overline{\mathcal{A}^*_1} \cap \left\{\mathbf{x} : \frac{G_2(\mathbf{x})}{G_3(\mathbf{x})} > T_2\right\}, \ 1 - G_2^N(\mathcal{A}^*_2) = \alpha^*_{2|2},$$

and

$$\mathcal{A}^*_3 = \mathcal{X}^N - (\mathcal{A}^*_1 \cup \mathcal{A}^*_2).$$

The corresponding error probabilities are denoted by

$$\alpha^*_{l|m}(\varphi_N), \ m, l = \overline{1,3}.$$

We must prove that the test determined by the sets $\mathcal{A}^*_1$, $\mathcal{A}^*_2$ and $\mathcal{A}^*_3$ is optimal in the sense that, for each other test defined by the sets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ with the corresponding error probabilities $\beta_{l|m}$, $m, l = \overline{1,3}$, if $\beta_{1|1} \leq \alpha^*_{1|1}$, then $\max(\beta_{1|2}, \beta_{1|3}) \geq \max(\alpha^*_{1|2}, \alpha^*_{1|3})$, and if $\beta_{2|2} \leq \alpha^*_{2|2}$ then $\beta_{2|3} \geq \alpha^*_{2|3}$.

Proof: Let $\Phi_{\mathcal{A}^*_m}$ and $\Phi_{\mathcal{B}_m}$ be the indicator functions of the decision regions $\mathcal{A}^*_m$ and $\mathcal{B}_m$. For all $\mathbf{x} = (x_1, x_2, ..., x_N) \in \mathcal{X}^N$, the following inequality is correct

$$(\Phi_{\mathcal{A}^*_1}(\mathbf{x}) - \Phi_{\mathcal{B}_1}(\mathbf{x}))\,(G_1(\mathbf{x}) - \max(T_1 G_2(\mathbf{x}), T_1 G_3(\mathbf{x}))) \geq 0.$$

Multiplying and then summing over $\mathcal{X}^N$ we obtain

$$\Phi_{\mathcal{A}^*_1}(\mathbf{x}) G_1(\mathbf{x}) - \Phi_{\mathcal{A}^*_1}(\mathbf{x}) \max(T_1 G_2(\mathbf{x}), T_1 G_3(\mathbf{x}))$$

$$-\Phi_{\mathcal{B}_1}(\mathbf{x}) G_1(\mathbf{x}) + \Phi_{\mathcal{B}_1}(\mathbf{x}) \max(T_1 G_2(\mathbf{x}), T_1 G_3(\mathbf{x})) \geq 0,$$

$$\sum_{\mathbf{x}:\ \mathbf{x} \in \mathcal{X}^N} \Big[\Phi_{\mathcal{A}^*_1}(\mathbf{x}) G_1(\mathbf{x}) - \Phi_{\mathcal{A}^*_1}(\mathbf{x}) \max(T_1 G_2(\mathbf{x}), T_1 G_3(\mathbf{x}))$$

$$-\Phi_{\mathcal{B}_1}(\mathbf{x}) G_1(\mathbf{x}) + \Phi_{\mathcal{B}_1}(\mathbf{x}) \max(T G_2(\mathbf{x}), T_1 G_3(\mathbf{x}))] \geq 0,$$

$$\sum_{\mathbf{x}:\ \mathbf{x} \in \mathcal{A}^*_1} [G_1(\mathbf{x}) - T_1 \max(G_2(\mathbf{x}), G_3(\mathbf{x}))] - \sum_{\mathbf{x}:\ \mathbf{x} \in \mathcal{B}_1} [G_1(\mathbf{x}) - T_1 \max(G_2(\mathbf{x}), G_3(\mathbf{x}))] \geq 0,$$

$$1 - \alpha^*_{1|1} - T_1 \max(\alpha^*_{1|2}, \alpha^*_{1|3}) - (1 - \beta_{1|1}) + T_1 \max(\beta_{1|2}, \beta_{1|3}) \geq 0,$$

$$-\beta_{1|1} + \alpha^*_{1|1} \leq T_1[-\max(\alpha^*_{1|2}, \alpha^*_{1|3}) + \max(\beta_{1|2}, \beta_{1|3})].$$

We see now that from $\beta_{1|1} \leq \alpha^*_{1|1}$, it follows that $\max(\beta_{1|2}, \beta_{1|3}) \geq \max(\alpha^*_{1|2}, \alpha^*_{1|3})$.

The proof of the other case is similar. The following inequality takes place for all $\mathbf{x} \in \mathcal{X}^N$

$$(\Phi_{\mathcal{A}^*_2}(\mathbf{x}) - \Phi_{\mathcal{B}_2}(\mathbf{x}))\,(G_2(\mathbf{x}) - T_2 G_3(\mathbf{x})) \geq 0.$$

Multiplying and after that summing over $\mathcal{X}^N$ we get

$$\Phi_{\mathcal{A}^*_2}(\mathbf{x}) G_2(\mathbf{x}) - \Phi_{\mathcal{A}^*_2}(\mathbf{x}) T_2 G_3(\mathbf{x}) - \Phi_{\mathcal{B}_2}(\mathbf{x}) G_2(\mathbf{x}) + \Phi_{\mathcal{B}_2}(\mathbf{x}) T_2 G_3(\mathbf{x}) \geq 0,$$

$$\sum_{\mathbf{x}:\ \mathbf{x} \in \mathcal{X}^N} \Big[\Phi_{\mathcal{A}^*_2}(\mathbf{x}) G_2(\mathbf{x}) - \Phi_{\mathcal{A}^*_2}(\mathbf{x}) T_2 G_3(\mathbf{x})) - \Phi_{\mathcal{B}_2}(\mathbf{x}) G_2(\mathbf{x}) + \Phi_{\mathcal{B}_2}(\mathbf{x}) T_2 G_3(\mathbf{x})\Big] \geq 0,$$

$$\sum_{\mathbf{x}:\ \mathbf{x}\in\mathcal{A}_2^*} [G_2(\mathbf{x}) - T_2 G_3(\mathbf{x})] - \sum_{\mathbf{x}:\ \mathbf{x}\in\mathcal{B}_2} [G_2(\mathbf{x}) - T_2 G_3(\mathbf{x})] \geq 0,$$

$$1 - \alpha^*_{2|2} - T_2\alpha^*_{2|3} - (1 - \beta_{2|2}) + T_2\beta_{2|3} \geq 0,$$

$$-\beta_{2|2} + \alpha^*_{2|2} \leq T_2(\beta^*_{2|3} - \alpha^*_{2|3}).$$

It is clear that if $\beta_{2|2} \leq \alpha^*_{2|2}$, then $\beta_{2|3} \geq \alpha^*_{2|3}$.

The theorem is proved.

3. Conclusion

In this paper we generalized Neyman-Pearson criterion of optimality for many continuous hypotheses. When distributions of X are discrete the Lemma can be reformulated with use of randomization as it is noted in [3], [4] and [7].

Bayesian testing was considered for the case of two and more hypotheses in [4], [5]. It is desirable have intention to consider multyhypotheses Bayesian testing for the model consisting of many objects.

Acknowledgement

This work was supported in part by SCS of MES of RA under Thematic Program No SCS 13–1A295.

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Submitted 22.08.2013, accepted 10.10.2013.

Նեյման-Պիրսոնի սկզբունքը բազմակի վարկածների տեստավորման վերաբերյալ

Ե. Հարությունյան և Փ. Հակոբյան

Ամփոփում

Այս աշխատանքում ներկայացված է Նեյման-Պիրսոնի դասական լեմմայի ընդհանրացումը երկուսից ավելի վարկածների վերաբերյալ:

О принципе Неймана-Пирсона при проверке многих гипотез

Е. Арутюнян и П. Акопян

Аннотация

Цель настоящей статьи представить новое обобщение классической Леммы Неймана-Пирсона для более чем двух гипотез.