

Re-Sourcing Mathematics Teaching Through Professional Development

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In any education system, curriculum alone cannot fulfill reform demands imposed onto teachers to help learners make meaning of subject contents they learn. This paper revisits the notion of resources as explored in a mathematics teacher professional development project that focused on algebra learning. As a way to promote the use of manipulatives in teaching and learning algebra, the project introduced algebra tiles to participating teachers and investigated how the tiles facilitated the enaction of algebraic meanings. The participating teachers learned different ways of helping learners interpret and solve algebraic problems, with the use of algebraic tiles.

The need to make adequate interpretations of algebraic problems

Before we go into the discussion of the use of material resources, which is the core of this paper, we will first attend to the need for encouraging the enaction of adequate mathematical interpretations of mathematical problems by adopting a vignette from Miranda (2004) which represents an algebraic task (Figure 1) that was assigned to a group of Grade 11 learners. The learners were working in small groups and this vignette represents only what happened in one of the small groups.

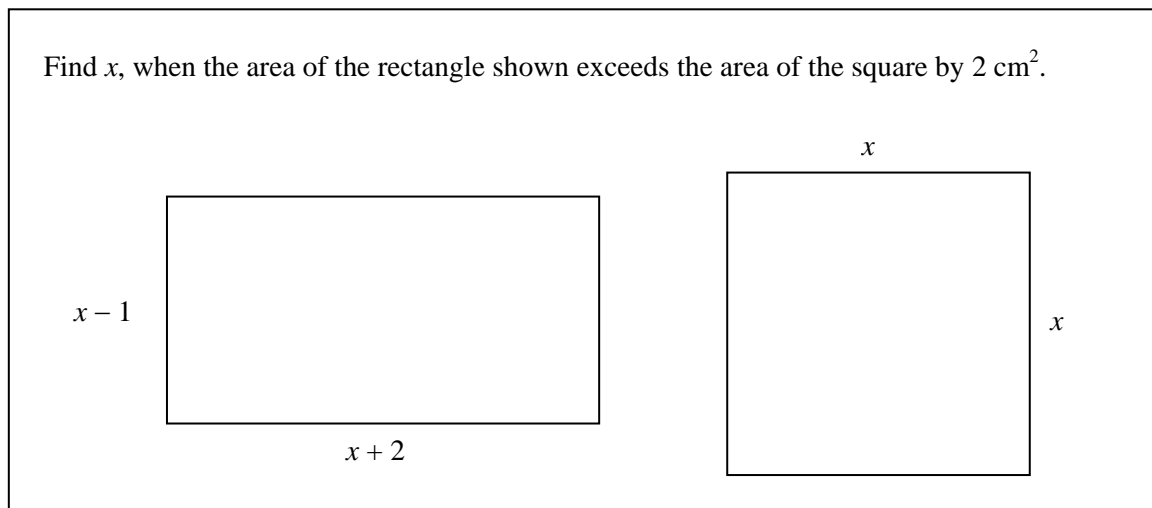


Figure 1: Area problem involving a rectangle and a square

The group members all expressed that this was an easy problem and that they could solve it quickly. They together constructed the (incorrect) equation $(x - 1)(x + 2) = x^2 - 2$ and tried to solve it. The group was later perturbed by the solution obtained which showed that $x = 0$. One learner suggested “that can’t be” and they all reasoned that x could not be zero because x is supposed to be a length.

When the teacher passed by, checking on this particular group's solution, she told them that their solution was wrong. The teacher read out the problem to the group and told them to "think harder" about it and left them. The discussion of the above problem in Miranda (2004) revealed how the learners were not able to come up with the solution to this seemingly easy problem.

One would expect learners not to have much difficulty in solving problems of this kind especially with the provision of the geometric visual of the objects involved in the problem. Yet one wonders what these learners were attending to in such a way that their interpretations of the problem itself did not mediate a successful solution method. Miranda's (2004) deeper analysis of the problem above suggests that one of the factors that could have hindered learners' adequate interpretation of the problem is the word "exceed". However, she notes that there could be other factors and that what is important to ask is: What pedagogic approaches teachers could take to help learners make more meaningful interpretations of mathematical problems and their contexts? How differently, for example, would learners approach this problem if they had access to other concepts and materials that allow for mathematical connections between algebra and geometry? In this paper we argue for the adoption of manipulatives, along with many other useful approaches, as a way to enrich learners' learning experiences as they think mathematically about algebraic problems.

The vignette above, indicates how the teacher's response to the learners' attempts to solve the problem does not go beyond "try-harder" suggestions and how learners fail to succeed in solving routine algebraic problems. This reflects dominant practice in Namibian mathematics classrooms in which there is very little teaching for understanding (Amoonga & Kasanda, 2010) and discussions of learners' problem solving methods and solutions rarely occur. It also opens space for working with teachers on expanding their pedagogical repertoires, through, for example, the use of concrete materials and multiple representations.

This paper reports on a project that explored this pedagogical challenge with mathematics teachers in Namibia. Firstly, the idea of teaching resources is introduced. Secondly, a re-conceptualization of the term resources is adopted with a focus on the notion of transparency among educational material resources. Thirdly and finally material resources known as algebra tiles are discussed as explored in an action research project and how these may mediate the construction of adequate interpretations of algebraic problems.

Background

In any education system, curriculum alone as a document cannot fulfill the current reform demands imposed onto teachers in order to help learners make meaning of the subject content they are supposed to learn. For example, in Namibia, a mathematics teacher is expected to be creative and innovative enough to ensure that the teaching of mathematics is as much learner-centered as possible. However, many teachers of mathematics are unable to imagine what a learner-centred lesson could look like, let alone practice it (van Graan, 1999). One of the main tenets argued for by those who call for learner-centred teaching is that learners should be actively engaged with the subject content as they try to relate it to their life experiences (Norman & Sphorer, 1996). In many Namibian classrooms, engaging learners with the mathematical content is only done through textbook exercises in which the learners are expected to complete lists of exercises and have them checked right or wrong by the teachers. There is little focus on helping the learners to make sense of the mathematics content either through debate and discussion (Miranda, 2004) or through the use of concrete learning aids (Miranda, 2007).

This paper revisits the notion of teaching and learning resources as explored in a mathematics teacher professional development project¹ that focused on the teaching and learning of algebra. The project took place in Northern Namibia involving 10 secondary mathematics teachers from different schools in weekly in-service training sessions. The initial purpose of the larger study was to explore, together with the participant teachers, learner meaning-making of algebra as they solve mathematical problems. But it

¹ This teacher professional development project was part of a larger research study (Miranda, 2009a), which explored innovative means of professional development for mathematics teachers with a particular focus on how to help learners realise the need to make adequate interpretations of algebra.

became apparent that the teachers encountered algebra in particular ways, and that they could only interpret learners' solution methods to problems as either right or wrong. So, as part of a re-plan² it became apparent that the teachers needed to first experience different ways of making meaning of algebra before they investigated and tried to interpret learners' meaning-making of algebra. The implications this had on the study was that, even though learners also participated in the study, their involvement was monitored in parallel to, but not through, the teachers' involvement, as was initially planned.

One way of getting the teachers to experience the notion of meaning-making of algebra, was to engage them with both interpretive and hands-on algebraic problems, in which they explored how such activities open up possibilities for a diverse pedagogic approaches to algebra. For example, the purpose of the sessions, from which this paper emerges, was to investigate how the use of material resources may allow for the meaning-making of algebraic concepts and the explanation of some of the algorithms related to algebraic problem solving methods. In Namibian mathematics classrooms, there is little awareness of the need to use material resources for enhancing mathematics learning. In this professional development project, mathematics teachers were engaged in activities of developing and adopting material resources, for mathematics teaching. The paper only discusses how some of those resources facilitated the development of various ways of helping learners construct adequate interpretations of algebraic situations. Overall, the participants were also able to notice the differences and similarities between what they knew and what they came to know about the pedagogy of algebra while engaging with the resources provided by the project.

Re-conceptualising the notion of resources

Adler (2000) calls for a re-conceptualisation of resources as an important theme in mathematics education. She suggests that the term resource could be thought of "as the verb *re-source*, to source again or differently" (p. 207). 'Resource', she adds, may also be considered "as both a noun and a verb, as both object and action" (p. 207). Adler further classifies educational resources in three main categories, namely human resources, cultural resources and material resources. Firstly, human resources include the teachers themselves and the pedagogic content knowledge that they embody. Secondly, cultural resources include resources such as language, time, and other culturally available tools or concepts. Thirdly, material resources are, for example, technologies, curricular documents, textbooks, and other tangible objects that may be incorporated into the teaching and learning process. This paper will focus on the third type of resources, material resources, which appear to be lacking or underutilised in many African mathematics classrooms.

Citing Black and Atkin (1996), Adler (2000) argues that even though, human resources serve as a critical means to the successful implementation of curricular innovation and reform, such a success is equally dependent on the "availability of supportive material resources" (p. 205). At the same time Adler cautions that while bearing in mind that limited resources may have a negative impact on learners' mathematical experiences and performance, it should not be assumed that an increase of material resources will amount to better pedagogic practices. Even if it leads to significantly better practices, this will not happen in unproblematic and linear ways (Adler, Reed, Lelliot, & Setati, 2002). Also, one must realise that "resources are not self explanatory objects with mathematics shining clearly through them" (Adler, 2000, p. 207). Adler further argues that mathematics education needs to "shift from broadening a view of *what* such resources are to *how* resources function as an extension of the mathematics teacher in the teaching [and] learning process" (p. 207).

Alongside this, and drawing on the work of Lave and Wenger, Adler introduces the term transparency that is needed in the use of any material resources for mathematical teaching and learning. Defining transparency in terms of how the resources are contextualised and used, Adler contends that resources need to be visible and invisible at the same time. On one hand, they need to be seen (visible) "so that they can be used [touched, felt, manipulated] and so extend the practices" (p. 214). On the other hand, resources need to be seen through (invisible) "so that they allow smooth entry into the practice" (p. 214).

² This was an action research guided study which followed a spiral process of plan-observe-reflect-re-plan. For a description of this process, see Miranda (2009a).

In other words, both teachers and learners need to move beyond the manipulation of material resources and see the mathematics through them, but not to be stuck with the materiality of the resources.

Constructing mathematical meaning through manipulatives

This paper reports on how material resources known as algebra tiles (Kitt & Leitz, 2000; Miranda, 2010) may serve as useful teaching and learning resources for making meaning of algebra. The term *manipulative* will be used in this paper in order to distinguish algebra tiles from other material resources such as chalkboards that are used in the teaching and learning of mathematics. As the word implies, manipulatives are either virtual or tangible materials that can be manipulated in shape or size as we use them to make meaning of our learning environment. For instance a computer is not a manipulative resource, but it may contain programs with virtual manipulatives within it, for example materials within a Geometer Sketchpad activity.

One way of making mathematics meaningful to learners is to make explicit the close relationship among all its domains and how each can be used in making sense of the other. For example, the use of geometric diagrams and concrete materials may serve as a good source for bringing some geometric aspects into algebra. This also stresses the use of multiple representations in teaching and learning mathematics which has been, for a long time now, one of the foci in mathematics education. It has been argued that using multiple representations allows learners to understand mathematics concepts from different perspectives (see e.g., Arcavi, 1999; Duval, 1999, 2006; National Council of Teachers of Mathematics, 2000).

The need to use concrete materials has a long history in the field of mathematics education (Szendrei, 1996). Exploring the role of manipulatives in a mathematics classroom, Szendrei (1996) argues that such materials “help pupils develop and understand the concepts, procedures, and other aspects of mathematics” (p. 427). She further alludes to the argument that adopting manipulatives in the teaching and learning process may also give equal opportunities to all learners to develop mathematical thinking in areas that are not readily supported by the abstract manipulation of mathematical structures. However, cautions Szendrei, both teachers and teacher educators should be vigilant enough not to regard concrete materials as some ‘miracle drugs’ that will fix all learning problems that learners experience with mathematics. The role of such materials must be clearly explained in order to avoid confusion and frustrations as the learners handle the materials.

Namibia is one of the many African countries, in which the use of manipulatives in mathematics classrooms is not a common practice. Unlike in some developed countries, the use of manipulatives is not explicitly required in most of the African school curricula (Namukasa, 2005). Therefore there are no rigorous standardised measures taken in order to ensure that possibilities for learning and teaching school mathematics through the use of manipulatives are taken advantage of. In a study that partly promoted the learning of mathematics through the use of semiotic artifacts, Namukasa (2005) observes that African, specifically Ugandan, learners appear less inclined than their North American (Canadian) counterparts to use manipulatives when learning mathematics. This, she supposes, may be due to the fact that using manipulatives is not prescribed by the curricular documents; rather it is based on individual teacher’s decisions on whether to or not to make manipulatives part of their teaching. Not making the use of manipulatives part of the school mathematics curriculum can therefore contribute to the teachers’ reluctance to go out of their way and try out different ways of adopting manipulatives in their teaching.

A particular research interest in the use of manipulatives in mathematics classrooms of developing countries has recently emerged among African mathematics educators. There has been a realisation of how little literature there is that explores specific cases concerned with this phenomenon. For example, in a research study that involved Ugandan pre-service mathematics teachers in the teaching of mathematics with manipulatives, Namukasa and Kaahwa (2007) introduced and investigated “the use of concrete, virtual and imagined materials/teaching aids in Ugandan mathematics teaching” (p. 3). Their findings suggest that there is need for mathematics education research to investigate the use of manipulatives in developing countries, especially those on the African continent. Miranda (2007) also explored the need for the use of manipulatives as felt and expressed by mathematics teachers, and yet teachers do not have enough experience to draw from in order to develop any kind of materials that may be pedagogically useful.

Taking further the discussion on the use of additional material resources, Adler et al. (2002) consider the educational implications that the acknowledgement of the need to use additional material resources has for teaching mathematics. For example, questions of provision, sufficiency and sustainability must always be accounted for, in order to make sure that all schools have access to the materials. Also, since the impact of the power of any material resources lies in the way that they are used, there is a need for the teachers to teach learners how to use the resources, and hence the need for focused teacher development arises. This, Adler et al. argue, will enable the teachers to realise that the resources do not have an automatic educational meaning but rather, the meaning emerges “through their use in the context of classroom practices and the subject [matter] being learned” (p. 69). In other words, the teachers should not just put the resources in front of the learners and expect them to be able to read the mathematics through them without proper guidance.

As a response to the need to incorporate tangible materials in the teaching and learning of mathematics, a teacher professional development project was established and monitored in Northern Namibia (Miranda, 2009a, 2009b). In addition to other activities of the teacher professional development project, the participating teachers were exposed to the manipulatives known as algebra tiles, for the first time. The purpose of introducing the manipulatives was partly to attend to the teachers’ expressed need of help with the development of teaching and learning materials and to investigate how the participants make algebraic meanings through their interactions with these materials.

Introducing algebra tiles

As a way of promoting the use of manipulatives in the teaching and learning of algebra, the project introduced the algebra tiles to the participants and investigated how the tiles facilitated the enaction of elaborated algebraic meanings (Miranda, 2009a). Algebra tiles are pieces of paper or cards that appear in geometric forms, usually rectangles and squares, of different sizes. The materials are commercially available but one can also make them by cutting out soft or poster paper. This is the most affordable way of constructing such tools especially in teaching environments such as the Namibian one.

In the first in-service session on the algebra tiles the teachers spent time learning how to use the tiles and getting more comfortable with using them to represent algebraic expressions. It was in the second and subsequent sessions that the manipulation of algebraic expressions through the use of algebra tiles was explored to a larger extent. Before the tiles were handed out, they were introduced by drawing three rectangles of different dimensions on the board. Each of these rectangles represents one type of algebra tile in the kit. The dimensions of these rectangles are pre-given and so the participants had to determine the area of each rectangle. This is illustrated in Figure 2.

Each kit of the algebra tiles contains a combination of many tiles; however, the group only explored three types of tiles, represented in Figure 2 by (a) a square measuring x units by x units; (b) a rectangle measuring x units by 1 unit and a square measuring 1 by 1. As the group worked with the tiles trying to figure out their dimensions and total areas, they as a community adopted a language to give each tile a name. These names shaped the discussions that emerged for the rest of the time the group worked with the tiles. For example, (a) was referred to as the *big square* – blue in colour, (b) the *blue strip* – also blue in colour and (c) the *small square* – white in colour.




Big square	Blue strip	Small square
<div style="text-align: center;">  </div> <p data-bbox="295 1926 558 1971">(a) Area = $x \times x = x^2$</p>	<div style="text-align: center;">  </div> <p data-bbox="654 1926 901 1971">(b) Area = $x \times 1 = x$</p>	<div style="text-align: center;">  </div> <p data-bbox="1013 1926 1236 1971">(c) Area = $1 \times 1 = 1$</p>

Figure 2: The representation of some of the algebra tiles

From the chalkboard drawing of the tiles, a transition was made to the tiles by holding them, one by one, and asking the participants to state which rectangle represented each kind of tile in the kit. Some conventional rules were also laid out for using these tiles in order to construct different geometric shapes. For instance, two or more tiles could only be placed next to one another if, and only if, the lengths of the adjacent sides were equal. For example, in Figure 3, (a) and (b) would be sensible constructions, but (c) would not be, since x is not always going to have the value of 3, because it is a variable.

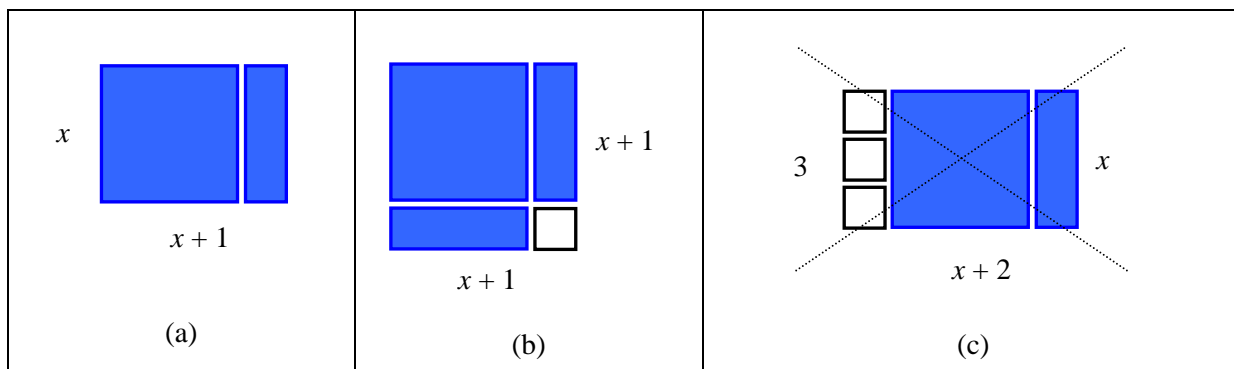


Figure 3: Sensible [(a) and (b)] and non-sensible [(c)] ways of aligning algebra tiles

Expanding algebraic expressions

After the group revised the rules for using the algebra tiles and their measurements as explored in the first session of the project, the teachers were invited to use the tiles to construct geometric shapes of given dimensions. The objective of this brief task was to use the tiles to determine the area of the rectangle that would have been formed. A handout of different expressions was given for the teachers to choose the one they would like to start with. However, the teachers were also encouraged to think of other different expressions they might want to explore.

The expression $(x + 2)(x + 3)$ was the first to be explored. It was explained to the teachers that what we are trying to do is explore the algebra tiles and question: In what ways can we use them to, for example, introduce algebra to the learners or help learners make algebraic meaning as they solve problems? If we look at the two factors $(x + 2)$ and $(x + 3)$, for example, we can also view them as the dimensions of the sides of the geometric shape [rectangle] that we want to form with the tiles. So, if the width is $(x + 2)$ and the length is $(x + 3)$, what is the area of that rectangle going to be?

The teachers took some time to renegotiate the dimensions of the sides of each tile before they started off building the required rectangles. This was done individually and then they all shared with one another what they had formed. Figure 4 is a representation of what three of the teachers constructed.

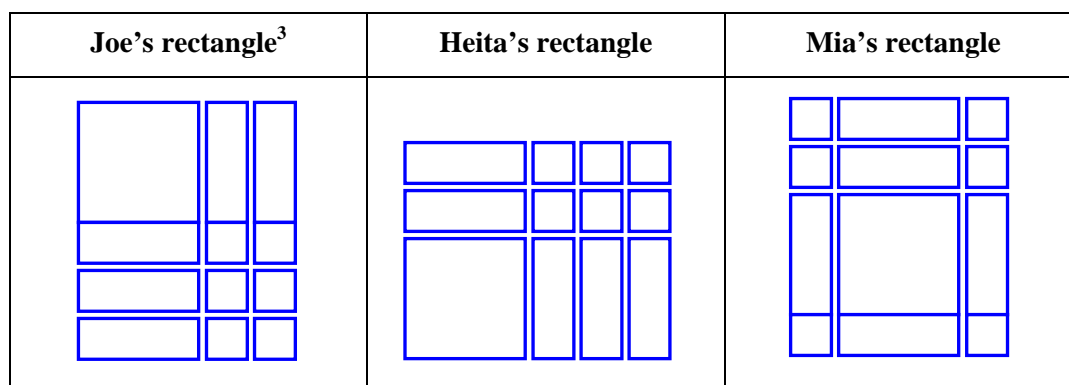


Figure 4: A representation of teachers' construction with the tiles

³ The names used here are not the real names of the teachers, but the pseudonyms as assigned in Miranda (2009a).

It was interesting that each teacher tried to build a shape that looked different from what the others had constructed, perhaps to have multiple interpretations of the same problem. For example, Mia could be heard remarking to Heita that “Mine is different from yours”. The next suggestion was to determine if they all had the same area, despite the different shapes formed by the tiles. It did not take Mia and Heita long to figure out the area of their rectangles, but Joe was finding it difficult to make sense of what he built:

- Miranda⁴: Did we get the area?
Mia: Yes.
Miranda: What is the area?
Mia: It is x squared, plus one, two, three, four, five x , plus six.
Miranda: Mmhhh, x squared, plus five x plus six. Did you get the same Joe? [As he looked puzzled, frowning]
Joe: I am lost.
Mia: The area?
Miranda: Yes, I mean the total area of the beautiful rectangle you constructed there.
Heita: What is this side? [Pointing at one of the sides of Joe’s rectangle]
Joe: x plus two?
Heita: And this one?
Joe: x plus three.
Heita: So, what is the whole area? [Joe still looked puzzled; the group tried another strategy to help him]

It was taking Joe some time here to figure out the total area of the rectangle that he constructed. It appeared Joe was attending to this construction in a way different from how the rest of the group could see it. Heita, in trying to help him, interrogated Joe on the dimensions of his rectangle. Even though Joe could answer these as “ x plus two” and “ x plus three” he could not immediately respond to Heita’s question: “So, what is the whole area?”

One amazing thing that was worth noticing was how much patience these teachers had with one another. After Joe admitted that he really did not understand what everybody meant, Mia and Heita paused, looking at him while giving him ample time to make sense of his construction. Then Joe looked at Miranda and asked: “What area now?” This time Mia responded with a question similar to Heita’s but phrased differently:

- Mia: What is the area of this ‘big square’ [pointing to the big blue square]?
Joe: It’s x squared.
Mia: What about the blue strips? What is the area of each one?
Joe: x ?
Mia: And how many do you have of those?
Joe: Five?
Miranda: And how many units [small squares] do you have?
Joe: I have six of them.
Heita & Mia: Now what is their total area, altogether?
Joe: x squared, five x , and six, Ooohhh! [Tapping his own head with the right hand]

At this point, Joe was laughing at himself when he finally realised that what he had in front of him was a rectangle with specific dimensions and hence a specific area. This, as he admitted, was a huge learning experience for him. What was even more interesting to the teachers is the connection between the area of

⁴ The first author as participant observer

this rectangle and what one would have obtained if they were to abstractly manipulate the given algebraic expressions. Mia commented: “This is actually what we would get if we were to expand”. Heita and Joe went ahead to symbolically expand $(x + 2)(x + 3)$ and ended up obtaining $x^2 + 5x + 6$. The group tried a couple of more examples while comparing the areas constructed with the physical objects – algebra tiles – always matched with the product of the symbolic expansions.

At the conclusion of the session, Mia realised that the way we typically teach learners to expand algebraic expressions makes little sense as compared to how the tiles clarify this action. What we usually tell learners to do is “multiply first terms together, then the outside terms, then inside terms and then the last terms together” (the First-Outside-Inside-Last, i.e. FOIL algorithm). For example in the case of $(x + 2)(x + 3)$, learners would be told (or expected) to first multiply x by x (giving x^2), then x by 3 plus 2 by x (giving $3x + 2x$) and then 2 by 3 (giving 6), which is exactly the same as what is obtained with the tiles. In this way, the algebra tiles allow us to visualise what we are unable to see when manipulating algebraic expressions. They also help in visualising the geometric shapes – rectangles – and their relationship to algebra that we seem to take for granted in most cases.

Factorising algebraic expressions using algebra tiles

The next time the professional learning group explored the algebra tiles, the session took a turn different from what was initially planned for. After the group reflected on what happened in the previous session of working with algebra tiles, one of the teachers (Mia) had a very interesting question concerning the applicability of algebra tiles to other aspects of algebra. This is how the conversation proceeded:

- Mia: Maybe before you go ahead- somebody asked me a question whether these tiles can be used, for example, in algebraic expressions that um...for example, quadratic expressions which cannot be factorized. How do the tiles help here? We are unable to use them to do other things.
- Miranda: To do what, like to factorize?
- Mia: Yes, to factorize. Let’s say I want to...for example, if I have x squared plus three x plus one. I could not factorize it with the tiles.
- Miranda: Okay, that is a very good question. Let’s see...what would we normally do to factorize that before we came to the tiles? Are you saying it would not factor at all?
- Mia: It would factor but not in the way that we factorize other expressions.
- Miranda: How would we factor it?
- Mia: We can complete the square.
- Miranda: Okay, oh, completing the square? I am sure we can do that with the tiles. Why don’t we try it and see what we get? We can do almost anything with these things.
- Mia: Oh, really?

Mia insisted that the group specifically look at the expression she mentioned above ($x^2 + 3x + 1$). Everybody was then requested to take out the algebraic tiles whose total area is equivalent to this particular expression (see Figure 5).

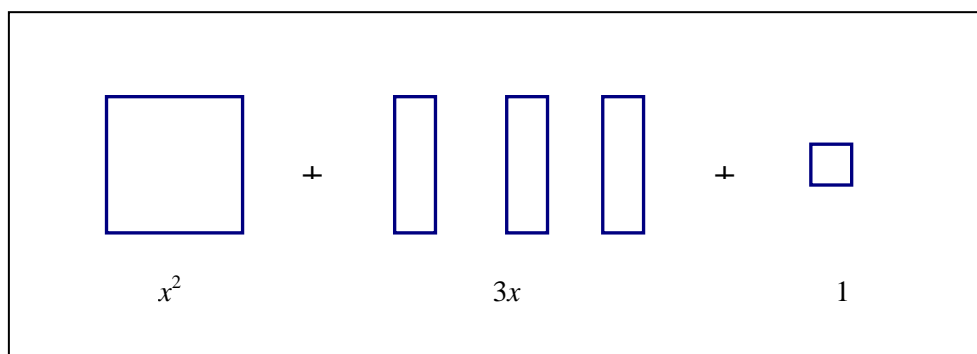


Figure 5: Tiles with area totalling $x^2 + 3x + 1$

After it was agreed that the tiles in Figure 5 had an area of $(x^2 + 3x + 1)$, the teachers tried to organise them into a rectangle which did not work because they obtained a rectangle measuring $(x)(x + 3)$ and then one unit remaining (see Figure 6).

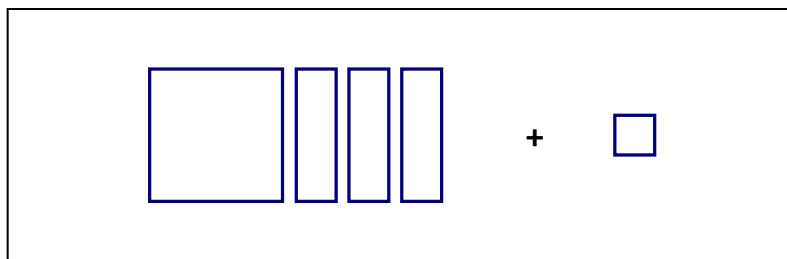


Figure 6: A representation of expressions that do not factor

The group was then reminded of Mia’s suggestion that since the expression $(x^2 + 3x + 1)$ would not factor right away they needed to complete a square. Then Miranda invited the group to consider the meaning of the phrase “completing the square,” especially in geometric terms. After some discussion, the group adopted an understanding that what this means is that the pieces *x squared*, *three x* and *one* should be used to form a rectangle that is a perfect square – in other words with all four sides equal in length. “How do we do that?” Miranda probed.

Another teacher, Seth, suggested that “we should add more pieces until all sides are equal”. Later it was realised that this would change the original expression. And this is exactly what happens when one symbolically completes the square as long as we keep track of how much more we add. Another teacher suggested: “first divide all that we have among the two sides [length and width], so that each side gets the same share of what we have”. This was done and is illustrated in the steps in Figure 7.

The piece *x squared* works out nicely because its length and width are already equal. In step one, a blue strip (x) is placed on each side – length and width. This leaves one blue strip unused, which must be further cut longitudinally into two equal parts with length x and width $\frac{1}{2}$ (step 2). In the third step, the remaining tile, with an area of 1 is added, leaving more open space to be filled.

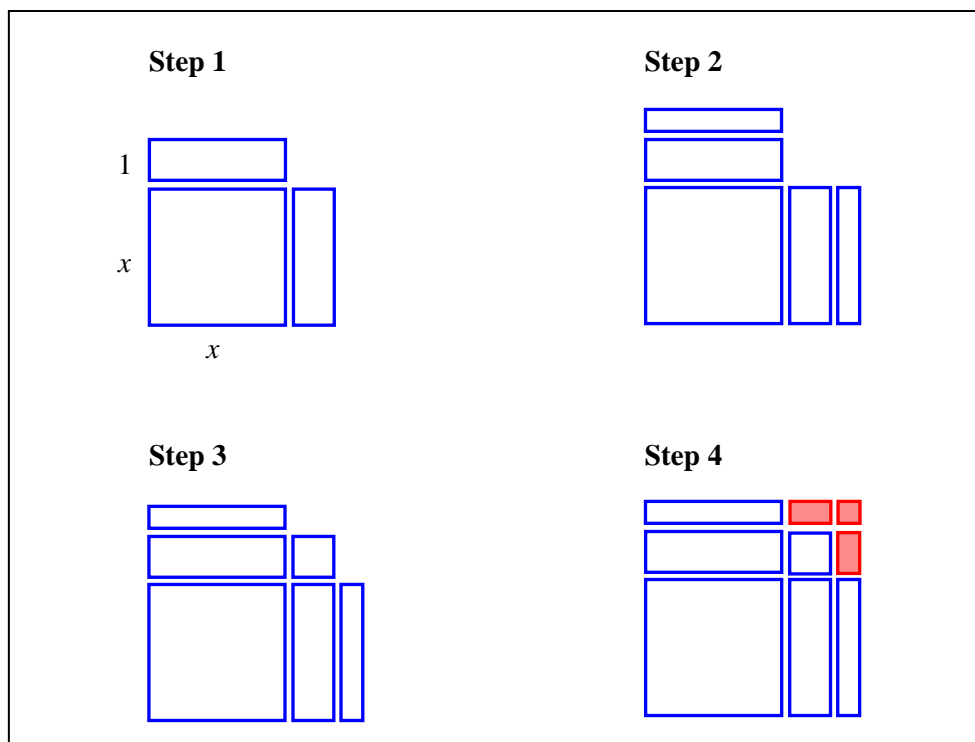


Figure 7: Completing a square geometrically

The next challenge was to figure out how much area needs to be added in order to fill the open space, hence completing the square. To answer this question, the group concentrated on the pieces that had been added (the shaded pieces in Figure 7). These measured $\frac{1}{2}$ by 1, $\frac{1}{2}$ by $\frac{1}{2}$, and 1 by $\frac{1}{2}$. These gave areas of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{2}$ respectively. In total these three have an area of $1\frac{1}{4} = \frac{5}{4}$. Interpreting this means that the three shaded pieces that were added in order to complete the rectangle into a square, had an area of $\frac{5}{4}$.

Next the group revised the well-known algorithm used to complete the square through symbolic abstraction. This is further represented in Figure 8.

Checking with symbolic abstraction methods

$x^2 + 3x + 1$

Using the well-known algorithm for completing the square, we would go through the following steps:

STEP 1: Half the coefficient of x : $\frac{3}{2}$

STEP 2: Square what you got in step 1: $\frac{9}{4}$

STEP 3: Subtract 1 from the answer in step 2: $\frac{9}{4} - 1 = \frac{5}{4}$

This would be the number we should add to our expression to complete a square and it is the same as the area of the tiles that we added to our rectangle to make it into a square.

Figure 8: *Completing a square symbolically*

Transparency with the use of algebra tiles

As noted earlier, transparency does not automatically come attached with any material resources. Rather, it becomes evident in the way the resources are being adapted and used in context. For example, one cannot claim that the tiles used in the activities discussed above were transparent in themselves. There was a need for explicit instructions on how the teachers could use the tiles to represent algebra. Hence, the transparency of the tiles lies in the way teachers were able to make connections between the geometric shapes of the tiles and their algebraic representations. The visibility of their geometric shapes, made the tiles visible and tangible, hence allowing for the facilitation of algebraic manipulations. This was however, not an advantage for all the teachers. For example, the tiles appeared too visible for Joe, because he could only see the geometric shapes of the tiles but could not read the algebra through them. His Aha moment of realising the link between the tiles and algebra surfaced when his colleagues intervened.

In addition, the transparency with the tiles enabled the teachers to move beyond the technicality of the tiles themselves by bringing up the ‘what if questions’ and questioning what more the tiles could help them do. For example, Mia suggested how the algebra tiles can be used to explain the FOIL strategy that is used when multiplying out factors of any algebraic expressions by opening their brackets. She further wondered if the tiles would help in factoring algebraic expressions that do not straightforwardly get factored by the FOIL strategy. Further manipulations of the tiles allowed for the realisation of the power and new understandings that teachers possess and that could now help them in interpreting learners’

possible meaning-making of algebra as opposed to the right vs. wrong interpretations that were displayed at the start of the project.

Apart from enabling the participants to use multiple representations of algebra, the algebraic tiles were helpful in the participants' learning of algebra in many different ways. It is not that these teachers did not, at the moment presented here, know how to expand or factorise algebraic expressions or to complete the square. They surely did, since they had been teaching this for quite a long time. However, in this study, the teachers were able to relearn those aspects of algebra again. They were learning how to expand and factorise expressions and complete the square differently. We say "differently" because even though they had learned these before, this was the first time they did this kind of learning with the algebra tiles where they paid attention to learners' ways of interpreting mathematical problems. So, in the middle of doing what was familiar to them through the actions of manipulating algebra tiles and within the inter-actions among themselves, their capabilities were stretched to a certain extent.

Having said that then, it becomes evident that on one hand the algebra tiles served as a way of re-sourcing the teachers with different perspectives and approaches of teaching and learning algebra. On the other hand the project provided (resourced) the teachers with materials and activities that could be used in exploring algebra, in addition to other resources they might already have. There was a point where the teachers were able to set aside the concrete algebra tiles and start to make drawings in their books. As they became more fluent with the diagrammatic representation of the tiles, they could solve more algebraic problems much quicker.

We now return to the vignette presented at the beginning of this paper's discussion of learners' attempt to find x if the area of a rectangle measuring $(x + 2)$ by $(x - 1)$ exceeds the area of a square of side x by 2 cm^2 . The point here is not to claim that algebra tiles would have helped the learners trying to solve this problem. Instead, we are trying to point to the range of possibilities of mathematical behaviours that could have emerged among the learners, should they have been exposed to algebra tiles prior to the area problem. Figure 9 presents one possible way the algebra tiles could have helped in the representation of the problem and hence the meaning making and comparison of the areas of the two shapes (rectangle and square).

Of course as mentioned earlier, there are many factors, such as language barriers, that should be considered when analysing learners' difficulties with problems of the kind in Figure 1. Yet one cannot ignore the range of possibilities that familiarity with algebra tiles could have opened up, and hence reducing the difficulties of mathematical interpretations.

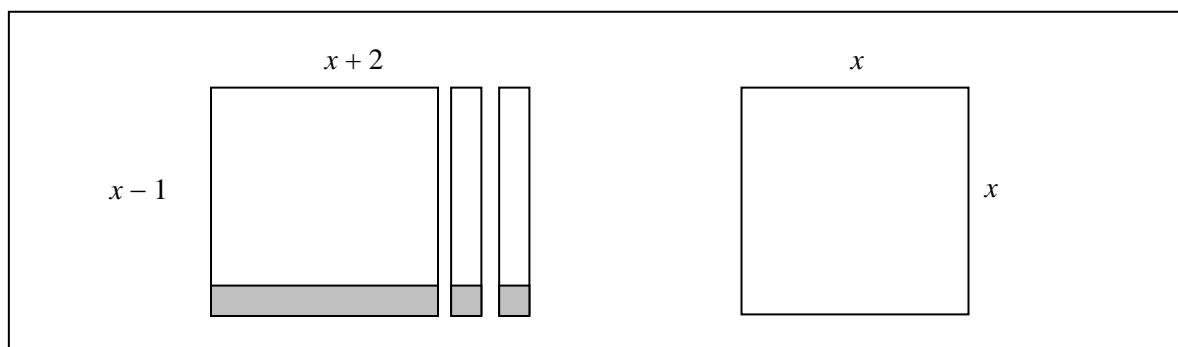


Figure 9: Possible representation and interpretation of problem in Figure 1

Conclusion

This paper is about learning algebra through the use of manipulatives called algebra tiles through teacher professional development. The teachers first learned about the tiles as a new means of teaching materials before they were able to utilise them. In their learning they were able to help each other make sense of the tiles and the different geometric constructions they did with the tiles. This can be seen for instance in the case where Mia and Heita assisted Joe to make sense of his own constructed tiles of the expression $(x + 2)(x + 3)$.

Apart from the geometric shape that they encompass the algebra tiles also have some sense of measurement in them. Such a measurement allows for flexibility because one can choose any parameters or variables to assign to the dimensions of each tile, depending on the nature of the problem being investigated and the meaning being construed. The paper also discussed the importance of ensuring that there is transparency with the use of any material resources by guiding the teachers how to use the materials. This will enable the teachers to realise that teaching and learning materials do not come with automatic transparency attached to them. Rather, there is need to teach learners and help them to make sense of the materials and use them to make mathematical meaning through them.

The participating teachers learned different ways of helping learners interpret and solve algebraic problems with the use of algebraic tiles. However, there is still need for curriculum to make specific stipulations with regard to the use of manipulatives as part of mathematics teaching and assessment.

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