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Weakly $\breve{\theta}$-$\mathcal{J}$-Closed Sets and Weakly $\breve{\theta}$-$\mathcal{J}$-continuous functions with respect to an Ideal Topological Spaces

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Abstract

In this paper, we introduce $\breve{\theta}$-$\mathcal{J}$-closed sets, $\breve{\theta}$-$\mathcal{J}_\alpha$-closed sets, $\breve{\theta}$-$\mathcal{J}$-continuous functions and $\breve{\theta}$-$\mathcal{J}_\alpha$-continuous functions and investigate their properties and its characterizations. After that we introduce weakly $\breve{\theta}$-$\mathcal{J}$-continuous functions and study the relationship between other types of continuous functions with suitable examples.

Keywords: $\breve{\theta}$-$\mathcal{J}$-cld, w$\breve{\theta}$-$\mathcal{J}$-cld, w$\breve{\theta}$-$\mathcal{J}$-openfunction, w$\breve{\theta}$-$\mathcal{J}$-continuous.

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1.Introduction

The advent of generalized closed sets, a new trend was created by Sundaram and Sheik John. Accordingly, a new notion called weakly closed sets were introduced by them. In this direction, many modifications of weakly closed sets are being introduced by the modern topologists due to their requirements. In this way, Ravi et al. [18] introduced weakly πg-closed sets and Sundaram and Nagaveni [24] introduced weakly g-closed sets in topological spaces. In 1961 Levine [14] obtained a decomposition of continuity. Later Professor Rose improved Levine's decomposition. In 1986 Tong [26] obtained a decomposition of continuity and proved that his decomposition is independent of Levine's. In 1989, Tong [27] improved upon his earlier decomposition and obtained yet another decomposition of continuity. In 1990, Ganster and Reilly [8] obtained a decomposition of continuity improving the first result of Tong. A. Acikgoz and et al. [1], introduced on α-I-continuous and α-I-open functions. J. Antony Rex Rodrigo and et al. [2], the introduced the mildly-I-locally closed sets and decompositions of ⋆-continuity .

K. Kuratowski [11], introduced topology. S. Jafari and N. Rajesh [12], introduced the generalized closed sets with respect to an ideal. N. Levine [13], introduced the generalized closed sets in topology. O. Njastad [17], introduced the on some classes of nearly open sets. R. Devi, K. Balachandran and H. Maki [4], introduced the semi-generalized closed maps and generalized semi-closed maps. Devi, R., Balachandran, K. and Maki, H.[5], introduced the on generalized α -continuous maps and α -generalized continuous maps. Devi, R., Balachandran, K. and Maki, H.[6], introduced the semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces.

Dontchev, J.[7], introduced the on generalizing semi-preopen sets. Levine, N.[15], introduced the semi-open sets and semi-continuity in topological spaces. A. S. Mashhour, et al. [16], introduced the α -continuous and α -open mappings. Rajamani, M. and Viswanathan, K.[19], introduced the on αgs-continuous maps in topological spaces. V. Renukadevi [20], introduced the note on IR-closed and AIR-sets. Sundaram, P. and et al. [22], introduced the semi-generalized continuous maps and semi-$T_{1/2}$-spaces. Sundaram, P.[23], introduced the study on generalizations of continuous maps in topological spaces. P. Sundaram and M. Rajamani [25], introduced some decompositions of regular generalized continuous maps in topological spaces. Veera Kumar, M. K. R. S.[28], introduced the between semi-closed sets and semi pre-closed sets.

In this paper, we introduce $\breve{\theta}$-$\mathscr{J}$-closed sets, $\breve{\theta}$-$\mathcal{J}_\alpha$-closed sets, $\breve{\theta}$-$\mathcal{J}$-continuous functions and $\breve{\theta}$-$\mathcal{J}_\alpha$-continuous functions and investigate their properties and its characterizations. After that we introduce weakly $\breve{\theta}$-$\mathscr{J}$-continuous functions and study the relationship between other types of continuous functions with suitable examples.

2. Preliminaries

An ideal $\mathcal{J}$ on a topological space (briefly, TPS) (X, τ) is a nonempty collection of subsets of X which satisfies
(1) $A\in\mathcal{J}$ and $B\subseteq A\Rightarrow B\in\mathcal{J}$ and
(2) $A\in\mathcal{J}$ and $B\in\mathcal{J}\Rightarrow A\cup B\in\mathcal{J}$.
Given a topological space (X, τ) with an ideal $\mathcal{J}$on X if $\wp(X)$ is the set of all subsets of X, a set operator $(\bullet)^\star$: $\wp(X)\rightarrow\wp(X)$, called a local function [10] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A\subseteq X$, $A^\star(\mathcal{J},\tau)=\{x\in X : U\cap A\notin\mathcal{J}$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau : x\in U\}$. A Kuratowski closure operator $cl^\star(\bullet)$ for a topology $\tau^\star(\mathcal{J},\tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A\cup A^\star(\mathcal{J},\tau)$ [10]. We will simply write $A^\star$ for $A^\star(\mathcal{J},\tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{J},\tau)$. If $\mathcal{J}$ is an ideal on X, then (X, τ, $\mathcal{J}$) is called an ideal topological space(briefly, ITPS). A subset A of an ideal topological space (X, τ, $\mathcal{J}$) is $\star$-closed (briefly, $\star$-cld) [10] if $A^\star\subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathcal{J}))$ is denoted by $int^\star(A)$.

Definition 2.1 A subset K of a TPS X is called:
(i) semi-open set [9] if $K\subseteq cl(int(K))$;
(ii) regular open set [21] if $K = int(cl(K))$;
The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.2 A subset K of a TPS X is called
(i) g-closed set (briefly, g-cld) [13] if $cl(K)\subseteq V$ whenever $K\subseteq V$ and V is open.
(ii) semi-generalized closed (briefly, sg-cld)[8] if $scl(K)\subseteq V$ whenever $K\subseteq V$ and V is semi-open.
(iii) generalized semi-closed (briefly, gs-cld)[29] if $scl(K)\subseteq V$ whenever $K\subseteq V$ and V is open.
The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3 A subset K of a ITPS X is called
(i) $\mathcal{J}_g$-closed (briefly, $\mathcal{J}_g$-cld) set [9] if $K^*\subseteq V$ whenever $K\subseteq V$ and V is open.
The complements of the above-mentioned closed sets are called their respective open sets.

Definition 2.4 A subset A of a topological space X is called:
(i) a weakly g-closed (briefly,wg-cld) set [24] if $cl(int(A))\subseteq U$ whenever $A\subseteq U$ and U is open in X.
(ii) a weakly πg-closed (briefly, wπg-cld) set [18] if $cl(int(A))\subseteq U$ whenever $A\subseteq U$ and U is π-open in X.

(iii) a regular weakly generalized closed (briefly, rwg-cld) set [18] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is regular open in X.

3. Weakly $\breve{\theta}$-$\mathcal{J}$-Closed Sets

We introduce the following definition:

Definition 3.1 A subset K of X is called

(i) $\breve{\theta}$-$\mathcal{J}$-closed (briefly, $\breve{\theta}$-$\mathcal{J}$-cld) if K* $\subseteq$ V whenever K $\subseteq$ V and V is sg-open.
The complement of $\breve{\theta}$-$\mathcal{J}$-cld is called $\breve{\theta}$-$\mathcal{J}$-open.
The family of all $\breve{\theta}$-$\mathcal{J}$-cld in X is denoted by $\breve{\theta}$-$\mathcal{J}$C(X).

(ii) $\breve{\theta}$-$\mathcal{J}_\alpha$-closed (briefly, $\breve{\theta}$-$\mathcal{J}_\alpha$-cld) if α cl(K*) $\subseteq$ V whenever K $\subseteq$ V and V is sg-open.
The complement of $\breve{\theta}$-$\mathcal{J}_\alpha$-cld is called $\breve{\theta}$-$\mathcal{J}_\alpha$-open.

(iii) A function f : (X, τ, $\mathcal{J}$) $\rightarrow$ (Y, σ) is called $\breve{\theta}$-$\mathcal{J}$-continuous if the inverse image of every closed set in Y is $\breve{\theta}$-$\mathcal{J}$-cld set in X.

(iv) $\star$-continuous if f^{-1}(V) is a $\star$-cld set in X, for every closed set in Y.

(v) A function f : (X, τ, $\mathcal{J}$) $\rightarrow$ (Y, σ) is called $\breve{\theta}$-$\mathcal{J}_\alpha$-continuous if f^{-1}(V) is a $\breve{\theta}$-$\mathcal{J}_\alpha$-cld set in X, for every closed set in Y.

(vi) A subset A of an ideal topological space (X, τ, $\mathcal{J}$) is called a weakly $\breve{\theta}$-$\mathcal{J}$-closed (briefly, w$\breve{\theta}$-$\mathcal{J}$-cld) set if (int(A))*$\subseteq$V whenever A$\subseteq$V and V is sg-open in X.

Theorem 3.2 Every $\breve{\theta}$-$\mathcal{J}$-cld set is w$\breve{\theta}$-$\mathcal{J}$-cld but not conversely.

Example 3.3 Let X = {p, q, r} and τ = {ϕ, {p, q}, X} with $\mathcal{J}$ = {ϕ}. Then the set {p} is w$\breve{\theta}$-$\mathcal{J}$-cld set but it is not a $\breve{\theta}$-$\mathcal{J}$-cld in X.

Theorem 3.4 Every w$\breve{\theta}$-$\mathcal{J}$-cld set is wg-cld but not conversely.

Proof

Let H be any w$\breve{\theta}$-$\mathcal{J}$-cld set and V be any open set containing H. Then V is a sg-open set containing H. We have (int(H))*$\subseteq$V. Thus, H is wg-cld.

Example 3.5 Let X = {p, q, r} and τ = {ϕ, {p}, X} with $\mathcal{J}$ = {ϕ}. Then the set {p, q} is wg-cld but it is not a w$\breve{\theta}$-$\mathcal{J}$-cld.

Theorem 3.6 Every w$\breve{\theta}$-$\mathcal{J}$-cld set is wπg-cld but not conversely.

Proof

Let H be any w$\breve{\theta}$-$\mathcal{J}$-cld set and V be any π-open set containing H. Then V is a sg-open set containing H. We have (int(H))*$\subseteq$V. Thus, H is wπg-cld.

Example 3.7 In Example 3.5, the set {p, r} is wπg-cld but it is not a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld.

Theorem 3.8 Every w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld set is rwg-cld but not conversely.
Proof
Let H be any w$\breve{\theta}$-$\mathcal{J}$-cld set and V be any regular open set containing H. Then V is a sg-open set containing H. We have (int(H))*$\subseteq$V. Thus, H is rwg-cld.

Example 3.9 In Example 3.5, the set {p} is rwg-cld but it is not a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld.

Theorem 3.10 If a subset H of an ideal topological space X is both $\star$-cld and α g-cld, then it is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld in X.
Proof
Let H be a α g-cld set in X and V be any open set containing H. Then V$\supseteq\alpha$ cl(H) = H$\cup$ (int(H*))*. Since H is$\star$-cld, V$\supseteq$ (int(H))*and hence w$\breve{\theta}$-$\mathcal{J}$-closed in X.

Theorem 3.11 If a subset H of an ideal topological space X is both open and w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld, then it is $\star$-cld.
Proof
Since H is both open and w$\breve{\theta}$-$\mathcal{J}$-cld, H$\supseteq$ (int(H))* = H* and hence H is $\star$-cld in X.

Corollary 3.12 If a subset H of an ideal topological space X is both open and w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld, then it is both regular open and regular closed in X.

Theorem 3.13 Let X be an ideal topological space and H$\subseteq$ X be open. Then, H is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld if and only if H is $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld.
Proof Let H be $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld. By Proposition 3.2, it is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld.
Conversely, let H be w$\breve{\theta}$-$\mathcal{J}$-cld. Since H is open, by Theorem 3.11, H is $\star$-cld. Hence H is $\breve{\theta}$-$\mathcal{J}$-cld.

Theorem 3.14 A set H is w$\star$-cld if and only if (int(H))*–H contains no non-empty sg-cld set.
Proof
Necessity. Let G be a sg-cld set such that G$\subseteq$ (int(H))*–H. Since G^c is sg-open and H$\subseteq G^c$, from the definition of w$\breve{\theta}$-$\mathcal{J}$-closedness it follows that (int(H))*$\subseteq G^c$. i.e., G$\subseteq$ $((\text{int}(H))^*)^c$. This implies that G$\subseteq$ ((int(H))*) $\cap$ $((\text{int}(H))^*)^c = \phi$.
Sufficiency. Let H$\subseteq$J, where J is *-cld and sg-open set in X. If (int(H))* is not contained in J, then (int(H))*$\cap J^c$ is a non-empty sg-cld subset of (int(H))*–H, we obtain a contradiction. This proves the sufficiency and hence the theorem.

Theorem 3.15 Let X be an ideal topological space and H$\subseteq$ Y $\subseteq$X . If H is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld in X, then H is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld relative to Y.

Proof

Let $H \subseteq Y \cap J$ where J is sg-open in X. Since H is $w\breve{\theta}$-$\mathcal{I}$-cld in X, $H \subseteq J$ implies $(int(H))^* \subseteq J$. That is $Y \cap ((int(H))^*)^c \subseteq Y \cap J$ where $Y \cap (int(H))^*$ is closure of interior of H in Y. Thus, H is $w\breve{\theta}$-$\mathcal{I}$-cld relative toY.

Theorem 3.16 If a subset H of an ideal topological space X is nowhere dense, then it is $w\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{I}}$-cld.

Proof

Since $int(H) \subseteq int(H^*)$ and H is nowhere dense, $int(H) = \phi$. Therefore $(int(H))^* = \phi$ and hence H is $w\breve{\theta}$-$\mathcal{I}$-cld in X.

The converse of Theorem 3.16 need not be true as seen in the following example.

Example 3.17 Let X = {p, q, r} and τ = {ϕ, {p}, {q, r}, X} with $\boldsymbol{\mathcal{I}}$ = {ϕ}. Then the set {p} is $w\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{I}}$-cldset but not nowhere dense in X.

Remark 3.18

The following examples show that $w\breve{\theta}$-$\mathcal{I}$-closedness and semi-closedness are independent.

Example 3.19

In Example 3.3, we have the set {p, r} is $w\breve{\theta}$-$\mathcal{I}$-cldset but not semi-cld in X.

Example 3.20

Let X = {p, q, r} and τ = {ϕ, {p}, {q}, {p, q}, X} with $\mathcal{I}$ = {ϕ}. Then the set {p} is semi-cld set but not $w\breve{\theta}$-$\mathcal{I}$-cld in X.

Remark 3.21

From the above discussions and known results in [18]. We obtain the following diagram, where $A \rightarrow B$ represents A implies B but not conversely.

Diagram

$\star$-cld$\rightarrow$$w\breve{\theta}$-$\mathcal{I}$-closed $\rightarrow$wg-closed $\rightarrow$ wπg-closed $\rightarrow$rwg-closed

Definition 3.22 A subset H of an ideal topological space X is called $w\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{I}}$-open set if H^c is $w\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{I}}$-cld in X.

Proposition3.23

(i) Every $\breve{\theta}$-$\mathcal{I}$-open set is $w\breve{\theta}$-$\mathcal{I}$-open but not conversely.

(ii) Every g-open set is $w\breve{\theta}$-$\mathcal{I}$-open but not conversely.

Theorem 3.24 A subset H of an ideal topological space X is $w\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{I}}$-open if $J \subseteq int(H^*)$ whenever $J \subseteq H$ and J is sg-cld.

Proof

Let H be any w$\breve{\theta}$-$\mathcal{J}$-open. Then H^c is w$\breve{\theta}$-$\mathcal{J}$-cld. Let J be a sg-cld set contained in H. Then J^c is a sg-open set containing H^c. Since H^c is w$\breve{\theta}$-$\mathcal{J}$-cld, we have $(\mathrm{int}(H^c))^* \subseteq J^c$. Therefore $J \subseteq \mathrm{int}(H^*)$.

Conversely, we suppose that $J \subseteq \mathrm{int}(H^*)$ whenever $J \subseteq H$ and J is sg-cld. Then J^c is a sg-open set containing H^c and $J^c \supseteq (\mathrm{int}(H^*))^c$. It follows that $J^c \supseteq (\mathrm{int}(H^c))^*$. Hence H^c is w$\breve{\theta}$-$\mathcal{J}$-cld and so A is w$\breve{\theta}$-$\mathcal{J}$-open.

4. Weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous functions

Definition 4.1 Let X and Y be two an ideal topological space. A function $f : X \to Y$ is called weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous (briefly, w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous) if $f^{-1}(V)$ is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open set in X for each open set Vof Y**.**

Example 4.2 Let X = Y = {p, q, r}, $\tau = \{\phi, \{p\}, \{q, r\}, X\}$ with $\boldsymbol{\mathcal{J}} = \{\phi\}$ and $\sigma = \{\phi, \{p\}, Y\}$. The function $f : (X, \tau, \boldsymbol{\mathcal{J}}) \to (Y, \sigma)$ defined by f(p) = q, f(q) = r and f(r) = p is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous, because every subset of Y is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld in X.

Theorem 4.3 Every $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous function is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous.

Proof

It follows from Proposition 3.23 (i).

The converse of Theorem 4.3 need not be true as seen in the following example.

Example 4.4 Let X = Y = {p, q, r}, $\tau = \{\phi, \{p\}, \{q, r\}, X\}$ with $\boldsymbol{\mathcal{J}} = \{\phi\}$ and $\sigma = \{\phi, \{q\}, Y\}$. Let $f : (X, \tau, \boldsymbol{\mathcal{J}}) \to (Y, \sigma)$ be the identity function. Then f is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous but not $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous.

Theorem 4.5 A function $f : X \to Y$ is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous if and only if $f^{-1}(V)$ is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld set in X for each closed set Vof Y.

Proof

Let V be any closed set of Y. According to the assumption $f^{-1}(V^c) = X \setminus f^{-1}(V)$ is w$\breve{\theta}$-$\mathcal{J}$-open in X, so $f^{-1}(V)$ is w$\breve{\theta}$-$\mathcal{J}$-cld in X.

The converse can be proved in a similar manner.

Definition 4.6 An ideal topological space X is said to be locally $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-indiscrete if every $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open set of X is $\star$-cld in X.

Theorem 4.7 Let $f : X \to Y$ be a function. If f is $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous and X is locally $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-indiscrete, then f is $\star$-continuous.

Proof. Let W be an open in Y. Since f is $\breve{\theta}$-$\mathcal{J}$-continuous, $f^{-1}(W)$ is $\breve{\theta}$-$\mathcal{J}$-open in X. Since X is locally $\breve{\theta}$-$\mathcal{J}$-indiscrete, $f^{-1}(W)$ is $\star$-cld in X. Hence f is $\star$-continuous.

Theorem 4.8 Let f : X → Y be a function. If f is contra $\breve{\theta}$-$\mathcal{J}$-continuous and X is locally $\breve{\theta}$-$\mathcal{J}$-indiscrete, then f is w$\breve{\theta}$-$\mathcal{J}$-continuous.

Proof

Let f : X → Y be contra $\breve{\theta}$-$\mathcal{J}$-continuous and X is locally $\breve{\theta}$-$\mathcal{J}$-indiscrete. By Theorem 4.7, f is ⋆-continuous, then f is w$\breve{\theta}$-$\mathcal{J}$-continuous.

Proposition 4.9 If f : X → Y is perfectly continuous and w$\breve{\theta}$-$\mathcal{J}$-continuous, then it is R-map.

Proof

Let W be any regular open subset of Y. According to the assumption, $f^{-1}(W)$ is both open and ⋆-cld in X. Since $f^{-1}(W)$ is⋆-cld, it is wg-closed. We have $f^{-1}(V)$ is both open and wg-closed. Hence, it is regular open in X, so f is R-map.

Definition 4.10 An ideal topological space X is called $\breve{\theta}$-$\mathcal{J}$-compact if every cover of X by $\breve{\theta}$-$\mathcal{J}$-open sets has finite subcover.

Definition 4.11 An ideal topological space X is weakly $\breve{\theta}$-$\mathcal{J}$-compact (briefly, w$\breve{\theta}$-$\mathcal{J}$-compact) if every w$\breve{\theta}$-$\mathcal{J}$-open cover of X has a finite subcover**.**

Remark 4.12 Every w$\breve{\theta}$-$\mathcal{J}$-compact space is $\breve{\theta}$-$\mathcal{J}$-compact.

Theorem 4.13 Let f : X → Y be surjective w$\breve{\theta}$-$\mathcal{J}$-continuous function. If X is w$\breve{\theta}$-$\mathcal{J}$-compact, then Y is compact.

Proof

Let $\{A_i : i \in I\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in I\}$ is a w$\breve{\theta}$-$\mathcal{J}$-open cover in X. Since X is w$\breve{\theta}$-$\mathcal{J}$-compact, it has a finite subcover, say$\{f^{-1}(A_1), f^{-1}(A_2),\ldots, f^{-1}(A_n)\}$. Since f is surjective $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of Y and hence Y is compact.

Definition 4.14 An ideal topological space X is weakly $\breve{\theta}$-$\mathcal{J}$-connected (briefly, w$\breve{\theta}$-$\mathcal{J}$-connected) if X cannot be written as the disjoint union of two non-empty w$\breve{\theta}$-$\mathcal{J}$-open sets.

Theorem 4.15 If an ideal topological space X is w$\breve{\theta}$-$\mathcal{J}$-connected, then X is almost connected (resp. $\breve{\theta}$-$\mathcal{J}$-connected).

Proof

It follows from the fact that each regular open set (resp. $\breve{\theta}$-$\mathcal{J}$-open set) is w$\breve{\theta}$-$\mathcal{J}$-open.

Theorem 4.16 For an ideal topological space X, the following statements are equivalent:

i. X is w$\breve{\theta}$-$\mathcal{J}$-connected.

ii. The empty set ϕ and X are only subsets which are both w$\breve{\theta}$-$\mathcal{J}$-open and w$\breve{\theta}$-$\mathcal{J}$-cld.

iii. Each w$\breve{\theta}$-$\mathcal{J}$-continuous function from X into a discrete space Y which has at least two points is a constant function.

Proof

(i) $\Rightarrow$ (ii). Let S $\subseteq$ X be any proper subset, which is both w$\breve{\theta}$-$\mathcal{J}$-open and w$\breve{\theta}$-$\mathcal{J}$-cld. Its complement X \ S is also w$\breve{\theta}$-$\mathcal{J}$-open and w$\breve{\theta}$-$\mathcal{J}$-cld. Then X = S $\cup$ (X \ S) is a disjoint union of two non-empty w$\breve{\theta}$-$\mathcal{J}$-open sets which is a contradiction with the fact that X is w$\breve{\theta}$-$\mathcal{J}$-connected. Hence, S = ϕ or X.

(ii) $\Rightarrow$ (i). Let X = A $\cup$ B where A $\cap$ B = ϕ, A $\neq\phi$, B $\neq\phi$ and A, B are w$\breve{\theta}$-$\mathcal{J}$-open. Since A = X \ B, A is w$\breve{\theta}$-$\mathcal{J}$-closed. According to the assumption A = ϕ, which is a contradiction.

(ii) $\Rightarrow$ (iii). Let f : X $\to$ Y be a w$\breve{\theta}$-$\mathcal{J}$-continuous function where Y is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is w$\breve{\theta}$-$\mathcal{J}$-closed and w$\breve{\theta}$-$\mathcal{J}$-open for each y $\in$Y and X = $\cup\{f^{-1}(\{y\}) \mid y \in Y\}$. According to the assumption, $f^{-1}(\{y\}) = \phi$ or $f^{-1}(\{y\})$ = X. If $f^{-1}(\{y\}) = \phi$ for all y $\in$Y, f will not be a function. Also there is no exist more than one y $\in$Y such that$f^{-1}(\{y\})$ = X. Hence, there exists only one y $\in$Y such that $f^{-1}(\{y\})$ = X and $f^{-1}(\{y_1\}) = \phi$ where $y \neq y_1 \in$ Y. This shows that f is a constant function.

(iii) $\Rightarrow$ (ii). Let S$\neq\phi$ be both w$\breve{\theta}$-$\mathcal{J}$-open and w$\breve{\theta}$-$\mathcal{J}$-closed in X. Let f : X $\to$ Y be a w$\breve{\theta}$-$\mathcal{J}$-continuous function defined by f(S) = {a} and f(X \ S) = {b} where a $\neq$ b. Since f is constant function,we get S = X.

Theorem 4.17 Let f : X $\to$ Y be a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous surjective function. If X is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-connected, then Y is connected.

Proof

We suppose that Y is not connected. Then Y = A $\cup$ B where A $\cap$ B = ϕ, A $\neq\phi$, B $\neq\phi$ and A, B are open sets in Y. Since f is w$\breve{\theta}$-$\mathcal{J}$-continuous surjective function, X = $f^{-1}(A) \cup f^{-1}(B)$ are disjoint union of two non-empty w$\breve{\theta}$-$\mathcal{J}$-open subsets. This is contradiction with the fact that X is w$\breve{\theta}$-$\mathcal{J}$-connected.

5. Weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-Open Functions and Weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-Closed Functions

Definition 5.1 Let X and Y be an ideal topological space. A function f : X $\to$ Y is called weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open (briefly, w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open) if f(V) is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open set in Y for each open set V of X**.**

Remark 5.2 Every $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open function is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open but not conversely.

Example 5.3 Let X = Y = {p, q, r, s}, τ = {ϕ, {p}, {p, q, s}, X} with $\boldsymbol{\mathcal{J}}$ = {ϕ} and σ = {ϕ, {p}, {q, r}, {p, q, r}, Y}. Let f : (X, τ, $\boldsymbol{\mathcal{J}}$) $\to$ (Y, σ) be the identity function. Then f is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open but not $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open**.**

Definition5.4 Let X and Y be an ideal topological space. A function f : X $\to$ Y is called weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-closed (briefly, w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld) if f(V) is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld set in Y for each closed set V of X**.**

It is clear that an open function is w$\breve{\theta}$-$\mathcal{J}$-open and a closed function is w$\breve{\theta}$-$\mathcal{J}$-cld.

Theorem 5.5 Let X and Y be an ideal topological space. A function f : X $\to$ Y is w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-closed if and only if for each subset B of Y and for each open set G containing $f^{-1}(B)$ there exists a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open set F of Y such that B $\subseteq$ F and$f^{-1}(F) \subseteq G$**.**

Proof

Let B be any subset of Y and let G be an open subset of X such that$f^{-1}(B) \subseteq G$. Then F = Y \ f(X \ G) is w$\breve{\theta}$-$\mathcal{J}$-open set containing B and $f^{-1}(F) \subseteq G$.

Conversely, let U be any closed subset of X. Then $f^{-1}(Y \setminus f(U)) \subseteq X \setminus U$ and X \ U is open. According to the assumption, there exists a w$\breve{\theta}$-$\mathcal{J}$-open set F of Y such that Y \ f(U) $\subseteq$ F and $f^{-1}(F) \subseteq X \setminus U$. Then $U \subseteq X \setminus f^{-1}(F)$. From $Y \setminus F \subseteq f(U) \subseteq f(X \setminus f^{-1}(F)) \subseteq Y \setminus F$ it follows that f(U) = Y \ F, so f(U) is w$\breve{\theta}$-$\mathcal{J}$-cld in Y. Therefore, f is a w$\breve{\theta}$-$\mathcal{J}$-cldfunction.

Remark 5.6 The composition of two w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld functions need not be a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld as we can see from the following example.

Example 5.7 Let X = Y = Z = {p, q, r}, τ = {ϕ, {p}, {p, q}, X} with $\boldsymbol{\mathcal{J}}$= {ϕ} and σ = {ϕ, {p}, {q, r}, Y} and η = {ϕ, {p, q}, Z} with $\boldsymbol{\mathscr{J}}$ **=** {ϕ}. We define f : (X, τ, $\boldsymbol{\mathcal{J}}$) $\to$ (Y, σ) by f(p) = r, f(q) = q and f(r) = p and let g : (Y, σ) $\to$ (Z, η, $\boldsymbol{\mathscr{J}}$) be the identity function. Hence both f and g are w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld functions. Hence the composition of two w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld functions need not be a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld.

Theorem 5.8 Let X, Y and Z be an ideal topological space. If f : X $\to$ Y is a $\star$-cld function and g : Y $\to$ Z is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld function, then gof : X $\to$ Z is a w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld function.

Theorem 5.9 A set K of X is $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open if and only if F $\subseteq$ int(K) whenever F is sg-cld and F $\subseteq$ K.

Proof

Suppose that F $\subseteq$ int(K) such that F is sg-cld and F $\subseteq$ K. Let $K^c \subseteq G$ where G is sg-open. Then $G^c \subseteq K$ and G^c is sg-cld. Therefore $G^c \subseteq int(K)$ by hypothesis. Since $G^c \subseteq int(K)$, we have $(int(K))^c \subseteq G$. i.e., $(K^c)^* \subseteq G$, since $(K^c)^* = (int(K))^c$. Thus, K^c is $\breve{\theta}$-$\mathcal{J}$-cld. i.e., K is $\breve{\theta}$-$\mathcal{J}$-open.

Conversely, suppose that K is $\breve{\theta}$-$\mathcal{J}$-open such that F $\subseteq$ K and F is sg-cld. Then F^c is sg-open and $K^c \subseteq F^c$. Therefore, $(K^c)^* \subseteq F^c$ by definition of $\breve{\theta}$-$\mathcal{J}$-cld and so F$\subseteq$ int(K), since $(K^c)^* = (int(K))^c$.

Lemma 5.10 For an $x \in X$, $x \in \breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cl(K) if and only if $V \cap K \neq \phi$ for every $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open set V containing x.

Proof

Let $x \in \breve{\theta}$-$\mathcal{J}$-cl(K) for any $x \in X$. To prove $V \cap K \neq \phi$ for every $\breve{\theta}$-$\mathcal{J}$-open set V containing x. Prove the result by contradiction. Suppose there exists a $\breve{\theta}$-$\mathcal{J}$-open set V containing x such that $V \cap K = \phi$. Then $K \subseteq V^c$ and V^c is $\breve{\theta}$-$\mathcal{J}$-cld. We have $\breve{\theta}$-$\mathcal{J}$-cl(K) $\subseteq V^c$. This shows that $x \notin \breve{\theta}$-$\mathcal{J}$-cl(K) which is a contradiction. Hence $V \cap K \neq \phi$ for every $\breve{\theta}$-$\mathcal{J}$-open set V containing x.

Conversely, let $V \cap K \neq \phi$ for every $\breve{\theta}$-$\mathcal{J}$-open set V containing x. To prove $x \in \breve{\theta}$-$\mathcal{J}$-cl(K). We prove the result by contradiction. Suppose $x \notin \breve{\theta}$-$\mathcal{J}$-cl(K). Then there exists a $\breve{\theta}$-$\mathcal{J}$-cld set F containing K such that $x \notin F$. Then $x \in F^c$ and F^c is $\breve{\theta}$-$\mathcal{J}$-open. Also, $F^c \cap K = \phi$, which is a contradiction to the hypothesis. Hence $x \in \breve{\theta}$-$\mathcal{J}$-cl(K).

Proposition 5.11

A function $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ is $\breve{\theta}$-$\mathcal{J}$-continuous if and only if $f^{-1}(U)$ is $\breve{\theta}$-$\mathcal{J}$-open in X, for every open set U in Y.

Proof

Let $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ be $\breve{\theta}$-$\mathcal{J}$-continuous and U be an open set in Y. Then U^c is closed in Y and since f is $\breve{\theta}$-$\mathcal{J}$-continuous, $f^{-1}(U^c)$ is $\breve{\theta}$-$\mathcal{J}$-cld in X. But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $\breve{\theta}$-$\mathcal{J}$-open in X.

Conversely, assume that $f^{-1}(U)$ is $\breve{\theta}$-$\mathcal{J}$-open in X, for each open set U in Y. Let F be a closed set in Y. Then F^c is open in Y and by assumption, $f^{-1}(F^c)$ is $\breve{\theta}$-$\mathcal{J}$-open in X. Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $\breve{\theta}$-$\mathcal{J}$-cld in X and so f is $\breve{\theta}$-$\mathcal{J}$-continuous.

Theorem 5.12 If $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ is $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous and pre-sg-closed and if A is an $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open (or $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld) subset of Y, then $f^{-1}(H)$ is $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-open (or $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-cld) in X**.**

Proof

Let H be an $\breve{\theta}$-$\mathcal{J}$-open set in Y and F be any sg-closed set in X such that $F \subseteq f^{-1}(H)$. Then $f(F) \subseteq H$. By hypothesis, f(F) is sg-closed and H is $\breve{\theta}$-$\mathcal{J}$-open in Y. Therefore, $f(F) \subseteq int(H)$ and so $F \subseteq f^{-1}(int(H))$. Since f is $\breve{\theta}$-$\mathcal{J}$-continuous and int(H) is open in Y, $f^{-1}(int(H))$ is $\breve{\theta}$-$\mathcal{J}$-open in X. Thus $F \subseteq int(f^{-1}(int(H))) \subseteq int(f^{-1}(H))$. i.e., $F \subseteq int(f^{-1}(H))$ and $f^{-1}(H)$ is $\breve{\theta}$-$\mathcal{J}$-open in X. By taking complements, we can show that if H is $\breve{\theta}$-$\mathcal{J}$-cld in Y, $f^{-1}(H)$ is $\breve{\theta}$-$\mathcal{J}$-cld in X.

6. Conclusions

The new class of generalized closed sets called weakly $\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-closed sets and w$\breve{\boldsymbol{\theta}}$-$\boldsymbol{\mathcal{J}}$-continuous functions are very useful for new research in topological spaces. This may leads some new applications in real life problems.

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