Mathematical Problems of Computer Science 49, 26–34, 2018.

On a Problem of Wang Concerning the Hamiltonicity of Bipartite Digraphs

Samvel Kh. Darbinyan and Iskandar A. Karapetyan

Institute for Informatics and Automation Problems of NAS RA
e-mail: amdarbin@ipia.sci.am, isko@ipia.sci.am

Abstract

R. Wang (Discrete Mathematics and Theoretical Computer Science, vol. 19(3), 2017) proposed the following problem.

Problem. Let D be a strongly connected balanced bipartite directed graph of order $2a \geq 8$. Suppose that $d(x) \geq 2a-k$, $d(y) \geq a+k$ or $d(y) \geq 2a-k$, $d(x) \geq a+k$ for every pair of vertices $\{x, y\}$ with a common out-neighbour, where $2 \leq k \leq a/2$. Is D Hamiltonian?

In this paper, we prove that if a digraph D satisfies the conditions of this problem, then

(i) D contains a cycle factor,

(ii) for every vertex $x \in V(D)$ there exists a vertex $y \in V(D)$ such that x and y have a common out-neighbour.

Keywords: Digraph, cycle, Hamiltonian cycle, Bipartite balanced digraph, Perfect matching.

1. Introduction

In this paper, we consider finite directed graphs (digraphs) without loops and multiple arcs. A digraph D is called Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle that includes every vertex of D. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The order of a digraph D is the number of its vertices. A cycle factor in D is a collection of vertex-disjoint cycles $C_1, C_2, \ldots, C_l$ such that $V(C_1) \cup V(C_2) \cup \ldots \cup V(C_l) = V(D)$. A digraph D is bipartite if there exists a partition X, Y of $V(D)$ into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$.

There are a number of conditions that guarantee that a bipartite digraph is Hamiltonian (see, e.g., [1]-[11]). Let us recall the following degree conditions that guarantee that a balanced bipartite digraph is Hamiltonian.

Theorem 1.1. (Adamus, Adamus and Yeo [8]) *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. Then D is Hamiltonian provided one of the following holds:*

(a) $d(u) + d(v) \geq 3a + 1$ for every pair of non-adjacent distinct vertices u and v of D;

(b) D is strongly connected and $d(u) + d(v) \geq 3a$ for every pair of non-adjacent distinct vertices u and v of D;

(c) the minimal degree of D is at least $(3a+1)/2$;
(d) D is strongly connected, and the minimal degree of D is at least $3a/2$.

Observe that Theorem 1.1 imposes a degree condition on all pairs of non-adjacent vertices. In the following theorems a degree condition requires only for some pairs of non-adjacent vertices.

Theorem 1.2. (J. Adamus [9]) *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 6$. If $d(x)+d(y) \geq 3a$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Notice that Theorem 1.2 improves Theorem 1.1.

Some sufficient conditions for the existence of Hamiltonian cycles in a bipartite tournament are described in the survey paper [3] by Gutin. A characterization for hamiltonicity for semicomplete bipartite digraphs was obtained independently by Gutin [2] and Häggkvist and Manoussakis [4].

Theorem 1.3. (Wang [10]) *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 1$. Suppose that, for every pair of vertices $\{x, y\}$ with a common out-neighbour, either $d(x) \geq 2a-1$ and $d(y) \geq a+1$ or $d(y) \geq 2a-1$ and $d(x) \geq a+1$. Then D is Hamiltonian.*

Before stating the next theorem we need to define a balanced bipartite digraph of order eight.

Example 1. Let $D(8)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, and the arc set $A(D(8))$ contains exactly the following arcs: y_0x_1, y_1x_0, x_2y_3, x_3y_2 and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i$, $i \in [0, 3]$, $y_0 \leftrightarrow x_2$, $y_0 \leftrightarrow x_3$, $y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$.

It is easy to see that

$$d(x_2) = d(x_3) = d(y_0) = d(y_1) = 7 \quad \text{and} \quad d(x_0) = d(x_1) = d(y_2) = d(y_3) = 3,$$

and the dominating pairs in $D(8)$ are: $\{y_0, y_1\}$, $\{y_0, y_2\}$,$\{y_0, y_3\}$,$\{y_1, y_2\}$, $\{y_1, y_3\}$, $\{x_0, x_2\}$, $\{x_0, x_3\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$. Note that every dominating pair satisfies the condition B_1. Since $x_0y_0x_3y_2x_2\ y_1x_0$ is a cycle in $D(8)$, it is not difficult to check that $D(8)$ is strong.

Observe that $D(8)$ is not Hamiltonian. Indeed, if C is a Hamiltonian cycle in $D(8)$, then C would contain the arcs x_1y_1 and x_0y_0. Therefore, C would contain the path $x_1y_1x_0y_0$ or the path $x_0y_0x_1y_1$, which is impossible since $N^-(x_0) = N^-(x_1) = \{y_0, y_1\}$.

Notice that the digraph $D(8)$ does not satisfy the conditions of Wang's theorem.

Theorem 1.4. (Darbinyan [11]) *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$. Suppose that $\max\{d(x), d(y)\} \geq 2a-1$ for every pair of vertices x, y with a common out-neighbour. Then D is Hamiltonian unless D is isomorphic to the digraph $D(8)$ (for definition of $D(8)$, see Example 1).*

For $a \geq 4$ Theorem 1.4 improves Wang's theorem.

A digraph D is called pancyclic if it contains cycles of every length k, $3 \leq k \leq |V(D)|$. A balanced bipartite digraph of order $2a$ is even pancyclic if it contains cycles of every length $2k$, $2 \leq k \leq a$.

There are various sufficient conditions for a digraph (undirected graph) to be Hamiltonian, they are also sufficient for the digraph (undirected graph) to be pancyclic.

Recently, the following results were proved.

Theorem 1.5. (Darbinyan [12]) *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ other than a directed cycle of length $2a$. If $max\{d(x), d(y)\} \geq 2a-1$ for every dominating pair of vertices $\{x, y\}$, then either D contains cycles of all even lengths less than or equal to $2a$ or D is isomorphic to the digraph $D(8)$.*

Theorem 1.6. (Meszka [13]) *Let D be a balanced bipartite digraph of order $2a \geq 4$ with partite sets X and Y. If $d(x) + d(y) \geq 3a + 1$ for every pair of distinct vertices $\{x, y\}$ either both in X or both in Y, then D contains cycles of all even lengths less than or equal to $2a$.*

Theorem 1.7. (Darbinyan [14]) *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. If $d(x) + d(y) \geq 3a$ for every pair of distinct vertices $\{x, y\}$ either both in X or both in Y, then D contains cycles of all even lengths less than or equal to $2a$.*

Theorem 1.8. (Adamus [15]) *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 6$. If $d(x) + d(y) \geq 3a$ for every pair of distinct vertices $\{x, y\}$ with a common in-neighbour or a common out-neighbour, then D contains cycles of all even lengths less than or equal to $2a$ or a directed cycle of length $2a$.*

Definition 1. *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. For any integer $k \geq 0$, we will say that D satisfies the condition B_k when*

$$d(x) \geq 2a-k, d(y) \geq a+k \quad or \quad d(x) \geq a+k, d(y) \geq 2a-k$$

for any dominating pair of vertices $\{x, y\}$ in D.

In [10], Wang proposed the following problem.

Problem (Wang [10]). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ satisfying the condition B_k with $2 \leq k \leq a/2$. Is D Hamiltonian?*

Before stating the next theorems we need to define a digraph of order ten.

Example 2. Let $D(10)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4\}$ satisfying the following conditions: The induced subdigraph $\langle\{x_1, x_2, x_3, y_0, y_4\}\rangle$ is a complete bipartite digraph with partite sets $\{x_1, x_2, x_3\}$ and $\{y_0, y_4\}$; $\{x_1, x_2, x_3\} \rightarrow \{y_1, y_2, y_3\}$; $x_4 \leftrightarrow y_4$; $x_0 \leftrightarrow y_0$, $x_3 \leftrightarrow y_1$ and $x_i \leftrightarrow y_{i+1}$ for all $i \in [1, 3]$. $D(10)$ contains no other arcs.

It is easy to check that the digraph $D(10)$ is strongly connected and satisfies the condition B_0, but the underlying undirected graph of $D(10)$ is not 2-connected, and $D(10)$ has no cycle of length 8. (It follows from the facts that $d(x_0) = d(x_4) = 2$, and x_0 (x_4) is on 2-cycle). Since $x_1y_1x_3y_3x_2y_2x_1$ is a cycle of length 6, $x_0 \leftrightarrow y_0$ and $x_4 \leftrightarrow y_4$, it is not difficult to check that any digraph obtained from $D(10)$ by adding a new arc the one end-vertex of which is x_0 or x_4 contains a cycle of length eight. Moreover, if to $A(D)$ we add some new arcs of the type y_ix_j, where $i \in [1,3]$ and $j \in [1,3]$, then we always obtain a digraph, which does not satisfy the condition B_0.

Theorem 1.9. ([16], [17]). *Let D be a balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. Suppose that D satisfies the condition B_0, i.e., $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$. Then D contains cycles of all lengths $2, 4, \dots, 2a-2$ unless D is isomorphic to the digraph $D(10)$.*

Clearly, the existence of a cycle factor is a necessary condition for a digraph to be Hamiltonian. In this note we prove the following theorem.

Theorem 1.10. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ satisfying the condition B_k with $2 \leq k \leq a/2$. Then D contains a cycle factor.*

2. Terminology and Notation

Terminology and notation not described below follow [1]. If $xy \in A(D)$, then we say that x dominates y or y is an out-neighbour of x and x is an in-neighbour of y.

Let x, y be distinct vertices in a digraph D. The pair $\{x, y\}$ is called dominating if there is a vertex z in D such that $xz \in A(D)$ and $yz \in A(D)$. In this case we say that x is a partner of y and y is a partner of x. If $x \in V(D)$ and $A = \{x\}$ we sometimes will write x instead of $\{x\}$. $A \to B$ means that every vertex of A dominates every vertex of B. The notation $x \leftrightarrow y$ denotes that $xy \in A(D)$ and $yx \in A(D)$.

Let $N^+(x)$, $N^-(x)$ denote the set of out-neighbours, respectively the set of in-neighbours of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$, $N^-(x, A) = A \cap N^-(x)$ and $N^+(A) = \cup_{x \in A} N^+(x)$, $N^-(A) = \cup_{x \in A} N^-(x)$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$).

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \dots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. Given a vertex x of a directed path P or a directed cycle C, we denote by x^+ (respectively, by x^-) the successor (respectively, the predecessor) of x (on P or C), and in case of ambiguity, we precise P or C as a subscript (that is $x_P^+ \dots$).

A digraph D is strongly connected (or, just, strong) if there exists an (x, y)-path in D for every ordered pair of distinct vertices x, y of D. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both).

Let H be a non-trivial proper subset of vertices of a digraph D. An (x, y)-path P is an H-bypass if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap H = \{x, y\}$.

Let D be a balanced bipartite digraph with partite sets X and Y. A matching from X to Y is an independent set of arcs with origin in X and terminus in Y. (A set of arcs with no common end-vertices is called independent). If D is balanced, one says that such a matching is perfect if it consists of precisely $|X|$ arcs.
The underlying undirected graph of a digraph D is denoted by $UG(D)$, it contains an edge xy if $xy \in A(D)$ or $yx \in A(D)$ (or both).

3. Main Result

Theorem 1.10 is the main result of this paper.

Proof of theorem 1.10. Let D be a digraph satisfying the conditions of the theorem. Ore in [18] (Section 8.6) has shown that a balanced bipartite digraph D with partite sets X and Y has a cycle factor if and only if D contains a perfect matching from X to Y and a perfect matching from Y to X.

Therefore, by the well-known Köning-Hall theorem (see, e.g., [19]) to show that D contains a perfect matching from X to Y, it suffices to show that $|N^+(S)| \geq |S|$ for every set $S \subseteq X$. Let $S \subseteq X$. If $|S| = 1$ or $|S| = a$, then $|N^+(S)| \geq |S|$ since D is strongly connected. Assume that $2 \leq |S| \leq a - 1$. We claim that $|N^+(S)| \geq |S|$. Suppose that this is not the case, i.e., $|N^+(S)| \leq |S| - 1 \leq a - 2$. From this and strongly connectedness of D it follows that there are two vertices $x, y \in S$ and a vertex $z \in N^+(S)$ such that $\{x, y\} \to z$, i.e., $\{x, y\}$ is a dominating pair. Therefore, by condition B_k, $d(x) \geq 2a - k$ and $d(y) \geq a + k$ or $d(x) \geq a + k$ and $d(y) \geq 2a - k$. Without loss of generality, we assume that $d(x) \geq 2a - k$ and $d(y) \geq a + k$. Then

$$2a - k \leq d(x) \leq 2|N^+(S)| + a - |N^+(S)| = a + |N^+(S)|.$$

Therefore, $|N^+(S)| \geq a - k$ and $|S| \geq a - k + 1$.
Proposition 1. Let $\{u, v\}$ be a dominating pair of vertices of D. Then from condition B_k and $2 \leq k \leq a/2$ it follows that $d(u) \geq a + k$ and $d(vu) \geq a + k$, i.e., if a vertex z has a partner in D , then $d(z) \geq a + k$.

We claim that each vertex in $Y \setminus N^+(S)$ has no partner in D. Indeed, let u be an arbitrary vertex in $Y \setminus N^+(S)$. Since $|S| \geq a - k + 1$, we have

$$d(u) \leq |S| + 2(a - |S|) = 2a - |S| \leq a + k - 1,$$

which contradicts Proposition 1. This means that u has no partner in D.

Without loss of generality, assume that

$$S = \{x_1, x_2, \ldots, x_s\} \quad \text{and} \quad N^+(S) = \{y_1, y_2, \ldots, y_t\}.$$

Recall that every vertex y_i with $t+1 \leq i \leq a$ has no partner in D. Note that $s \geq t+1$, $a-s \leq a-t-1$ and there is no arc from a vertex of $\{x_1, x_2, \ldots, x_s\}$ to a vertex of $\{y_{t+1}, y_{t+2}, \ldots, y_a\}$. From this and strongly connectedness of D it follows that there is a vertex x_{i_1} such that $y_{t+1}x_{i_1} \in A(D)$. Since y_{t+1} has no partner, it follows that $d^-(x_{i_1}, Y \setminus \{y_{t+1}\}) = 0$. Therefore, $d(x_{i_1}) \leq a + 1 \leq a + k - 1$ since $k \geq 2$. By Proposition 1, this means that the vertex

x_{i_1} also has no partner. Since D is strongly connected, there is a vertex $y_{i_2} \in Y$ such that $x_{i_1}y_{i_2} \in A(D)$. Then $d^-(y_{i_2}, X \setminus \{x_{i_1}\}) = 0$, because of the fact that x_{i_1} has no partner. Therefore, $d(y_{i_2}) \leq a+1$ and hence, y_{i_2} also has no partner. Continuing this process, as long as possible, as a result we obtain a path $P = y_{t+1}x_{i_1}y_{i_2}x_{i_2} \ldots x_{i_l}y_{i_l}$ or a cycle $C = y_{t+1}x_{i_1}y_{i_2}x_{i_2} \ldots x_{i_l}y_{t+1}$. It is not difficult to see that all the vertices of this path (cycle) have no partners. If the former case holds, then x_1 is in P, which is a contradiction since x_1 has a partner (namely x_2).

If the second case holds, then, since every vertex of C has no partner in D, it follows that there is no arc from a vertex of $V(D) \setminus V(C)$ to a vertex of $V(C)$, which contradicts that D is strongly connected. This completes the proof of the existence of a perfect matching from X to Y. The proof for a perfect matching in the opposite direction is analogous. This completes the proof of the theorem. □

4. Remarks

Now using Theorem 1.10, we prove the following results (Lemmas 3.1-3.3).

Lemma 3.1. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y satisfying the condition B_k, $2 \leq k \leq a/2$. If D is not Hamiltonian, then every vertex $u \in V(D)$ has a partner in D.*

Proof of Lemma 3.1: Let D be a digraph satisfying the conditions of the lemma. For a proof by contradiction, suppose that there is a vertex x in D, which has no partner. By Theorem 1.10, D has a cycle factor, say C_1, C_2, ... , C_l. Then $l \geq 2$ since D is not Hamiltonian. Without loss of generality, we assume that $x \in V(C_1)$. It follows that $d^-(x^+_{C_1}) = 1$. Therefore, $d(x^+_{C_1}) \leq a+1$. By Proposition 1, this means that the vertex $x^+_{C_1}$ also has no partner. Similarly, we obtain that $d(x^{++}_{C_1}) \leq a+1$ (where $x^{++}_{C_1}$ denotes the successor of $x^+_{C_1}$ on C_1) and hence, $x^{++}_{C_1}$ also has no partner in D. Continuing this process, we conclude that every vertex of C_1 has no partner in D. This implies that there is no arc from a vertex of $A(V(D) \setminus V(C_1)$ to a vertex of $V(C_1))$, which contradicts that D is strongly connected. The lemma is proved. □

Lemma 3.2. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y satisfying the condition B_k, $2 \leq k \leq a/2$. If D is not a cycle, then D contains a non-Hamiltonian cycle of length at least four.*

Proof of Lemma 3.2: Let D be a digraph satisfying the conditions of the lemma. For a proof by contradiction, suppose that D contains a non-Hamiltonian cycle of length at least four.

If D is Hamiltonian, then it is not difficult to show that D contains a non-Hamiltonian cycle of length at least 4. So we suppose, from now on, that D is not Hamiltonian and contains no cycle of length at least 4. By Theorem 1.10, D contains a cycle factor. Let $C_1, C_2, \ldots, C_t$ be a minimal cycle factor of D (i.e., t is as small as possible). Then the length of every C_i is equal to two and $t = a$. Let $C_i = x_iy_ix_i$, where $x_i \in X$ and $y_i \in Y$. By Lemma 3.1, every vertex of D has a partner. This means that for every vertex $x \in V(D)$, $d(x) \geq a+k$ and $d^-(x) \geq k \geq 2$, $d^+(x) \geq k \geq 2$. Without loss of generality, we assume that $\{x_1, x_j\}$ with $j \neq 1$ is a dominating pair and $d(x_1) \geq 2a-k$.

Let Z be the subset of Y with the maximum cardinality, such that every vertex of Z together with x_1 forms a cycle of length two. Without loss of generality, we assume that $Z = \{y_1, y_2, \dots, y_l\}$. Then $2a - k \leq d(x_1) \leq 2l + a - l = a + l$. Hence, $l \geq a - k$. Since D contains no cycle of length four, it follows that the vertices y_1 and x_i, $2 \leq i \leq l$, are not adjacent. Therefore,

$$a + k \leq d(y_1) \leq 2a - 2l + 2 \leq 2k + 2,$$

i.e., $k \geq a - 2$. Since $a/2 \geq k \geq a - 2$, we have $a \geq 2k \geq 2a - 4$, $a \leq 4$. If $a = 4$, then $k = a/2 = 2$ and $l = a - k = 2$. It is easy to see that $d(x_1) = d(y_1) = 6$, the vertices y_1 and x_i, $3 \leq i \leq 4$, form a cycle of length two and $x_1y_3 \in A(D)$ or $y_3x_1 \in A(D)$. Now it easy to see that D contains a cycle of length four. Lemma 3.2 is proved. □

For the next lemma we need the following lemma due to Bondy.

Bypass Lemma (Lemma 3.17, Bondy [20]). *Let D be a strong non-separable (i.e., $UG(D)$ is 2-connected) digraph, and let H be a non-trivial proper subdigraph of D. Then D contains an H-bypass.*

Remark: One can prove Bypass Lemma using the proof of Theorem 5.4.2 [1].

Now we will prove the following lemma.

Lemma 3.3. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y satisfying the condition B_k, where $2 \leq k \leq a/2$. Then the following statements hold:*

(i) the underlying undirected graph $UG(D)$ is 2-connected;

(ii) if C is a cycle of length m, $2 \leq m \leq 2a - 2$, then D contains a C-bypass.

Proof of Lemma 3.3. (i) Suppose, on the contrary, that D is a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y satisfies the condition B_k but $UG(D)$ is not 2-connected. Then $V(D) = E \cup F \cup \{u\}$, where E and F are non-empty subsets, $E \cap F = \emptyset$, $u \notin E \cup F$ and there is no arc between E and F. Since D is strong, it follows that there are vertices $x \in E$ and $y \in F$ such that $\{x, y\} \to u$, i.e., $\{x, y\}$ is a dominating pair. Without loss of generality, we assume that $x, y \in X$. Then $u \in Y$. By condition B_k, it is easy to see that

$$3a \leq d(x) + d(y) \leq 4 + 2|E \cap Y| + 2|F \cap Y| \leq 2a + 2,$$

which is a contradiction. This proves that $UG(D)$ is 2-connected.

(ii) The second claim of the lemma is an immediate consequence of the first claim and Bypass Lemma. Lemma 3.3 is proved. □

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, 2001.

[2] G. Gutin, "Criterion for complete bipartite digraphs to be Hamiltonian", *Vestsi Akad, Navuk BSSR Ser. Fiz.-Mat. Navuk* vol. 1, pp. 109-110, 1984.

[3] G. Gutin, "Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey". *J. Graph Theory* vol. 19, no. 4, pp. 481-505, 1995.

[4] R. Häggkvist and Y. Manoussakis, "Cycles and paths in bipartite tournaments with spanning configurations". *Combinatorica*, vol. 9, no. 1, pp. 33-38, 1989.

[5] Y. Manoussakis and I. Millis, "A sufficient condition for maximum cycles in bipartite digraphs", *Discrete Math.*, vol. 207, pp. 161-171, 1999.

[6] J. Adamus and L. Adamus, "A degree condition for cycles of maximum length in bipartite digraphs", *Discrete Math.*, vol. 312, pp. 1117-1122, 2012.

[7] D. Amar and Y. Manoussakis, "Cycles and paths of many lengths in bipartite digraphs", *J. Combin. Theory Ser.* **B 50**, pp. 254-264, 1990.

[8] J. Adamus, L. Adamus and A. Yeo, "On the Meyniel condition for hamiltonicity in bipartite digraphs", *Discrete Math. and Theoretical Computer Science,*vol. 16, no. 1, pp. 293-302, 2014.

[9] J. Adamus, "A degree sum condition for hamiltonicity in balanced bipartite digraphs", *Graphs and Combinatorics,* vol. 33, no. 1, pp. 43-51, 2017.

[10] R. Wang, "A sufficient condition for a balanced bipartite digraph to be Hamiltonian". *Discrete Mathematics and Theoretical Computer Science,* vol. 19, no. 3, 2017.

[11] S. Kh. Darbinyan, "Sufficient conditions for Hamiltonian cycles in bipartite digraphs", arXiv:1604.08733v1 [matm.CO] 29, 15 pages, Apr 2016.

[12] S. Kh. Darbinyan, "Sufficient conditions for a balanced bipartite digraph to be even pancyclic", *Discrete Applied Mathematics*, vol. 238, pp. 70-76, 2018.

[13] M. Meszka, "New sufficient conditions for bipancyclicity of balanced bipartite digraphs", *Discrete Math.*, Submitted for publication.

[14] S. Kh. Darbinyan, "A theorem on even pancyclic bipartite digraphs", arXiv:1801.05177v1 [matm.CO] 16, 12 pages, Jan 2018.

[15] J. Adamus, "A Meyniel-type condition for bipancyclicity in balanced bipartite digraphs", arXiv:1708.04674v2 [matm.CO], 7 pages, 22 Aug 2017.

[16] S. Kh. Darbinyan, "On pre-Hamiltonian cycles in balanced bipartite digraphs", *Mathematical problems of Computer Science*, vol. 46, pp. 7-17, 2014.

[17] S. Kh. Darbinyan and I. A. Karapetyan, "A Sufficient condition for pre-Hamiltonian Cycles in Bipartite Digraphs", *CSIT 2017 Revised Selected Papers,IEEE conference proceeding,*, pp. 101-109, DOI:10.1109/CSITTechnol.2017.8312150.

[18] O. Ore, "Theory of graphs", American Mathematical Society, Providence, R.I., American Mathematical Society Colloquium Publications, vol. 38, 1962.

[19] C. Berge, *Graphs and hypergraphs*, North-Holland, Amsterdam, 1973.

[20] J. A. Bondy, *Basic Graph Theory: Paths and Circuits*, In Handbook of combinatorics **1-2**, Elsevier, Amsterdam, 1995.

Submitted 20.09.2017, accepted 16.01.2018.

Կոմնորոշված երկմասնյա գրաֆի համիլտոնյանության վերաբերյալ Վանգի խնդրի մասին

Ս. Դարբինյան և Ի. Կարապետյան

Ամփոփում

Վանգը (Discrete Mathematics and Theoretical Computer Science, vol. 19(3) 2017) առաջարկել է հետևյալ խնդիրը:

Խնդիր: Դիցուք D-ն ուժեղ կապակցված $2a$-գագաթանի $2a \geq 8$ կողնորոշված երկմասնյա հավասարակշռված գրաֆ է, որում գագաթների ցանկացած $\{x,y\}$ հաղթող զույգի համար տեղի ունեն հետևյալ անհավասարությունները. $d(x) \geq 2a-k$, և $d(y) \geq a+k$ կամ $d(x) \geq a+k$ և $d(y) \geq 2a-k$, որտեղ $2 \leq k \leq a/2$: Արդյոք D-ն համիլտոնյան է:

Ներկա աշխատանքում ապացուցված է, որ եթե D գրաֆը բավարարում է Վանգի խնդրի պայմաններին, ապա (i) D գրաֆը պարունակում է ցիկլ-ֆակտոր և առնվազն չորս երկարությամբ ոչ-համիլտոնյան ցիկլ, (ii) D գրաֆի ցանկացած x գագաթի համար գոյություն ունի այնպիսի մի y գագաթ, որ $\{x,y\}$ - ը հաղթող զույգ է:

О задаче Ванга о гамильтоновости двудольных орграфов

С. Дарбинян и И. Карапетян

Аннотация

Ванг (Discrete Mathematics and Theoretical Computer Science, vol. 19(3) 2017) предложила следующую задачу.

Задача: Пусть D -$2a$ -вершинный $2a \geq 8$ сильносвязный сбалансированный двудольный орграф, в котором для любой пары доминирующих вершин $\{x,y\}$, $d(x) \geq 2a-k$, $d(y) \geq a+k$ или $d(y) \geq a+k$, $d(y) \geq 2a-k$, где $2 \leq k \leq a/2$. Является ли D гамильтоновым?

В настоящей работе доказано, что если орграф D удовлетворяет условиям задачи Ванга, то (i) D содержит цикл-фактор и не-гамильтоновый цикл длины по крайней мере 4, (ii) для каждой вершины x существует такая вершина y, что $\{x,y\}$ является доминирующим паром.