Mathematical Problems of Computer Science 38, 46–48, 2012.

On Long Cycles in Digraphs with the Meyniel-type Conditions

S.Kh. Darbinyan, I.A. Karapetyan

Institute for Informatics and Automation Problems,
Armenian National Academy of Sciences
E-mails: samdarbin@ipia.sci.am, isko@ipia.sci.am

We shall assume that the reader is familiar with the standard terminology on directed graphs (digraphs) and use Bang-Jensen and Gutin [1] as reference for undefined terms. In this paper we consider finite digraphs without loops and multiple arcs. The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A\rangle$. We will denote the complete bipartite digraph with partite sets of cardinalities p, q by $K^*_{p,q}$.

Meyniel [11] proved the following theorem: If D is a strong digraph on $n \geq 2$ vertices and $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D, then D is hamiltonian (see also [1], [5] and [12]).

Thomassen [14] (for $n = 2k+1$) and Darbinyan [7] (for $n = 2k$) proved: If D is a digraph on $n \geq 5$ vertices with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$, then D is hamiltonian (unless some extremal cases).

In each above mentioned theorems (as well as, in well know theorems Ghouila-Houri [10], Woodall [15]) imposes a degree condition on all pairs of non-adjacent vertices (on all vertices). Bang-Jensen, Gutin, Li, Guo and Yeo [2, 3] obtained sufficient conditions for hamiltonisity of digraphs in which degree conditions requiring only for some pairs of non-adjacent vertices. Namely, they proved the following theorems (in all three theorems D is a strong digraph on $n \geq 2$ vertices).

Theorem A [2]. If $min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices x, y with a common in-neighbour, then D is hamiltonian.

Theorem B [2]. If $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for every pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is hamiltonian.

Theorem C [3]. If $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is hamiltonian.

Note that Theorem C generalizes Theorem B. In [9, 13, 6, 8] it was shown that if the strong digraph D satisfies the condition of the theorem of Ghouila-Houri [10] (Woodall [15], Meyniel [11], Thomassen and Darbinyan [14, 7]), then D is pancyclic (unless some extremal cases, which are characterized). It is not difficult to check that the digraphs $K^*_{n/2,n/2}$ and $K^*_{n/2,n/2} - \{e\}$, where n is even and e is an arc of $K^*_{n/2,n/2}$, satisfy the conditions of Theorem A (B, C) and has no cycle of odd length. Moreover, if in Theorems A (B, C) the digraph D has no pair of non-adjacent vertices with a common in-neighbour and a common out-neighbour, then D is a locally semicomplete digraph, and in [4], Bang-Jensen, Gutin and Volkmann characterize those strong locally semicomplete digraphs which are not pancyclic.

It is natural to set the following problem:

Problem. Characterize those digraphs which satisfy the conditions of Theorem A (B, C), but are not pancyclic.

To investigate that a given digraph D is pancyclic, in [9, 13, 6, 8] it was proved the existence of cycles of length $|V(D)|-1$ and $|V(D)|-2$, and then using the constructions of these cycles it was proved that D is pancyclic with some exceptions.

We prove three results which provide some support for the above Problem.

Theorem 1. Let D be a strong digraph on n vertices with minimum semi-degree at least two. If D satisfies the conditions of Theorem A, then either D contains a cycle of length $n-1$ or n is even and D is isomorphic to complete bipartite digraph $K^*_{n/2,n/2}$ or $K^*_{n/2,n/2}-\{e\}$, where e is an arc of $K^*_{n/2,n/2}$.

Theorem 2. Let D be a strong digraph on $n \geq 4$ vertices, which is not directed cycle of length n. If D satisfies the conditions of Theorem B, then either D contains a cycle of length $n-1$ or n is even and D isomorphic to complete bipartite digraph $K^*_{n/2,n/2}$.

Note that Theorem 1 is sharp, in the sense that for all $n \geq 6$ there is a strong digraph D on n vertices which has minimum semi-degree one and satisfies the condition of Theorem 1, but contain no cycle of length $n-1$. To see this, it is sufficient to consider the digraph $D_{n,m}$ which was defined in [13] (see also [1].p.300). When $m = n-1$, then $D_{n,m}$ has minimum semi-degree one and satisfies the conditions of Theorem 1 but has no cycle of length $n-1$.

We believe Theorem 2 can be generalized to the following

Conjecture. Let D be a strong digraph on $n \geq 4$ vertices. If D satisfies the conditions of Theorem C, then D contains a cycle of length $n-1$ maybe except some digraphs which has a "simple" characterization.

Support for the conjecture we prove the following.

Theorem 3. Let D be a strong digraph with $n \geq 2$ vertices, which is not directed cycle. If D satisfies the conditions of Theorem C, then D contains a cycle of length $n-2$ or $n-1$.

References

[1] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, 2000.

[2] J. Bang-Jensen, G. Gutin, H. Li, Sufficient conditions for a digraph to be hamiltonian, J. Graph Theory 22 (2) (1996) 181-187.

[3] J. Bang-Jensen, Y. Guo, A.Yeo, A new sufficient condition for a digraph to be hamiltonian, Discrete Applied Math., 95 (1999) 77-87.

[4] J. Bang-Jensen, Y. Guo, L. Volkmann, A classification of locally semicomplete digraphs. 15th British Combinatorial Conference (Stirling, 1995). Discrete Math.. 167/168 (1997) 101-114.

[5] J.A. Bondy, C. Thomassen, A short proof of Meyniel's theorem, Discrete Math. 19 (1977) 195-197.

[6] S.Kh. Darbinyan, Pancyclic and panconnected digraphs, Ph. D. Thesis, Institute Mathematici Akad. Nauk BSSR, Minsk, 1981 (see also Pancyclicity of digraphs with the Meyniel condition, Studia Sci. Math. Hungar., 20 (1-4) (1985) 95-117, in Russian).

[7] S.Kh. Darbinyan, A sufficient condition for the Hamiltonian property of digraphs with large semidegrees, Akad. Nauk Armyan. SSR Dokl. 82 (1) (1986) 6-8 (see also arXiv: 1111.1843v1 [math.CO] 8 Nov 2011).

[8] S.Kh. Darbinyan, On the pancyclicity of digraphs with large semidegrees, Akad. Nauk Armyan. SSR Dokl. 83 (3) (1986) 99-101 (see also arXiv: 1111.1841v1 [math.CO] 8 Nov 2011).

[9] R. Häggkvist, C. Thomassen, On pancyclic digraphs, J. Combin. Theory Ser. B 20 (1976) 20-40.

[10] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit hamiltonien, C. R. Acad. Sci. Paris Ser. A-B 251 (1960) 495-497.

[11] M. Meyniel, Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe oriente, J. Combin. Theory Ser. B 14 (1973) 137-147.

[12] M. Overbeck-Larisch, Hamiltonian paths in oriented graphs, J. Combin. Theory Ser. B 21 (1) (1976) 76-80.

[13] C. Thomassen, An Ore-type condition implying a digraph to be pancyclic, Discrete Math. 19 (1) (1977) 85-92.

[14] C. Thomassen, Long cycles in digraphs, Proc. London Math. Soc. (3) 42 (1981) 231-251.

[15] D.R. Woodall, Sufficient conditions for circuits in graphs, Proc. London Math. Soc. 24 (1972) 739-755.