Pell Even Sum Cordial Labeling of Graphs

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Abstract

Let $G = (V, E)$ be a simple graph and let P_i be Pell numbers. For a bijection $f: V(G) \to \{P_0, P_1, \dots, P_{|V|-1}\}$, assign the label 1 for the edge $e = uv$ if $f(u) + f(v)$ is even and label 0 otherwise. Then f is said to be a Pell even sum cordial labeling of G if $\left|e_f(0) - e_f(1)\right| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with 0 and 1 respectively. If any graph admits Pell even sum cordial labeling, it is called Pell even sum cordial graph. In this study, we show that star, comb, bistar, jewel, crown, bipartite graph $K_{m,m}$, flower graph, helm, wheel, triangular book, $K_2 + mK_1$ are Pell even sum cordial.

Keywords: Cordial Labeling, Pell Numbers, Pell Even Sum Cordial Labeling.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected simple graph. We refer to Harary [5] for graph theory concepts and refer Gallian [4] for the literature on graph labeling. The Pell numbers are defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ where $P_0 = 0$ and $P_1 = 1$. The Pell number sequence is given as $0,1,2,5,12,\ldots$Cahit [2] is credited for inventing cordial labeling. Shiama [11] defined Pell labeling and shown that path, cycle, star, double star, coconut tree, bistar and $B_{m,n,k}$ are Pell graphs. Muthu Ramakrishnan and Sutha[8] proposed Pell graceful labeling as an extension of Fibonacci graceful labeling and demonstrated that cycle, path, olive tree, comb graph are Pell graceful graphs whereas complete graph and wheel graphs are not Pell graceful. Indira et al. [6] suggested some algorithms for the existence of Pell labeling in quadrilateral snake, extended duplicate graph of quadrilateral graph. Sriram et al. [12] investigated the Pell labeling for the joins of square of a path. Muthu Ramakrishnan and Sutha [7] also demonstrated that bistar, subdivision of bistar, caterpillar graphs, Jelly fish graph, star graph, coconut tree are pell graceful. Sharon Philomena and Thirusangu [10] shown that $< K_{1,n} : 2 >$ is a Pell graph. Avudainayaki and Selvam [1] shown that the extended duplicate graph of arrow graph and splitting graph of path admits harmonious and Pell labeling. Celine Mary et al. [3] demonstrated through an algorithm that inflation of alternate triangular snake graph of odd length in which the alternate block starts from the second vertex is a Pell graph. Muthu Ramakrishnan and Sutha [9] proposed the Pell square graceful labeling and proved that subdivision of the edges of a pathP_n in $P_n \odot K_1, < S_n: m >$, Olive tree, twig graph are Pell square graceful. Inspired by the concepts discussed above, the *Pell even sum cordial labeling* is being introduced here. It is defined as follows.

Definition 1.1 Let$G = (V, E)$ be a simple graph and let P_i be Pell numbers. For a bijection$f: V(G) \to \{P_0, P_1, \ldots, P_{|V|-1}\}$, assign the label 1 for the edge $e = uv$ if $f(u) + f(v)$ is even and label 0 otherwise. Then f is said to be a Pell even sum cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with 0 and 1 respectively. If any graph admits Pell even sum cordial labeling, it is called Pell even sum cordial graph.

Definition 1.2 The Comb $P_n \odot K_1$ is the graph created by adding a pendent edge to each vertex of a path.

Definition 1.3 The bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of $K_{1,n}$.

Definition 1.4 The jewel graph J_n is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i: 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i: 1 \leq i \leq n\}$.

Definition 1.5 The crown $C_n \odot K_1$ is the graph obtained from a cycle by attaching a pendent edge to each vertex of the cycle.

Definition 1.6 A complete bipartite is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph.

Definition 1.7 The helm is the graph obtained from a wheel graph by adjoining a pendent edge at each node of the cycle.

Definition 1.8 The Flower Fl_n is the graph obtained from a helm by attaching each pendent vertex to the apex of the helm.

Definition 1.9 The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in G_1, v \in G_2\}$.

Definition 1.10 One edge union of cycles of same length is called a book. The common edge is called the base of the book. If we consider t copies of cycles of length $n \geq 3$, then the book is denoted by B_n^t. If $n = 3,4,5$ or 6, the book B is called book with triangular, rectangular, pentagonal or hexagonal pages respectively.

Definition 1.11 The Wheel W_n can be defined as the graph join $K_1 + C_{n-1.}$

Definition 1.12 $K_{1,k}$is a tree with one internal node and k leaves.

2. Pell Even Sum Cordial Graphs

In this section, we prove that star, comb, bistar, jewel, crown, bipartite graph $K_{n,n}$, flower graph, helm, wheel, triangular book, $K_2 + mK_1$ are Pell even sum cordial.

Theorem 2.1. For $n \geq 2$, the comb is Pell even sum cordial graph

Proof. Recall that comb $G = (V, E)$ is a graph obtained from the path with vertices $u_1, u_2, \dots, u_n$ by joining a vertex v_i to each u_i where $1 \leq i \leq n$. Actually $|V(G)| = 2n$ and $|E(G)| = 2n - 1$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_{2n-1}\}$ defined by $f(u_i) = P_{2i-1}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-2}$ for $1 \leq i \leq n$. The induced edge labels are given $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n-1$ and $f^*(u_i v_i) = 0$ for $1 \leq i \leq n$. From this, $e_f(0) = n, e_f(1) = n - 1$ and so $|e_f(0) - e_f(0)| \leq 1$.

Therefore, for $n \geq 2$, the comb is a Pell even sum cordial graph.

Theorem 2.2. For $n \geq 2$, the bistar $B_{m,m}$ is a Pell even sum cordial.

Proof. Consider the bistar $G = B_{m,m}$. Then $V(G) = \{u, v, u_i, v_i: 1 \leq i \leq m\}, E(G) = \{uv, uu_i, vv_i : 1 \leq i \leq m\}, |V(G)| = 2m + 2$ and $|E(G)| = 2m + 1$. Consider

$f: V(G) \to \{P_0, P_1, \dots, P_{2m+1}\}$ defined by $f(u) = P_0, f(v) = P_1, f(u_i) = P_{i+1}$ for $1 \leq i \leq m-1$ and $f(v_i) = P_{m+i}$ for $1 \leq i \leq m$. The induced edge labels are given by $f^*(uv) = 0$.

For $1 \leq i \leq m-1, f^*(uu_i) = \begin{cases} 1 & if \quad i \equiv 1 \ (mod\ 2); \\ 0 & if \quad i \equiv 0 \ (mod\ 2). \end{cases}$

For $1 \leq i \leq m, f^*(vv_i) = \begin{cases} 1 & if \quad i \equiv 1 \ (mod\ 2); \\ 0 & if \quad i \equiv 0 \ (mod\ 2). \end{cases}$

When m is odd, $e_f(1) = m+1, e_f(0) = m$ and on the otherhand, when m is even, $e_f(1) = m$, $e_f(0) = m+1$. Therefore, for $n \geq 2$, bistar $B_{m,m}$ is Pell even sum cordial.

Theorem 2.3. For $n \geq 1$, the jewel graph J_n is Pell even sum cordial.

Proof. Consider the jewel graph $G = J_n$. Here $V(G) = \{u, v, x, y, u_i: 1 \leq i \leq n\}, E(G) = \{ux, uy, xy, xv, yv, uu_i, vv_i: 1 \leq i \leq n\}, |V(G)| = n+4$ and $|E(G)| = 2n+5$.

Let $f: V(G) \to \{P_0, P_1, \dots, P_{n+3}\}$ be defined by $f(u) = P_0, f(v) = P_1, f(x) = P_2$ and $f(u_i) = P_{i+3}$ for $1 \leq i \leq n$. Then the induced edge labels are given by,$f^*(uv) = 1, f^*(uy) = 1, f^*(xy) = 0, f^*(vx) = 0$ and $f^*(vy) = 1$

For $1 \leq i \leq n$,

$f^*(uu_i) = \begin{cases} 1 & if \quad i \equiv 0 \ (mod\ 2); \\ 0 & if \quad i \equiv 1 \ (mod\ 2). \end{cases}$

$f^*(vv_i) = \begin{cases} 1 & if \quad i \equiv 1 \ (mod\ 2); \\ 0 & if \quad i \equiv 0 \ (mod\ 2). \end{cases}$

From the above, $e_f(0) = n+3, e_f(1) = n+2$ and so$\left|e_f(0) - e_f(0)\right| \leq 1$. Therefore, for $n \geq 1$, the jewel graph J_n is Pell even sum cordial.

Theorem 2.4 For $n \geq 3$, the crown $C_n \odot K_1$ is Pell even sum cordial.

Proof. Consider the crown $G = C_n \odot K_1$. Here $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_i u_{i+1}: 1 \leq i \leq n-1;\ u_n u_1, u_i v_i : 1 \leq i \leq n\}, |V(G)| = 2n = |E(G)|$.

Consider $f: V(G) \to \{P_0, P_1, \dots, P_{2n-1}\}$ defined by $f(u_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-1}$ for $1 \leq i \leq n$. Then the induced edge labels are given by $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n, f^*(u_n u_1) = 1$ and $f^*(u_i v_i) = 0$ for $1 \leq i \leq n$.

From the above, $e_f(0) = n+3, e_f(1) = n+2$ and so $\left|e_f(0) - e_f(0)\right| \leq 1$. Therefore, for $n \geq 1$, the crown $C_n \odot K_1$ is Pell even sum cordial.

Theorem 2.5 For $n \geq 2$, the complete bipartite graph $K_{n,n}$ is Pell even sum cordial.

Proof. Let $G = K_{n,n}$. Let the partitions of vertex set be $V_1 = \{u_1, u_2, \dots, u_n\}$and $V_2 = \{v_1, v_2, \dots, v_n\}$. Hence $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$and $E(G) = \{u_i v_{\boldsymbol{i}} : 1 \leq i \leq n\}$. Then $|V(G)| = 2n$, $|E(G)| = n^2$.

Consider $f: V(G) \to \{P_0, P_1, \dots, P_{2n-1}\}$defined by $f(u_i) = P_{i-1}$for $1 \leq i \leq n$ and$f(v_i) = P_{(n-1)+i}$ for $1 \leq i \leq n$. The induced edge labels are given by, For $1 \leq i \leq n$,

$f^*(u_i v_i) = \begin{cases} 1 \ \ if \ \ \ i \equiv 1{,}0 \ (mod \ 2); \\ 0 \ \ \ \ \ \ \ \ \ \ \ otherwise. \end{cases}$

When n is odd, $e_f(0) = \frac{n^2+1}{2}, e_f(1) = \frac{n^2-1}{2}$ and when, n is even, $e_f(0) = e_f(1) = \frac{n^2}{2}$. Therefore, for $n \geq 2$, the complete bipartite graph $K_{n,n}$ is Pell even sum cordial.

Theorem 2.6 For $n \geq 3$, the flower graph Fl_n is Pell even sum cordial.

Proof. Let $G = Fl_n$. Then $V(G) = \{v, u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_i u_i, vu_i : 1 \leq i \leq n \, ; \, v_n v_1 \, ; \, v_i v_{i+1} : 1 \leq i \leq n-1\}$. Then,$|V(G)| = 2n+1, |E(G)| = 4n$. Consider $f: V(G) \to \{P_0, P_1, \dots P_{2n}\}$ defined by $f(v_n) = P_{2n}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(u_i) = P_{2i-1}$ for $1 \leq i \leq n$. The induced edge labels are given by $f^*(v_i v_{i+1}) = 1, f^*(v_i u_i) = 0, f^*(vu_i) = 0, f^*(vv_i) = 1$ and $f^*(v_n v_1) = 1$ for $1 \leq i \leq n$.

From the above, $e_f(0) = e_f(1) = 2n$ and so $|e_f(0) - e_f(0)| \leq 1$. Therefore, the flower graph Fl_n is Pell even sum cordial for $n \geq 3$.

Theorem 2.7 For $n \geq 3$ and n is even, the Helm H_n is Pell even sum cordial.

Proof. Let $G = H_n$. Then $V(G) = \{v, u_i, v_i : 1 \leq i \leq n\}, E(G) = \{u_i u_{i+1}: 1 \leq i \leq n-1 \, ; \, u_i v_i : 1 \leq i \leq n \, ; vu_i : 1 \leq i \leq n-1 \, ; \, u_1 u_n\}$, $|V(G)| = 2n+1, |E(G)| = 3n$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_{2n}\}$ defined by $f(u_i) = P_i$ for $1 \leq i \leq n-1, f(v_i) = P_{n+i}$ for $1 \leq i \leq n$ and $f(v) = 0$. Then the induced edge labels are given by,

For $1 \leq i \leq n-1$,

$f^*(u_i u_{i+1}) = 0, f^*(v_i u_i) = 1, f^*(u_n u_1) = 0$ and $f^*(vu_i) = \begin{cases} 0 \ \ if \ \ \ i \equiv 1 \ (mod \ 2); \\ 1 \ \ if \ \ \ i \equiv 0 \ (mod \ 2). \end{cases}$

From the above, $e_f(0) = e_f(1) = \frac{3n}{2}$ and so $|e_f(0) - e_f(0)| \leq 1$. Therefore, for an even integer ≥ 3 , the Helm H_n is Pell even sum cordial.

Theorem 2.8 For $n \geq 4$, the wheel graph W_n is Pell even sum cordial.

Proof. Let $G = W_n$. Then $V(G) = \{u_0, u_1, u_2, \dots, u_n\}$ and $(G) = \{u_i u_{i+1}: 1 \leq i \leq n-1 \, ; \, u_0 v_i : 1 \leq i \leq n-1\}, |V(G)| = n+1$ and $|E(G)| = 2n-2$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_{2n}\}$ defined by $f(u_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(u_0) = P_1$. Then the induced edge labels are given by $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n-1, f^*(u_n u_1)$ =1 and $f^*(u_0 u_i) = 0$ for $1 \leq i \leq n$. From this, $e_f(0) = e_f(1) = n-1$
and so $|e_f(0) - e_f(0)| \leq 1$. Therefore, for $n \geq 4$, the wheel graph W_n is Pell even sum cordial.

Theorem 2.9 For $n \geq 2$, the star graph $K_{1,n}$ is Pell even sum cordial.

Proof. Let $G = K_{1,n}$. Then $V(G) = \{v, u_1, u_2, \dots, u_n\}$ and $E(G) = \{vv_i : 1 \leq i \leq n\}$, $|V(G)| = n+1$ and $|E(G)| = n$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_n\}$ defined by $f(v) = P_0, f(u_i) = P_i$ for $1 \leq i \leq n-1$. The induced edge labels are given by, for $1 \leq i \leq n, f^*(vu_i) = \begin{cases} 1 \ \ if \ \ \ i \equiv 0 \ (mod \ 2); \\ 0 \ \ if \ \ \ i \equiv 1 \ (mod \ 2). \end{cases}$

When n is *odd*, $e_f(0) = \frac{n+1}{2}, e_f(1) = \frac{n-1}{2}$ whereas when n is *even*,$e_f(0) = \frac{n}{2} = e_f(1)$.Therefore, for $n \geq 2$, the star graph $K_{1,n}$ is Pell even sum cordial.

Theorem 2.10 For $n \geq 3$, the triangular book TB_n is Pell even sum cordial.
Proof. Let $G = TB_n$. Then $V(G) = \{v_1, v_2, \dots, v_{n-1}, v_n\}$, $E(G) = \{v_1 v_i , v_n v_i , v_n v_1 : 2 \leq i \leq n-1\}$, $|V(G)| = n$ and $|E(G)| = 2n-3$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_{n-1}\}$ defined by $f(v_1) = P_0$, $f(v_n) = P_1$ and $f(v_i) = P_{i+1}$ for $1 \leq i \leq n-2$. Then the induced edge labels are given by $f^*(v_1 v_n) = 0$.
For $1 \leq i \leq n-2$,

$$f^*(v_1 v_i) = \begin{cases} 1 & if \quad i \equiv 0\ (mod\ 2); \\ 0 & if \quad i \equiv 1\ (mod\ 2). \end{cases}$$

$$f^*(v_n v_i) = \begin{cases} 1 & if \quad i \equiv 1\ (mod\ 2); \\ 0 & if \quad i \equiv 0\ (mod\ 2). \end{cases}$$

From the above, $e_f(0) = n-1, e_f(1) = n-2$ and so $\left|e_f(0) - e_f(0)\right| \leq 1$. Therefore, for $n \geq 3$, the triangular book TB_n is Pell even sum cordial.
Theorem 2.11. For $m \geq 3$, the graph $K_2 + mK_1$ is Pell even sum cordial.
Proof. Let $G = K_2 + mK_1$. Then $(G) = \{u_1, u_2, v_1, v_2 \dots, v_m\}$, $E(G) = \{u_1 u_2, u_1 v_i, u_2 v_i : 1 \leq i \leq m\}$, $|V(G)| = m+2$ and $|E(G)| = 2m+1$. Consider $f: V(G) \to \{P_0, P_1, \dots, P_{m+1}\}$ defined by $f(u_1) = P_0, f(u_2) = P_1$ and $f(v_i) = P_{i+1}$ for $1 \leq i \leq m$. Then the induced edge labels are given by $f^*(u_1 u_2) = 0$.
For $1 \leq i \leq m$,

$$f^*(u_1 v_i) = \begin{cases} 1 & if \quad i \equiv 1\ (mod\ 2); \\ 0 & if \quad i \equiv 0\ (mod\ 2). \end{cases}$$

$$f^*(u_2 v_i) = \begin{cases} 1 & if \quad i \equiv 0\ (mod\ 2); \\ 0 & if \quad i \equiv 1\ (mod\ 2). \end{cases}$$

From the above, $e_f(0) = m+1, e_f(0) = m$ and so $\left|e_f(0) - e_f(0)\right| \leq 1$. Therefore, for $m \geq 3$, the graph $K_2 + mK_1$ is Pell even sum cordial.

3. Conclusions

This article introduces a novel graph labeling technique called Pell even sum cordial labeling and it is demonstrated that several typical graphs are Pell even sum cordial. More Pell even sum graphs will be provided in the following article.

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