Ratio Mathematica Volume 45, 2023

Forcing vertex square free detour number of a graph

G. Priscilla Pacifica *
K. Christy Rani†

Abstract

Let G be a connected graph and S a square free detour basis of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique square free detour basis of S containing T. A forcing subset for S of minimum order is a minimum forcing subset of G. The forcing square free detour number of G is $fdn_{\square f_u}(G) = min\{fdn_{\square f_u}(S_u)\}$, where the minimum is taken over all square free detour bases S in G. In this paper, we introduce the forcing vertex square free detour sets. The general properties satisfied by these forcing subsets are discussed and the forcing square free detour number for a certain class of standard graphs are determined. We show that the two parameters $dn_{\square f_u}(G)$ and $fdn_{\square f_u}(G)$ satisfy the relationship $0 \leq fdn_{\square f_u}(G) \leq dn_{\square f_u}(G)$. Also, we prove the existence of a graph G with $fdn_{\square f_u}(G) = \alpha$ and $dn_{\square f_u}(G) = \beta$, where $0 \leq \alpha \leq \beta$ and $\beta \geq 2$ for some vertex u in G.
Keywords: forcing square free detour number; forcing vertex square free detour set; forcing vertex square free detour number.
2020 AMS subject classifications: 05C12, 05C69[1]

*Assistant Professor, Department of mathematics, St. Mary's College(Autonomous), Thoothukudi - 628 001; priscillamelwyn@gmail.com; e-mail@address.
†Research Scholar, Reg. No.: 20122212092002, Department of Mathematics, St. Mary's College(Autonomous), Thoothukudi - 628 001, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamilnadu, India ; christy.agnes@gmail.com
[1]Received on June 5, 2022. Accepted on August 25, 2022. Published on January 30, 2023. DOI: 10.23755/rm.v45i0.1030. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

In a simple connected graph $G = (V, E)$ of order at least two, a longest xy path is called an $x - y$ detour path for any two vertices x and y in G. A set S of vertices of G is a detour set of G if each vertex v of G lies on an $x - y$ detour path for some elements x and y in S. The minimum cardinality of a detour set of G is the detour number of G and is denoted by $dn(G)$. A subset T of a minimum detour set S of G is a forcing detour subset for S if S is the unique detour basis containing T. A forcing detour subset for S of minimum cardinality is a minimum forcing detour subset of S. The forcing detour number $fdn(S)$ of S, is the minimum cardinality of a forcing detour subset for S. The forcing detour number $fdn(G)$ of G, is $min\{fdn(S)\}$, where the minimum is taken over all detour bases S in G. This notion of forcing detour number was initiated by Chartrand et al. [2003]. From then onwards there have been many relevant studies about the forcing detour concept in the past few decades. The connected detour parameter was defined by Santhakumaran and Athisayanathan [2009]. Also, Santhakumaran and Athisayanathan [2010] focussed on edge of a graph and introduced forcing edge detour number and the forcing weak edge detour number. Further results on connected detour number was studied by Ali and Ali [2019]. Titus and Balakrishnan [2016] concentrated on a vertex of a graph and compiled the findings on forcing vertex detour monophonic number arising from the chordless path. Ramalingam et al. [2016] extended this concept to triangle free and extracted forcing results. The square free detour concept was introduced by Rani and Pacifica [2022]. Though there are several novel forcing parameters, this paper aims at filling some significant research gaps and brings forth the results on the forcing vertex square free detour number of a graph. We determine the bounds for it and compare the relationship with the vertex square free detour number of a graph. These ideas have interesting application in channel assignment problem and in radio technologies. Moreover, this concept can be applied in security based community design. For the basic terminologies we refer to Chartrand et al. [1993].

2 Preliminaries

Theorem 2.1. *Let u be any vertex of a connected graph G.*

(i) Every end-vertex of G other than the vertex u (whether u is an end-vertex or not) belongs to every u-square free detour set.

(ii) No cut-vertex of G belongs to any $dn_{\square_{fu}}$-set.

Theorem 2.2. *For any vertex u in a non-trivial connected graph G of order n, $1 \leq dn_{\square_{fu}}(G) \leq n - 1$.*

Theorem 2.3. *Let G be a connected graph.*

(i) If G is the complete graph K_n, then $dn_{\square_{f_u}}(G) = 1$ for every vertex u in K_n.

(ii) If G is the complete bipartite graph $K_{m,n}(2 \le m \le n)$ with partitions X and Y, then

$$dn_{\square_{f_u}}(G) = \begin{cases} m-1 & \text{if } u \in X \\ n-1 & \text{if } u \in Y \end{cases}$$

(iii) If G is the cycle C_n, then $dn_{\square_{f_u}}(G) = 1$for every vertex u in C_n.

(iv) For a wheel $W_n(n \ge 4)$,

$$dn_{\square_{f_u}}(W_n) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil & \text{if } u \in K_1.\ n \ge 4 \\ 2 & \text{if } u \in C_{n-1},\ n \ge 6 \\ 1 & \text{if } u \in C_{n-1},\ 4 \le n \le 5 \end{cases}$$

3 Forcing vertex square free detour number of a graph

Even though every connected graph contains a vertex square free detour set, some connected graph may contain several vertex square free detour sets. For each minimum vertex square free detour set S_u in a connected graph there is always some subset T of S_u that uniquely determines S_u as the minimum vertex square free detour set containing T such forcing subsets are considered in this section.

Definition 3.1. *Let u be any vertex of a connected graph G and S_u be a $dn_{\square_{f_u}}$ -set of G. A subset $F_u \subseteq S_u$ is called a u-forcing subset for S_u if S_u is the unique $dn_{\square_{f_u}}$ -set consisting of F_u. The u-forcing subset for S_u of minimum cardinality is a minimum u-forcing subset of S_u. The forcing u-square free detour number of S_u, denoted by $fdn_{\square_{f_u}}(S_u)$ is the order of a minimum u-forcing subset for S_u. The forcing u-square free detour number of G is $fdn_{\square_{f_u}}(G) = min fdn_{\square_{f_u}}(S_u)$ where the minimum is considered over all $dn_{\square_{f_u}}$ -sets S_u in G.*

Example 3.1. *For the graph G shown in Figure 1, the only $dn_{\square_{f_{v_1}}}$ -sets are $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_4\}$ and $\{v_3, v_5\}$. Hence $fdn_{\square_{f_u}}(G) = 2$. Also $dn_{\square_{f_u}}(G) = 2$ and $fdn_{\square_{f_u}}(G) = 1$ for $u = v_3$ and v_4 in G. Moreover v_5 and v_4 are the unique vertex square free detour sets for vertices v_2 and v_5 respectively and so $fdn_{\square_{f_u}}(G) = 0$ for $u = v_2, v_5$.*

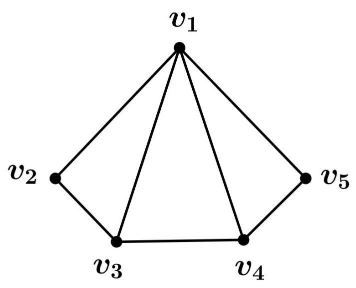

Figure 1: G

The following theorem follows from the definitions of vertex square free detour number and forcing vertex square free detour number of a graph G.

Theorem 3.1. *For any vertex u in a connected graph G, $0 \leq fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$.*

Proof. Let u be any vertex of G. From the definition of $fdn_{\square_{fu}}(G)$, we find that $fdn_{\square_{fu}}(G) \geq 0$, consider a $dn_{\square_{fu}}$-set S_u in G. We have $fdn_{\square_{fu}}(G) = min\{fdn_{\square_{fu}}(S_u) : S_u$ is a $dn_{\square_{fu}}$-set in G $\}$ and so $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$. Hence $0 \leq fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$.

□

Now, we characterize the graph G for which the bounds in Theorem 3.1 are reached and the graph for which $fdn_{\square_{fu}}(G) = 1$.

Theorem 3.2. *Let u be any vertex of a connected graph G. Then*

(i) $fdn_{\square_{fu}}(G) = 0$ if and only if G has a unique $dn_{\square_{fu}}$-set,

(ii) $fdn_{\square_{fu}}(G) = 1$ if and only if G has at least two $dn_{\square_{fu}}$-sets one of which is a unique $dn_{\square_{fu}}$-set containing any one of its elements,

(iii) $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G)$ if and only if no $dn_{\square_{fu}}$-set of G is the unique $dn_{\square_{fu}}$-set containing any of its proper subsets.

Proof. (i) Let $fdn_{\square_{fu}}(G) = 0$. Then, $fdn_{\square_{fu}}(S_u) = 0$ where S_u is any $dn_{\square_{fu}}$-set by definition and so the empty set is the minimum u-forcing subset for S_u. Since the empty set ϕ is a subset of every set we have S_u is the unique minimum u-forcing subset of G. The converse of this Theorem is obvious.

(ii) Let $fdn_{\square_{fu}}(G) = 1$. Then by (i), G has at least two $fdn_{\square_{fu}}$-sets. Also, since $fdn_{\square_{fu}}(G) = 1$, there is a singleton subset F of a $dn_{\square_{fu}}$-set S_u of G such that F is not a subset of any other $dn_{\square_{fu}}$-set of G. Thus S_u is the unique $dn_{\square_{fu}}$-set containing one of its elements. The converse is obvious.

(iii) Let $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}$. Then $fdn_{\square_{fu}}(S_u) = dn_{\square_{fu}}(G)$ for every $dn_{\square_{fu}}$-set S_u in G. By Theorem 2.1, $dn_{\square_{fu}}(G) \geq 1$ and so $fdn_{\square_{fu}}(G) = 1$. Also by (i), G has at least two $dn_{\square_{fu}}$-sets and hence the empty set ϕ is not a u-forcing subset of any $dn_{\square_{fu}}$ -set S_u of G is unique $dn_{\square_{fu}}$-set which consists of its proper subsets.

$\square$

Theorem 3.3. *Let $G = (V, E)$ be a connected graph and let S_u^* be the set of all u-square free detour vertices of G. Then $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G) - |S_u^*|$.*

Proof. Let S_u be any square free detour basis of G. Then $dn_{\square_{fu}}(S_u) = |S_u|, S_u^* \subseteq S_u$ and S_u is the unique square free detour basis containing $S_u - S_u^*$. Thus $fdn_{\square_{fu}}(G) \leq |S_u - S_u^*| = |S_u| - |S_u^*| = dn_{\square_{fu}}(G) - |S_u^*|$. $\square$

Remark 3.1. *The bound in Theorem 3.3 is sharp. For the graph G given in Figure 2, $fdn_{\square_{fv_1}}(G) = 0, |S_{v_1}^*| = 2$ and $dn_{\square_{fv_1}}(G) = 2$. Also, the inequality in Theorem 3.3, can be strict. For the graph G given in Figure 1, $fdn_{\square_{fv_3}}(G) = 1, |S_{v3}^*| = 0$ and $dn_{\square_{fv_3}}(G) = 2$. Thus $fdn_{\square_{fv_3}}(G) < dn_{\square_{fv_3}}(G) - |S_{v3}^*|$.*

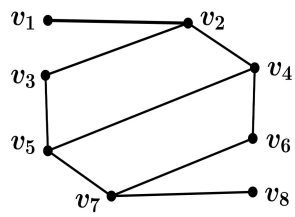

Figure 2: G

Theorem 3.4. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum u-forcing subsets in their respective minimum u-square free detour sets in G. Then $\cap_{F\in\Im}F$ is the set of all u-square free detour vertices of G.*

Proof. Let S_u^* be the set of all u-square free detour vertices of G. We claim that $S_u^* \subseteq \cap_{F\in\Im}F$. Let $x \in S_u^*$. Then x is a u-square free detour vertex of G so that x belongs to every u-square free detour set S_u of G. Let $T \subseteq S_u$ be any minimum u-forcing subset for any u-square free detour basis S_u of G. We claim that $x \notin T$.

If $x \in T$, then $T' = T - \{x\}$ is a proper subset of S_u is the unique u-square free detour containing T' so that T' is a u-forcing subset for S_u with $|T'| < |T|$, which is a contradiction to T is a minimum u-forcing subset for S_u. Thus $x \notin T$ and so $x \in F$, where F is the relative complement of T in S_u. Hence $x \in \cap_{F \in \Im} F$ so that $S_u^* \subseteq \cap_{F \in \Im} F$.
Conversely, let $x \in \cap_{F \in \Im} F$. Then x belongs to the relative complement of T in S_u for every T for every S_u such that $T \subset S_u$, where T is a minimum u-forcing subset for S_u. Since F is the relative complement of T in $S_u, F \subset S_u$ and so $x \in S_u$ for every S_u so that is a u-square free detour vertex of G. Thus $x \in S_u^*$ and so $\cap_{F \in \Im} F \subset S_u^*$. Hence $S_u^* \cap_{F \in \Im} F$.

□

Theorem 3.5. *Let u be any vertex of a connected graph G and let S_u be any $dn_{\square_{fu}}$ -set of G. Then*

(i) No u-square free detour vertex belongs to any minimum u-forcing subset of S_u.

(ii) No cut-vertex of G belongs to any minimum u-forcing subset of S_u.

Proof. (i) The proof follows from the first part of Theorem 3.4.

(ii) Since any minimum u-forcing subset of S_u is a subset of S_u, the proof follows from Theorem 2.1(ii).

□

Corolary 3.1. *Let u be any vertex of a connected graph G. If G contains l end-vertices, then $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G) - l + 1$.*

Proof. This follows from Theorems 2.1(i) and 3.5(i). □

Remark 3.2. *The bound in Corollary 3.1 is sharp. For a tree T with l end-vertices, $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G) - l + 1$ for any end-vertex u in T.*

Theorem 3.6. *Let G be any connected graph of order n. Then*

(i) If G is a tree with l end-vertices, then $fdn_{\square_{fu}}(G) = 0$ for every vertex l in G.

(ii) If G is the complete graph K_n, then

$$fdn_{\square_{fu}}(G) = \begin{cases} 0 & \text{if } n = 4 \\ n & \text{otherwise} \end{cases}$$

(iii) If G is the complete bipartite graph $K_{m,n}(2 \leq m \leq n)$, with partitions X and Y, $dn_{\square f_u}(K_{m,n}) = 0$ for every vertex u in $K_{m,n}$.

(iv) If G is the cycle C_n, then

$$fdn_{\square f_u}(G) = \begin{cases} 0 & \textit{if } n = 4 \\ 1 & \textit{otherwise} \end{cases}$$

(v) If G is the wheel W_n, then

$$fdn_{\square f_u}(W_n) = \begin{cases} 0 & \textit{if } n = 5, u \in K_1 \\ 1 & \textit{if } n = 4, u \in W_n \textit{ and } n \geq 6, u \in C_{n-1} \\ \left\lceil \dfrac{n-1}{3} \right\rceil & \textit{if } n \geq 6, u \in K_1. \end{cases}$$

Proof. (i) By the fact that $dn_{\square f_u}(G) = l - 1$ or $dn_{\square f_u}(G) = l$ when u is an end-vertex or not an end-vertex. Since the set of all end-vertices of a tree is the unique $dn_{\square f_u}$-set, the result follows from Theorem 3.2(i) that $fdn_{\square f_u}(G) = 0$.

(ii) By Theorem 2.3(i) for the complete graph K_4, S_u consists of the antipodal vertex of u. Since the set of antipodal vertex is unique for K_4, the result follows from Theorem 3.2(i) that $fdn_{\square f_u}(G) = 0$. For $K_n(n \neq 4)$ it follows from Theorem 2.3(i), that S_u consists of exactly one vertex of $V - \{u\}$. Thus there exist $n - 1$ distinct vertices other than u in K_n. Then the result follows from Theorem 3.2(ii) that $fdn_{\square f_u}(G) = 1$.

(iii) By Theorem 2.3(ii), for $K_{m,n}(2 \leq m \leq n)$ with partitions X, Y with $|X| = m$ and $|Y| = n$, we have $dn_{\square f_u}(G) = m - 1$ or $dn_{\square f_u}(G) = n - 1$ according to whether the vertex u lies in X or Y. Since the $dn_{\square f_u}$-set S_u is unique in both the cases, the result follows from Theorem 3.2(i) that $fdn_{\square f_u}(G) = 0$.

(iv) By Theorem 2.3(iii) for C_4, $dn_{\square f_u}$-set S_u consists of the antipodal vertex of u. Thus we observe that S_u is unique and so $fdn_{\square f_u}(G) = 0$ by Theorem 3.2(i). By Theorem 2.3(iii), for an even cycle $C_n(n \neq 4), S_u$ consists of exactly one vertex which is antipodal or adjacent to u. Also, for an odd cycle $C_n(n \geq 3) dn_{\square f_u}$-set S_u contains exactly one vertex which is adjacent to u. Since there exist two adjacent vertices for an odd cycle and in addition an antipodal vertex for an even cycle we have $fdn_{\square f_u}(G) = 1$.

(v) By Theorem 2.3(iv), for $W_n = K_1 + C_{n-1}(n \geq 5)$ a $dn_{\Box f_u}$-set S_u consists of $\left\lceil \frac{n-1}{3} \right\rceil$ vertices of the rim C_{n-1} of W_n, where u is a central vertex of W_n that is in K_1 called as hub. Thus there exist three different $dn_{\Box f_u}$-sets with $\left\lceil \frac{n-1}{3} \right\rceil$ vertices. Therefore, from Theorem 3.2(ii) the result follows that $fdn_{\Box f_u}(G) = \left\lceil \frac{n-1}{3} \right\rceil$ for the central vertex of $W_n(n \geq 5)$.

When u is a vertex on C_{n-1} of W_n by Theorem 2.3(iv), S_u contains exactly one adjacent or antipodal vertex on C_{n-1} with the hub of the wheel, according as $n-1$ is odd or $n-1$ is even. Thus there are two adjacent vertices for a vertex on an odd cycle and an antipodal vertex besides two adjacent vertices for a vertex on even cycle. Hence the result follows from Theorem 3.2(ii) that $fdn_{\Box f_u}(G) = 1$ for $u \in C_{n-1}(n \geq 6)$.

By Theorem 2.3(iv) for W_4, S_u consists of exactly one adjacent vertex for every $u \in W_4$. Thus there are $n-1$ such $dn_{\Box f_u}$-sets, for the vertices of W_4 are adjacent to each other. Hence by Theorem 3.2(ii), we have $fdn_{\Box f_u}(G) = 1$.

Also, for W_5, S_u contains two antipodal vertices of C_4 where u is the central vertex of W_5. Since there exist two different S_u with distinct pair of antipodal vertices of C_4, from Theorem 3.2(iii), the result follows that $fdn_{\Box f_u}(G) = dn_{\Box f_u}(G) = 2$. Furthermore, when u is a vertex on the rim of W_5, S_u consists of the antipodal vertex of u. Since there is only one antipodal vertex for any vertex u on C_4 of W_5, we have unique S_u for all vertices on the rim of W_5. Hence the result follows from Theorem 3.2(i) that $fdn_{\Box f_u}(G) = 0$.

□

Theorem 3.7. *For every pair α, β of positive integers with $0 \leq \alpha \leq \beta$ and $\beta \geq 2$, there exists a connected graph G with $fdn_{\Box f_u}(G) = \alpha$ and $dn_{\Box f_u}(G) = \beta$ for some vertex u in G.*

Proof. We consider two cases.
Case 1: Let $\alpha = 0$. Let G be any tree with $\beta + 1$ end-vertices. Then for an end-vertex u in $G, fdn_{\Box f_u}(G) = 0$ by Theorems 3.2(i) and 3.5(i).

Case 2: Let $\alpha \geq 1$. For each $i(1 \leq i \leq \alpha)$, let D_6^i be a Dutch Windmill graph consisting of i copies of $C_6 : u, p_i, q_i, r_i, s_i, t_i, u$. Let H be a graph obtained by adding α new vertices $r'_1, r'_2, ..., r'_\alpha$ and joining each $r'_i(1 \leq i \leq \alpha)$ to both the vertices q_i and s_i of $D_6^i(1 \leq i \leq \alpha)$.

Let $K_{1,\beta-\alpha}$ be the star with common vertex w_o and let $W = \{w_1, w_2, ..., w_{\beta-\alpha}\}$ be the set of all end-vertices of $K_{1,\beta-\alpha}$. Let G be the required graph produced by identifying the vertex u of D_6^i with the common vertex w_o of $K_{1,\beta-\alpha}$ as pictured in Figure 3.

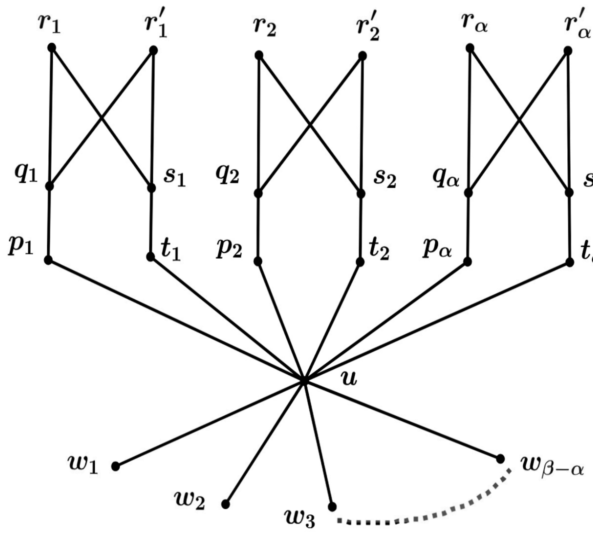

Figure 3: G

First we show that $dn_{\square_{fu}}(G) = \beta$ for some vertex u in G. By the fact that every $dn_{\square_{fu}}$-set of G contains W and exactly one vertex from each C_6 of $D_6^i(1 \leq i \leq \alpha)$. Then $dn_{\square_{fu}}(G)(\beta - \alpha) + \alpha$ and so $dn_{\square_{fu}}(G) = \beta$. Let $S_u = W \cup \{x_1, x_2, ..., x_\alpha\}$, where $x_j = p_j$ or t_j of $C_6^i(1 \leq i \leq \alpha)$. Clearly S_u is a $dn_{\square_{fu}}$-set of G and so $(G) \leq |S_u| = (\beta-\alpha)+\alpha = \beta$. Thus $dn_{\square_{fu}}(G) = \beta$.

Next we show that $fdn_{\square_{fu}}(G) = \alpha$. Since $dn_{\square_{fu}}(G) = \beta$, we find that every $dn_{\square_{fu}}$-set of G consists of W and exactly one vertex from each C_6 of $D_6^i(1 \leq i \leq \alpha)$. Let $F \subseteq S_u$ be any minimum u-forcing subset of S_u. Then By Theorem 3.5(ii),$F \subseteq S_u - W$ and so $|F| \leq \alpha$. If $|F| < \alpha$ then there is a vertex y_j of $C_6^i(1 \leq i \leq \alpha)$ distinct from $x_j(1 \leq i, j \leq \alpha)$. Then $S_u' = (S_u - x_j) \cup y_j$ is also a minimum u-forcing subset containing F such that $x_j \notin F$. Thus S_u is not a unique u-square free detour set which consists of F and so F is not a minimum u-forcing subset of S_u. Thus $fdn_{\square_{fu}}(G) = \alpha$.

□

4 Conclusion

In this paper, we computed the forcing vertex square free detour number of some standard graphs. We discussed the characteristics of the forcing vertex square free detour sets. Also, the relationship between the vertex square free detour number and the forcing vertex square free detour number has been exhibited. In future, this concept can be extended to edge related parameter. To derive similar results in the context of some other variants of detour number is the open area of research.

References

A. M. Ali and A. A. Ali. The connected detour numbers of special classes of connected graphs. *Journal of Mathematics*, 2019, 2019.

G. Chartrand, G. L. Johns, and S. Tian. Detour distance in graphs. In *Annals of discrete mathematics*, volume 55, pages 127–136. Elsevier, 1993.

G. Chartrand, G. L. Johns, and P. Zhang. The detour number of a graph. *Utilitas mathematica*, 64, 2003.

S. S. Ramalingam, I. K. Asir, and S. Athisayanathan. Vertex triangle free detour number of a graph. *Mapana Journal of Sciences*, 15(3):9–24, 2016.

K. C. Rani and G. P. Pacifica. Outer independent square free detour number of a graph. *Ratio Mathematica*, 44:309–315, 2022.

A. Santhakumaran and S. Athisayanathan. On the connected detour number of a graph. *Journal of Prime research in Mathematics*, 5:149–170, 2009.

A. Santhakumaran and S. Athisayanathan. Forcing edge detour number of an edge detour graph. *Journal of Prime Research in Mathematics*, 6:13–21, 2010.

P. Titus and P. Balakrishnan. The forcing vertex detour monophonic number of a graph. *AKCE International Journal of Graphs and Combinatorics*, 13(1): 76–84, 2016.